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POLYNOMIAL AND RATIONAL INTEGRABILITY OF POLYNOMIAL HAMILTONIAN SYSTEMS

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Abstract. Within the class of canonical polynomial Hamiltonian systems anti-symmetric under phase-space involutions, we generalize some results on the existence of Darboux polynomial and rational first integrals for "kinetic plus potential" systems to general systems.

1. Introduction and statement of the main results

This note concerns the integrability of canonical polynomial Hamiltonian systems. Usually the integrability of these kind of Hamiltonian systems is considered using Ziglin's approach [8] or differential Galois theory [5], but here we use the Darboux theory of integrability [1]. Our findings are generalisations of some results presented by Maciejewski et al. in [7, 6], and Garcia at el. in [2].

A natural class of canonical Hamiltonian systems is given by systems expressed as sum of the kinetic and potential terms

$$H(q,p)=\frac{1}{2}\sum_{i=1}^{m}\mu_i p_i^2+V(q), \tag{1.1}$$

where $q,p\in\mathbb{C}^m$, and $\mu_i\in\mathbb{C}$ for $i=1,\dots,m$. In what follows we observe that certain statements on polynomial Hamiltonians of the form (1.1) obtained in [2] generalize to time-reversible Hamiltonian systems with an arbitrary polynomial Hamiltonian $H(q,p)$. For such systems, under convenient assumptions, we deduce the existence of a second polynomial first integral independent of the Hamiltonian.

Further, we consider polynomial Hamiltonian systems together with anti-symmetric under involutions $(q,p)\to(-q,p)$. In this case we obtain a second polynomial or rational first integral independent of the Hamiltonian.

A canonical Hamiltonian system with m degrees of freedom and Hamiltonian $H(q,p)$ is given by

$$\frac{dq_i}{dt}=\frac{\partial H(q,p)}{\partial p_i},\quad \frac{dp_i}{dt}=-\frac{\partial H(q,p)}{\partial q_i},\quad \text{for } i=1,\dots,m, \tag{1.2}$$

where $q=(q_1,\dots,q_m)\in\mathbb{C}^m$ and $p=(p_1,\dots,p_m)\in\mathbb{C}^m$ are the generalized coordinates and momenta, respectively.

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We denote by X_H the associated Hamiltonian vector field in $\mathbb{C}^{2m}$ to the Hamiltonian system (1.2); i.e.,

$$X_H = \sum_{i=1}^{m} \frac{\partial H(q,p)}{\partial p_i}\frac{\partial}{\partial q_i} - \sum_{i=1}^{m} \frac{\partial H(q,p)}{\partial q_i}\frac{\partial}{\partial p_i}. \tag{1.3}$$

Let U be an open subset of $\mathbb{C}^{2m}$, such that its closure is $\mathbb{C}^{2m}$. Then, a function $I : U \to \mathbb{C}^{2m}$ constant on the orbits of the Hamiltonian vector field X_H contained in U is called a *first integral* of X_H, i.e. $X_H I \equiv 0$ on U. It is immediate that H is a first integral of the vector field X_H.

A non-constant polynomial $F \in \mathbb{C}[q,p]$ is a *Darboux polynomial* of the polynomial Hamiltonian vector field X_H if there exists a polynomial $K \in \mathbb{C}[q,p]$, called the *cofactor* of F, such that $X_H F = KF$. We say that F is a *proper* Darboux polynomial if its cofactor is not zero, i.e. if F is not a polynomial first integral of X_H.

One may check directly from the definition of a Darboux polynomial F that the hypersurface $F(q,p) = 0$ defined by a Darboux polynomial is invariant by the flow of X_H, i.e., if an orbit of the vector field X_H has a point on that hypersurface, then the whole orbit is contained in it.

The Darboux polynomials where introduced by Darboux [1] in 1878 for studying the existence of first integrals in the polynomial differential systems in $\mathbb{C}^m$. His original ideas have been developed by many authors; see the survey [3] and the paper [4] with the references therein on the recent result on the Darboux theory of integrability.

We say that a function $G(q,p)$ is *even* with respect to the variable q if $G(q,p) = G(-q,p)$, and we say that it is *odd* with respect to the variable q if $G(q,p) = -G(-q,p)$. An analogous definition applies for G being even or odd with respect to the variable p.

2. Involutions with respect to momenta

In general, a (smooth) involution is a (smooth) map f such that $f \circ f = Id$,, where Id is the identity. In our context, consider the involution given by the diffeomorphism $\tau : \mathbb{C}^{2m} \to \mathbb{C}^{2m}$, $\tau(q,p) := (q,-p)$. The vector field X_H on $\mathbb{C}^{2m}$ is said to be τ*–reversible* if $\tau_*(X_H) = -X_H$, where τ_* is the push–forward associated to the diffeomorphism τ. This is the case when

$$\frac{\partial H(q,-p)}{\partial p_i} = -\frac{\partial H(q,p)}{\partial p_i} \quad \text{and} \quad \frac{\partial H(q,-p)}{\partial q_i} = \frac{\partial H(q,p)}{\partial q_i}.$$

For instance, systems of the form (1.1) fulfill these conditions.

Theorem 2.1. *Consider a polynomial Hamiltonian $H(q,p)$ such that its corresponding Hamiltonian vector field* (1.3) *is τ-reversible. Let $F(q,p)$ be a proper Darboux polynomial of the Hamiltonian vector field X_H with a cofactor $K(q,p)$ which is an even function with respect to the variable p. Then $F(q,p)F(q,-p)$ is a polynomial first integral of X_H.*

To prove the above theorem, we need the following result.

Lemma 2.2. *Under the assumptions of Theorem 2.1, we have that $F(q,-p)$ is another proper Darboux polynomial of X_H with cofactor $-K(q,-p)$.*

Proof. Since
$$X_H F(q,p) = K(q,p)F(q,p),$$
we have
$$\tau_*(X_H F)(q,p) = \tau_*(K \cdot F)(q,p)\,.$$
In the relation above, the left hand side is
$$\begin{aligned}\tau_*(X_H F)(q,p) &= \tau_*(X_H)\tau_*(F)(q,p) = -X_H F\left(\tau^{-1}(q,p)\right)\\ &= -X_H F\left(\tau(q,p)\right) = -X_H F(q,-p)\end{aligned}\tag{2.1}$$
where we used that $\tau^{-1} = \tau$. The right hand side is
$$\begin{aligned}\tau_*(K \cdot F)(q,p) &= \left((K \cdot F) \circ \tau^{-1}\right)(q,p) = \left((K \cdot F) \circ \tau\right)(q,p)\\ &= (K \cdot F)(q,-p) = K(q,-p) \cdot F(q,-p)\end{aligned}\tag{2.2}$$
Since (2.1) equals (2.2) we obtain
$$X_H F(q,-p) = -K(q,-p)F(q,-p).$$
So $F(q,-p)$ is a proper Darboux polynomial of X_H with cofactor $-K(q,-p) \neq 0$, because $K(q,p) \neq 0$ due to the fact that $F(q,p)$ is a proper Darboux polynomial. ◻

Proof of Theorem 2.1. Under the assumptions of Theorem 2.1 we have $X_H F(q,p) = K(q,p)F(q,p)$ with $K(q,p) \neq 0$. By Lemma 2.2 we have that $X_H F(q,-p) = -K(q,-p)F(q,-p)$. Therefore,
$$\begin{aligned}X_H(F(q,p)F(q,-p)) &= X_H(F(q,p))F(q,-p) + F(q,p)X_H(F(q,-p))\\ &= K(q,p)F(q,p)F(q,-p) + F(q,p)(-K(q,-p)F(q,-p))\\ &= (K(q,p) - K(q,-p))F(q,p)F(q,-p).\end{aligned}$$
This last expression is zero due to the fact that the cofactor $K(q,p)$ is an even function in the variable p. So $F(q,p)F(q,-p)$ is a polynomial first integral of Hamiltonian vector field X_H. ◻

Corollary 2.3. *Consider a polynomial Hamiltonian $H(q,p)$ given by* (1.1). *Let $F(q,p)$ be a proper Darboux polynomial of the Hamiltonian vector field X_H. Then $F(q,p)F(q,-p)$ is a polynomial first integral of X_H.*

A proof of the above corollary can be found in [2, Theorem 3]; We omit it.

A Hamiltonian system is called *time-reversible* if for any integral curve $(q(t),p(t))$ of X_H we have $(q(-t),p(-t)) = (q(t),-p(t))$. In the configurations space this means that whenever we have a trajectory $q(t)$ then $q(-t)$ is also a trajectory. Note that time-reversibility is equivalent to the invariance of the flow under involutions acting on the independent variable (time) as well; i.e., $(q,p,t) \to (q,-p,-t)$. In this context, Theorem 2.1 may be extended as follows:

Theorem 2.4. *Let $H(q,p)$ be a time-reversible polynomial Hamiltonian system and assume that $F(q,p)$ is a proper Darboux polynomial of the Hamiltonian vector field X_H with a cofactor $K(q,p)$ such that $K \circ \tau = K$. Then $F \cdot (F \circ \tau)$ is a polynomial first integral of X_H.*

The proof of Theorem 2.4 is similar to the proof of Theorem 2.1. We omit it.

3. Involutions with respect to coordinates

Let $\hat{\tau}\colon \mathbb{C}^{2m} \to \mathbb{C}^{2m}$ be the involution $\hat{\tau}(q,p) = (-q,p)$. The vector field X_H or the Hamiltonian system (1.2) on $\mathbb{C}^{2m}$ is *$\hat{\tau}$-equivariant* if the Hamiltonian system (1.2) is invariant under $\hat{\sigma}$, that is $\hat{\tau}_*(X_H) = -X_H$. This is the case when

$$\frac{\partial H(-q,p)}{\partial p_i} = \frac{\partial H(q,p)}{\partial p_i} \quad \text{and} \quad \frac{\partial H(-q,p)}{\partial q_i} = -\frac{\partial H(q,p)}{\partial q_i}$$

Theorem 3.1. *Consider a polynomial Hamiltonian $H(q,p)$ such that its corresponding Hamiltonian vector field (1.3) is $\hat{\tau}$-equivariant. Let $F(q,p)$ be a proper Darboux polynomial of the Hamiltonian vector field X_H with a cofactor $K(q,p)$. Then the following statements hold.*

- (a) *If $K(q,p)$ is an even function with respect to q, then $F(-q,p)F(q,p)$ is a polynomial first integral of X_H.*
- (b) *If $K(q,p)$ is an odd function with respect to q, then $F(-q,p)/F(q,p)$ is a rational first integral of X_H.*

To prove the above theorem we need the following result:

Lemma 3.2. *Under the assumptions of Theorem 3.1 we have that $F(-q,p)$ is another proper Darboux polynomial of X_H with cofactor $-K(-q,p)$.*

Proof. From the definition of $\hat{\tau}_*$ it follows that $\hat{\tau}_*(X_H) = -X_H$. This implies that

$$\hat{\tau}_*(X_H F) = -X_H \hat{\tau}(F) = -X_H F(-q,p). \tag{3.1}$$

Moreover, we have that $X_H F = KF$ and thus

$$\hat{\tau}_*(X_H F) = \hat{\tau}_*(KF) = \hat{\tau}_*(K)\hat{\tau}_*(F) = K(-q,p)F(-q,p). \tag{3.2}$$

Combining equations (3.1) and (3.2) we obtain

$$X_H F(-q,p) = -K(-q,p)F(-q,p).$$

Therefore, $F(-q,p)$ is a proper Darboux polynomial of X_H with cofactor $-K(-q,p)$. We note that $K(-q,p) \neq 0$ due to the fact that $F(-q,p)$ is a proper Darboux polynomial and consequently $K(q,p) \neq 0$. □

Proof of Theorem 3.1. Under the assumptions of Theorem 3.1 we have $X_H F(q,p) = K(q,p)F(q,p)$ with $K(q,p) \neq 0$. By Lemma 3.2 we have that $X_H F(-q,p) = -K(-q,p)F(-q,p)$. Therefore,

$$\begin{aligned} X_H(F(-q,p)F(q,p)) &= X_H(F(-q,p))F(q,p) + F(-q,p)X_H(F(q,p)) \\ &= -K(-q,p)F(-q,p)F(q,p) + F(-q,p)K(q,p)F(q,p) \\ &= (-K(-q,p) + K(q,p))F(q,-p)F(q,p). \end{aligned}$$

If K is an even function in the variable q, the last expression is zero. So, in this case, $F(-q,p)F(q,p)$ is a polynomial first integral of the Hamiltonian vector field X_H. This completes the proof of statement (a).

On the other hand,

$$\begin{aligned} X_H(F(-q,p)/F(q,p)) &= \frac{X_H(F(-q,p))F(q,p) - F(-q,p)X_H(F(q,p))}{F(q,p)^2} \\ &= \frac{-K(-q,p)F(-q,p)F(q,p) - F(-q,p)K(q,p)F(q,p)}{F(q,p)^2} \end{aligned}$$

$$= -(K(-q,p) + K(q,p))\frac{F(-q,p)}{F(q,p)}.$$

If K is an odd function in the variable q, the last expression is zero. So, in this case, $F(-q,p)/F(q,p)$ is a rational first integral of the Hamiltonian vector field X_H. This completes the proof of the theorem. □

It is natural to extend Theorem 3.1 to involutions acting on the independent variable (time) of the form $(q,p,t) \to (-q,p,-t)$, under which the flow is invariant. In this case, whenever $(q(t),p(t))$ is an integral curve, so is $(-q(-t),p(-t))$.

Theorem 3.3. *Consider a polynomial Hamiltonian $H(q,p)$ such that its flow is invariant under $(q,p,t) \to (-q,p,-t)$. Let $F(q,p)$ be a proper Darboux polynomial of the Hamiltonian vector field X_H with a cofactor K. Then the following statements hold.*

- (a) *If K is such that $K \circ \hat{\tau} = K$, then $F \cdot (F \circ \hat{\tau})$ is a polynomial first integral of X_H.*
- (b) *If K is such that $K \circ \hat{\tau} = -K$, then $(F \circ \hat{\tau})/F$ is a rational first integral of X_H.*

The proof of Theorem 3.3 is the same as the proof of Theorem 3.1. we omit it.

Proposition 3.4. *Consider a polynomial Hamiltonian $H(q,p)$ given by* (1.1)*, where $V(q)$ is even. Let $F(q,p)$ be a proper Darboux polynomial of the Hamiltonian vector field X_H with cofactor K. Then the following statements hold.*

- (a) *If K is an even function in the variable q, then $F(-q,p)F(q,p)$ is a polynomial first integral of X_H.*
- (b) *If K is an odd function in the variable q, then $F(-q,p)/F(q,p)$ is a rational first integral of X_H.*

To prove Proposition 3.4 we recall the following result whose proof can be found in [2].

Lemma 3.5. *Let $F(q,p)$ be a proper Darboux polynomial of the Hamiltonian vector field X_H associated to the Hamiltonian H given by* (1.1)*. Then its cofactor is a polynomial of the form $K(q)$.*

Proof of Proposition 3.4. If $F(q,p)$ is a proper Darboux polynomial of the Hamiltonian vector field X_H, by Lemma 3.5 we have that its cofactor is of the form $K(q)$. Then, if K is an even function in the variable q then the Hamiltonian vector field X_H satisfies all the assumptions of Theorem 3.1(a), and consequently $F(-q,p)F(q,p)$ is a polynomial first integral of X_H. On the other hand, if K is an odd function in the variable q then the Hamiltonian vector field X_H satisfies all the assumptions of Theorem 3.1(b), and consequently $F(-q,p)/F(q,p)$ is a rational first integral of X_H. This completes the proof. □

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