

EXISTENCE AND UNIQUENESS OF SOLUTIONS TO FUNCTIONAL INTEGRO-DIFFERENTIAL FRACTIONAL EQUATIONS

MOULAY RCHID SIDI AMMI, EL HASSAN EL KINANI, DELFIM F. M. TORRES

ABSTRACT. Using a fixed point theorem in a Banach algebra, we prove an existence result for a fractional functional differential equation in the Riemann-Liouville sense. Dependence of solutions with respect to initial data and an uniqueness result are also obtained.

1. INTRODUCTION

Fractional Calculus is a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. The subject has its origin in the 1600s. During three centuries, the theory of fractional calculus developed as a pure theoretical field, useful only for mathematicians. In the last few decades, however, fractional differentiation proved very useful in various fields of applied sciences and engineering: physics (classic and quantum mechanics), chemistry, biology, economics, signal and image processing, calculus of variations, control theory, electrochemistry, viscoelasticity, feedback amplifiers, and electrical circuits [4, 6, 8, 9, 15, 18, 20, 22, 23]. The “bible” of fractional calculus is the book of Samko, Kilbas and Marichev [20].

Several definitions of fractional derivatives are available in the literature, including the Riemann-Liouville, Grunwald-Letnikov, Caputo, Riesz, Riesz-Caputo, Weyl, Hadamard, and Chen derivatives [11, 12, 13, 16, 17, 20]. The most common used fractional derivative is the Riemann-Liouville [10, 17, 20, 21], which we adopt here. It is worth to mention that functions that have no first order derivative might have Riemann-Liouville fractional derivatives of all orders less than one [17]. Recently, the physical meaning of the initial conditions to fractional differential equations with Riemann-Liouville derivatives has been discussed [7, 14, 16].

Using a fixed point theorem, like Schauder’s fixed point theorem, and the Banach contraction mapping principle, several results of existence have been obtained in the literature to linear and nonlinear equations, and recently also to fractional differential equations. The interested reader is referred to [1, 2, 3, 19].

Let $I_0 = [-\delta, 0]$ and $I = [0, T]$ be two closed and bounded intervals in \mathbb{R} ; $B(I, \mathbb{R})$ be the space of bounded real-valued functions on I ; and $C = C(I_0, \mathbb{R})$ be the space

2000 *Mathematics Subject Classification.* 26A33, 47H10.

Key words and phrases. Riemann-Liouville operator; fractional calculus; fixed point theorem; Lipschitz condition.

©2012 Texas State University - San Marcos.

Submitted January 27, 2012. Published June 20, 2012.

of continuous real valued functions on I_0 . Given a function $\phi \in C$, we consider the functional integro-differential fractional equation

$$\frac{d^\alpha}{dt^\alpha} \left[\frac{x(t)}{f(t, x(t))} \right] = g\left(t, x_t, \int_0^t k(s, x_s) ds\right) \quad \text{a.e., } t \in I, \quad (1.1)$$

subject to

$$x(t) = \phi(t), \quad t \in I_0, \quad (1.2)$$

where d^α/dt^α denotes the Riemann-Liouville derivative of order α , $0 < \alpha < 1$, and $x_t : I_0 \rightarrow \mathbb{C}$ is the continuous function defined by $x_t(\theta) = x(t + \theta)$ for all $\theta \in I_0$, under suitable mixed Lipschitz and other conditions on the nonlinearities f and g . For a motivation to study such type of problems we refer to [1]. Here we just mention that problems of type (1.1)–(1.2) seem important in the study of dynamics of biological systems [1].

Our main aim is to prove existence of solutions for (1.1)–(1.2). This is done in Section 3 (Theorem 3.1). Our main tool is a fixed point theorem that is often useful in proving existence results for integral equations of mixed type in Banach algebras [5], and which we recall in Section 2 (Theorem 2.2). We end with Section 4, by proving dependence of the solutions with respect to their initial values (Theorem 4.1) and, consequently, uniqueness to (1.1)–(1.2) (Corollary 4.2).

2. PRELIMINARIES

In this section we give the notations, definitions, hypotheses and preliminary tools, which will be used in the sequel. We deal with the (left) Riemann-Liouville fractional derivative, which is defined in the following way.

Definition 2.1 ([20]). The fractional integral of order $\alpha \in (0, 1)$ of a function $f \in L^1[0, T]$ is defined by

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds,$$

$t \in [0, T]$, where Γ is the Euler gamma function. The Riemann-Liouville fractional derivative operator of order α is then defined by

$$\frac{d^\alpha}{dt^\alpha} := \frac{d}{dt} \circ I^{1-\alpha}.$$

In this article, X denotes a Banach algebra with norm $\|\cdot\|$. The space $C(I, \mathbb{R})$ of all continuous functions endowed with the norm $\|x\| = \sup_{t \in I} |x(t)|$ is a Banach algebra. To prove the existence result for (1.1)–(1.2), we shall use the following fixed point theorem.

Theorem 2.2 ([5]). Let $B_r(0)$ and $\overline{B_r(0)}$ be, respectively, open and closed balls in a Banach algebra X centered at origin 0 and of radius r . Let $A, B : \overline{B_r(0)} \rightarrow X$ be two operators satisfying:

- (a) A is Lipschitz with Lipschitz constant L_A ,
- (b) B is compact and continuous, and
- (c) $L_A M < 1$, where $M = \|B(\overline{B_r(0)})\| := \sup\{\|Bx\|; x \in \overline{B_r(0)}\}$.

Then, either

- (i) the equation $\lambda[AxBx] = x$ has a solution for $\lambda = 1$, or
- (ii) there exists $x \in X$ such that $\|x\| = r$, $\lambda[AxBx] = x$ for some $0 < \lambda < 1$.

Throughout this article, we assume the following hypotheses:

- (H1) The function $f : I \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$ is Lipschitz and there exists a positive constant L such that for all $x, y \in \mathbb{R}$, $|f(t, x) - f(t, y)| \leq L|x - y|$ a.e., $t \in I$.
- (H2) The function $k : I \times C \rightarrow \mathbb{R}$ is continuous and there exists a function $\beta \in L^1(I, \mathbb{R}^+)$ such that $|k(s, y)| \leq \beta(s)\|y\|$ a.e., $s \in I, y \in C$.
- (H3) There exists a continuous function $\gamma \in L^\infty(I, \mathbb{R}^+)$ and a contraction $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with contraction constant < 1 and $\psi(0) = 0$ such that for $x \in C$ and $y \in \mathbb{R}$, $g(t, x, y) \leq \gamma(t)\psi(\|x\| + |y|)$ a.e., $t \in I$.
- (H4) The function k is Lipschitz with respect to the second variable with Lipschitz constant L_k : $|k(s, x) - k(s, y)| \leq L_k\|x - y\|$ for $s \in I, x, y \in C$.
- (H5) There exist L_1 and L_2 such that for $x_1, x_2 \in C, y_1, y_2 \in \mathbb{R}$ and $s \in I$,

$$|g(s, x_1, y_1) - g(s, x_2, y_2)| \leq L_1\|x_1 - x_2\| + L_2|y_1 - y_2|.$$

3. EXISTENCE OF SOLUTIONS

We prove existence of a solution to (1.1)–(1.2) under hypotheses (H1)–(H5).

Theorem 3.1. *Suppose that hypotheses (H1)–(H5) hold. Assume there exists a real number $r > 0$ such that*

$$r > \frac{\frac{FT^\alpha}{\Gamma(\alpha+1)} \sup_{s \in (0, T)} \gamma(s)(1 + \|\beta\|_{L^1})\psi(r)}{1 - \frac{LT^\alpha}{\Gamma(\alpha+1)} \sup_{s \in (0, T)} \gamma(s)(1 + \|\beta\|_{L^1})\psi(r)}, \quad (3.1)$$

where

$$1 - \frac{LT^\alpha}{\Gamma(\alpha+1)} \sup_{s \in (0, T)} \gamma(s)(1 + \|\beta\|_{L^1})\psi(r) > 0, \quad F = \sup_{t \in [0, T]} |f(t, 0)|.$$

Then (1.1)–(1.2) has a solution on I .

Proof. Let $X = C(I, \mathbb{R})$. Define an open ball $B_r(0)$ centered at origin and of radius $r > 0$, which satisfies (3.1). It is easy to see that x is a solution to (1.1)–(1.2) if and only if it is a solution of the integral equation

$$x(t) = f(t, x(t))I^\alpha \left(g \left(t, x_t, \int_0^t k(s, x_s) ds \right) \right).$$

In other terms,

$$x(t) = f(t, x(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g \left(s, x_s, \int_0^s k(\tau, x_\tau) d\tau \right) ds. \quad (3.2)$$

The integral equation (3.2) is equivalent to the operator equation $Ax(t)Bx(t) = x(t)$, $t \in I$, where $A, B : \overline{B_r(0)} \rightarrow X$ are defined by

$$Ax(t) = f(t, x(t)), \quad Bx(t) = I^\alpha \left(g \left(t, x_t, \int_0^t k(s, x_s) ds \right) \right).$$

We need to prove that the operators A and B verify the hypotheses of Theorem 2.2. To continue the proof of Theorem 3.1, we use two technical lemmas.

Lemma 3.2. *The operator A is Lipschitz on X .*

Proof. Let $x, y \in X$ and $t \in I$. By (H1) we have

$$|Ax(t) - Ay(t)| = |f(t, x) - f(t, y)| \leq L|x(t) - y(t)| \leq L\|x - y\|.$$

Then, $\|Ax - Ay\| \leq L\|x - y\|$ and it follows that A is Lipschitz on X with Lipschitz constant L . \square

Lemma 3.3. *The operator B is completely continuous on X .*

Proof. We prove that $B(\overline{B_r(0)})$ is an uniformly bounded and equicontinuous set in X . Let x be arbitrary in $\overline{B_r(0)}$. By hypotheses (H2)–(H4) we have

$$\begin{aligned} |Bx(t)| &\leq I^\alpha \left(\left| g\left(t, x_t, \int_0^t k(s, x_s) ds\right) \right| \right) \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| g\left(s, x_s, \int_0^s k(\tau, x_\tau) d\tau\right) \right| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \gamma(s) \psi\left(\|x_s\| + \int_0^s \beta(\tau) \|x_\tau\| d\tau\right) ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \gamma(s) (1 + \|\beta\|_{L^1}) \psi(r) ds \\ &\leq \frac{\sup_{s \in [0, T]} \gamma(s)}{\Gamma(\alpha)} (1 + \|\beta\|_{L^1}) \psi(r) \int_0^t (t-s)^{\alpha-1} ds \\ &\leq \frac{\sup_{s \in [0, T]} \gamma(s)}{\Gamma(\alpha+1)} (1 + \|\beta\|_{L^1}) \psi(r) T^\alpha. \end{aligned}$$

Taking the supremum over t , we get $\|Bx\| \leq M$ for all $x \in \overline{B_r(0)}$, where

$$M = \frac{\sup_{s \in [0, T]} \gamma(s)}{\Gamma(\alpha+1)} (1 + \|\beta\|_{L^1}) \psi(r) T^\alpha.$$

It results that $B(\overline{B_r(0)})$ is an uniformly bounded set in X . Now, we shall prove that $B(\overline{B_r(0)})$ is an equicontinuous set in X . For $0 \leq t_1 \leq t_2 \leq T$ we have

$$\begin{aligned} &|Bx(t_2) - Bx(t_1)| \\ &\leq \frac{1}{\Gamma(\alpha)} \left\{ \left| \int_0^{t_2} (t_2-s)^{\alpha-1} g\left(s, x_s, \int_0^s k(\tau, x_\tau) d\tau\right) ds \right. \right. \\ &\quad \left. \left. - \int_0^{t_1} (t_1-s)^{\alpha-1} g\left(s, x_s, \int_0^s k(\tau, x_\tau) d\tau\right) ds \right| \right\} \\ &\leq \frac{1}{\Gamma(\alpha)} \left\{ \left| \int_0^{t_1} (t_2-s)^{\alpha-1} g\left(s, x_s, \int_0^s k(\tau, x_\tau) d\tau\right) ds \right. \right. \\ &\quad \left. \left. + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} g\left(s, x_s, \int_0^s k(\tau, x_\tau) d\tau\right) ds \right. \right. \\ &\quad \left. \left. - \int_0^{t_1} (t_1-s)^{\alpha-1} g\left(s, x_s, \int_0^s k(\tau, x_\tau) d\tau\right) ds \right| \right\} \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left| (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right| \left| g\left(s, x_s, \int_0^s k(\tau, x_\tau) d\tau\right) \right| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \left| g\left(s, x_s, \int_0^s k(\tau, x_\tau) d\tau\right) \right| ds. \end{aligned}$$

On the other hand,

$$\begin{aligned} \left| g\left(s, x_s, \int_0^s k(\tau, x_\tau) d\tau\right) \right| &\leq \gamma(s) \psi\left(\|x\| + \left| \int_0^s k(\tau, x_\tau) d\tau \right| \right) \\ &\leq \gamma(s) \psi\left(\|x\| + \int_0^s \beta(\tau) \|x\| d\tau\right) \end{aligned}$$

$$\begin{aligned} &\leq \gamma(s)\|x\|(1 + \|\beta\|_{L^1})\psi(s) \\ &\leq \sup_s(\gamma(s)\psi(s))\|x\|(1 + \|\beta\|_{L^1}) \leq c, \end{aligned}$$

where c is a positive constant. Then

$$\begin{aligned} &|Bx(t_2) - Bx(t_1)| \\ &\leq \frac{c}{\Gamma(\alpha)} \int_0^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| ds + \frac{c}{\Gamma(\alpha)} \int_{t_1}^{t_2} |(t_2 - s)^{\alpha-1}| ds \\ &\leq \frac{c}{\Gamma(\alpha + 1)} |t_2^\alpha - t_1^\alpha - 2(t_2 - t_1)^\alpha|. \end{aligned}$$

Because the right hand side of the above inequality doesn't depend on x and tends to zero when $t_2 \rightarrow t_1$, we conclude that $B(\overline{B_r(0)})$ is relatively compact. Hence, B is compact by the Arzela-Ascoli theorem. It remains to prove that B is continuous. For that, let us consider a sequence x^n converging to x . Then,

$$\begin{aligned} &|Fx^n(t) - Fx(t)| \\ &\leq \left| f(t, x^n)I^\alpha \left(g \left(t, x_t^n, \int_0^t k(s, x_s^n) ds \right) \right) - f(t, x)I^\alpha \left(g \left(t, x_t, \int_0^t k(s, x_s) ds \right) \right) \right| \\ &\leq |f(t, x^n) - f(t, x)|I^\alpha \left(\left| g \left(t, x_t^n, \int_0^t k(s, x_s^n) ds \right) \right| \right) \\ &\quad + |f(t, x)| \left| I^\alpha \left(g \left(t, x_t^n, \int_0^t k(s, x_s^n) ds \right) \right) - I^\alpha \left(g \left(t, x_t, \int_0^t k(s, x_s) ds \right) \right) \right| \\ &\leq L\|x^n - x\| + |f(t, x)|I^\alpha \left(\left| g \left(t, x_t^n, \int_0^t k(s, x_s^n) ds \right) - g \left(t, x_t, \int_0^t k(s, x_s) ds \right) \right| \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} &I^\alpha \left(\left| g \left(t, x_t^n, \int_0^t k(s, x_s^n) ds \right) - g \left(t, x_t, \int_0^t k(s, x_s) ds \right) \right| \right) \\ &\leq I^\alpha \left(L_1|x_t^n - x_t| + L_2 \left| \int_0^t k(s, x_s^n) ds - \int_0^t k(s, x_s) ds \right| \right) \\ &\leq I^\alpha \left(L_1|x_t^n - x_t| + L_2 \int_0^t |k(s, x_s^n) - k(s, x_s)| ds \right) \\ &\leq I^\alpha \left(L_1|x_t^n - x_t| + L_2L_k \int_0^t |x_s^n - x_s| ds \right) \\ &\leq I^\alpha (L_1\|x_t^n - x_t\| + L_2L_kT\|x_s^n - x_s\|) \\ &\leq (L_1 + L_2L_kT)\|x_t^n - x_t\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\ &\leq \frac{1}{\Gamma(\alpha + 1)} (L_1 + L_2L_kT)\|x_t^n - x_t\|. \end{aligned}$$

Taking the norm, $\|Fx^n - Fx\| \leq L\|x^n - x\| + \frac{1}{\Gamma(\alpha+1)}|f(t, x)|(L_1 + L_2L_kT)\|x_t^n - x_t\|$. Hence, the right hand side of the above inequality tends to zero whenever $x^n \rightarrow x$. Therefore, $Fx^n \rightarrow Fx$. This proves the continuity of F . \square

Using Theorem 2.2, we obtain that either the conclusion (i) or (ii) holds. We show that item (ii) of Theorem 2.2 cannot be realizable. Let $x \in X$ be such that

$\|x\| = r$ and $x(t) = \lambda f(t, x(t)) I^\alpha \left(g \left(t, x_t, \int_0^t k(s, x_s) ds \right) \right)$ for any $\lambda \in (0, 1)$ and $t \in I$. It follows that

$$\begin{aligned}
 |x(t)| &\leq \lambda (|f(t, x(t)) - f(t, 0)| + |f(t, 0)|) I^\alpha \left(g \left(t, x_t, \int_0^t k(s, x_s) ds \right) \right) \\
 &\leq \lambda (L|x(t)| + F) I^\alpha \left(g \left(t, x_t, \int_0^t k(s, x_s) ds \right) \right) \\
 &\leq \lambda (L\|x\| + F) I^\alpha \left(g \left(t, x_t, \int_0^t k(s, x_s) ds \right) \right) \\
 &\leq (L\|x\| + F) I^\alpha \left(\gamma(t) \psi \left(\|x\| + \left| \int_0^t k(s, x_s) ds \right| \right) \right) \\
 &\leq (L\|x\| + F) I^\alpha (\gamma(t) \psi (\|x\| + \|\beta\|_{L^1} \|x\|)) \\
 &\leq \frac{L\|x\| + F}{\Gamma(\alpha)} \int_0^t \gamma(s) (1 + \|\beta\|_{L^1}) \psi (\|x\|) (t-s)^{\alpha-1} ds \\
 &\leq \frac{L\|x\| + F}{\Gamma(\alpha)} \sup_{s \in (0, T)} \gamma(s) (1 + \|\beta\|_{L^1}) \psi (\|x\|) \int_0^t (t-s)^{\alpha-1} ds \\
 &\leq \frac{L\|x\| + F}{\Gamma(\alpha+1)} \sup_{s \in (0, T)} \gamma(s) (1 + \|\beta\|_{L^1}) \psi (\|x\|) T^\alpha \\
 &\leq \left(\frac{LT^\alpha}{\Gamma(\alpha+1)} \sup_{s \in (0, T)} \gamma(s) (1 + \|\beta\|_{L^1}) \psi (\|x\|) \right) \|x\| \\
 &\quad + \frac{FT^\alpha}{\Gamma(\alpha+1)} \sup_{s \in (0, T)} \gamma(s) (1 + \|\beta\|_{L^1}) \psi (\|x\|).
 \end{aligned}$$

Passing to the supremum in the above inequality, we obtain

$$\|x\| \leq \frac{\frac{FT^\alpha}{\Gamma(\alpha+1)} \sup_{s \in (0, T)} \gamma(s) (1 + \|\beta\|_{L^1}) \psi (\|x\|)}{1 - \frac{LT^\alpha}{\Gamma(\alpha+1)} \sup_{s \in (0, T)} \gamma(s) (1 + \|\beta\|_{L^1}) \psi (\|x\|)}. \quad (3.3)$$

If we replace $\|x\| = r$ in (3.3), we have

$$r \leq \frac{\frac{FT^\alpha}{\Gamma(\alpha+1)} \sup_{s \in (0, T)} \gamma(s) (1 + \|\beta\|_{L^1}) \psi (r)}{1 - \frac{LT^\alpha}{\Gamma(\alpha+1)} \sup_{s \in (0, T)} \gamma(s) (1 + \|\beta\|_{L^1}) \psi (r)},$$

which is in contradiction to (3.1). Then the conclusion (ii) of Theorem 2.2 is not possible. Therefore, the operator equation $AxBx = x$ and, consequently, problem (1.1)–(1.2), has a solution on I . This ends the proof of Theorem 3.1. \square

Let us see an example of application of our Theorem 3.1. Let $I_0 = [-\pi, 0]$ and $I = [0, \pi]$. Consider the integro-differential fractional equation

$$\frac{d^\alpha}{dt^\alpha} \left(\frac{x(t)}{1 + \frac{\sin t}{12} |x(t)|} \right) = g \left(t, x_t \left(\frac{t}{2} \right), \int_0^t k(s, x_s) ds \right) \quad \text{a.e., } t \in I, \quad (3.4)$$

where $\alpha = 1/2$ and $f : I \times \mathbb{R} \rightarrow \mathbb{R}^+ \setminus \{0\}$, $g : I \times C \times \mathbb{R} \rightarrow \mathbb{R}$ and $k : I \times C \rightarrow \mathbb{R}$ are given by

$$f(t, x) = 1 + \frac{\sin t}{12} (|x(t)|), \quad g(t, x, y) = \gamma(t) (\|x\| + |y|), \quad k(t, x) = \frac{x}{4\pi},$$

with $\gamma(t) = t, \beta(t) = \frac{1}{4\pi}, T = \pi, L = \frac{1}{12}, F = \sup_t f(t, 0) = 1, \|\beta\|_{L^1} = \frac{1}{4}$ and $\sup_{t \in [0, \pi]} \gamma(t) = \pi$. It is easy to see that all hypotheses (H1)–(H5) are satisfied with $\psi(r) = \frac{r}{12}$ for all $r \in \mathbb{R}^+$. By Theorem 3.1, r satisfies

$$\frac{12 - FB}{LB} \simeq 0, 26 \leq r \leq \frac{12}{LB} \simeq 18, 34,$$

where $B = \frac{T^\alpha}{\Gamma(\alpha+1)} \sup_{s \in (0, T)} \gamma(s)(1 + \|\beta\|_{L^1})$. We conclude that if $r = 2$, then (3.4) has a solution in $B_2(0)$.

4. DEPENDENCE ON THE DATA AND UNIQUENESS OF SOLUTION

In this section we derive uniqueness of solution to (1.1)–(1.2).

Theorem 4.1. *Let x and y be two solutions to the nonlocal fractional equation (1.1) subject to (1.2) with $\phi = \phi_1$ and $\phi = \phi_2$, respectively. Then, we have*

$$\|x - y\| \leq \frac{(L\|y\| + F)(L_1 + L_2L_kT)^{\frac{T^\alpha}{\Gamma(\alpha+1)}}}{1 - L} \|\phi_1 - \phi_2\|.$$

Proof. Let x and y be two solutions of (1.1). Then, from (3.2), one has

$$\begin{aligned} & |x(t) - y(t)| \\ & \leq \left| f(t, x) I^\alpha \left(g \left(t, x_t, \int_0^t k(s, x_s) ds \right) \right) - f(t, y) I^\alpha \left(g \left(t, y_t, \int_0^t k(s, y_s) ds \right) \right) \right| \\ & \leq |f(t, x) - f(t, y)| I^\alpha \left(\left| g \left(t, x_t, \int_0^t k(s, x_s) ds \right) \right| \right) \\ & \quad + |f(t, y)| \left| I^\alpha \left(g \left(t, x_t, \int_0^t k(s, x_s) ds \right) \right) - I^\alpha \left(g \left(t, y_t, \int_0^t k(s, y_s) ds \right) \right) \right| \\ & \leq L \|x - y\| + |f(t, y)| I^\alpha \left(\left| g \left(t, x_t, \int_0^t k(s, x_s) ds \right) - g \left(t, y_t, \int_0^t k(s, y_s) ds \right) \right| \right) \end{aligned}$$

for $t > 0$. On the other hand, we have

$$\begin{aligned} & \left| g \left(t, x_t, \int_0^t k(s, x_s) ds \right) - g \left(t, y_t, \int_0^t k(s, y_s) ds \right) \right| \\ & \leq L_1 |x_t - y_t| + L_2 \int_0^t |k(s, x_s) - k(s, y_s)| ds \\ & \leq L_1 |\phi_1(t) - \phi_2(t)| + L_2 L_k T \|\phi_1 - \phi_2\| \\ & \leq (L_1 + L_2 L_k T) \|\phi_1 - \phi_2\|. \end{aligned}$$

Then

$$\begin{aligned} & |f(t, y)| I^\alpha \left(\left| g \left(t, x_t, \int_0^t k(s, x_s) ds \right) - g \left(t, y_t, \int_0^t k(s, y_s) ds \right) \right| \right) \\ & \leq |f(t, y)| (L_1 + L_2 L_k T) \|\phi_1 - \phi_2\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\ & \leq |f(t, y)| (L_1 + L_2 L_k T) \|\phi_1 - \phi_2\| \frac{T^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

Taking the supremum, we conclude that

$$\begin{aligned} & \|x - y\| \\ & \leq L\|x - y\| + |f(t, y)| (L_1 + L_2 L_k T) \|\phi_1 - \phi_2\| \frac{T^\alpha}{\Gamma(\alpha)} \\ & \leq L\|x - y\| + (|f(t, y) - f(t, 0)| + |f(t, 0)|) (L_1 + L_2 L_k T) \|\phi_1 - \phi_2\| \frac{T^\alpha}{\Gamma(\alpha + 1)} \\ & \leq L\|x - y\| + \left(L\|y\| + \sup_t |f(t, 0)| \right) (L_1 + L_2 L_k T) \frac{T^\alpha}{\Gamma(\alpha + 1)} \|\phi_1 - \phi_2\| \\ & \leq L\|x - y\| + (L\|y\| + F) (L_1 + L_2 L_k T) \frac{T^\alpha}{\Gamma(\alpha + 1)} \|\phi_1 - \phi_2\|. \end{aligned}$$

Therefore,

$$\|x - y\| \leq \frac{(L\|y\| + F)(L_1 + L_2 L_k T) \frac{T^\alpha}{\Gamma(\alpha + 1)}}{1 - L} \|\phi_1 - \phi_2\|.$$

□

Corollary 4.2. *The solution obtained in Theorem 3.1 is unique.*

Acknowledgments. This work was supported by *FEDER* through *COMPETE* — Operational Programme Factors of Competitiveness (“Programa Operacional Factores de Competitividade”) and by Portuguese funds through the *Center for Research and Development in Mathematics and Applications* (University of Aveiro) and the Portuguese Foundation for Science and Technology (“FCT — Fundação para a Ciência e a Tecnologia”), within project PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690. The authors were also supported by the project *New Explorations in Control Theory Through Advanced Research* (NECTAR) cofinanced by FCT, Portugal, and the *Centre National de la Recherche Scientifique et Technique* (CNRST), Morocco.

REFERENCES

- [1] S. Abbas; *Existence of solutions to fractional order ordinary and delay differential equations and applications*, Electron. J. Differential Equations **2011** (2011), no. 9, 11pp.
- [2] R. P. Agarwal, M. Benchohra, S. Hamani; *Boundary value problems for fractional differential equations*, Georgian Math. J. **16** (2009), no. 3, 401–411.
- [3] R. P. Agarwal, Y. Zhou, Y. He; *Existence of fractional neutral functional differential equations*, Comput. Math. Appl. **59** (2010), no. 3, 1095–1100.
- [4] R. Almeida, D. F. M. Torres; *Calculus of variations with fractional derivatives and fractional integrals*, Appl. Math. Lett. **22** (2009), no. 12, 1816–1820.
- [5] B. C. Dhage, *Nonlinear functional boundary value problems in Banach algebras involving Carathéodories*, Kyungpook Math. J. **46** (2006), no. 4, 527–541.
- [6] L. Gaul, P. Klein, S. Kempfle; *Damping description involving fractional operators*, Mech. Syst. Signal Process. **5** (1991), no. 2, 81–88.
- [7] C. Giannantonio; *The problem of the initial conditions and their physical meaning in linear differential equations of fractional order*, Appl. Math. Comput. **141** (2003), no. 1, 87–102.
- [8] R. Hilfer; *Applications of fractional calculus in physics*, World Sci. Publishing, River Edge, NJ, 2000.
- [9] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204, Elsevier, Amsterdam, 2006.
- [10] M. Kirane, S. A. Malik; *Determination of an unknown source term and the temperature distribution for the linear heat equation involving fractional derivative in time*, Appl. Math. Comput. **218** (2011), no. 1, 163–170.

- [11] M. Klimek; *Stationarity-conservation laws for fractional differential equations with variable coefficients*, J. Phys. A **35** (2002), no. 31, 6675–6693.
- [12] A. B. Malinowska, M. R. Sidi Ammi, D. F. M. Torres; *Composition functionals in fractional calculus of variations*, Commun. Frac. Calc. **1** (2010), no. 1, 32–40.
- [13] K. S. Miller, B. Ross; *An introduction to the fractional calculus and fractional differential equations*, A Wiley-Interscience Publication, Wiley, New York, 1993.
- [14] D. Mozyrska, D. F. M. Torres; *Modified optimal energy and initial memory of fractional continuous-time linear systems*, Signal Process. **91** (2011), no. 3, 379–385.
- [15] M. D. Ortigueira; *Fractional calculus for scientists and engineers*, Springer, New York, 2011.
- [16] I. Podlubny; *Fractional differential equations*, Mathematics in Science and Engineering, 198, Academic Press, San Diego, CA, 1999.
- [17] B. Ross, S. G. Samko, E. R. Love; *Functions that have no first order derivative might have fractional derivatives of all orders less than one*, Real Anal. Exchange **20** (1994/95), no. 1, 140–157.
- [18] J. Sabatier, O. P. Agrawal, J. A. Tenreiro Machado; *Advances in fractional calculus*, Springer, Dordrecht, 2007.
- [19] S. N. Salunkhe, M. G. Timol; *Existence theorem for first order ordinary functional differential equations with periodic boundary conditions*, Gen. Math. Notes **1** (2010), no. 2, 185–194.
- [20] S. G. Samko, A. A. Kilbas, O. I. Marichev; *Fractional integrals and derivatives*, translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.
- [21] S. G. Samko, M. N. Yakhshiboev; *A modification of Riemann-Liouville fractional integro-differentiation applied to functions on \mathbf{R}^1 with arbitrary behavior at infinity*, Izv. Vyssh. Uchebn. Zaved. Mat. **1992**, no. 4, 96–99.
- [22] H. M. Srivastava, R. K. Saxena; *Operators of fractional integration and their applications*, Appl. Math. Comput. **118** (2001), no. 1, 1–52.
- [23] J. A. Tenreiro Machado, V. Kiryakova, F. Mainardi; *Recent history of fractional calculus*, Commun. Nonlinear Sci. Numer. Simul. **16** (2011), no. 3, 1140–1153.

MOULAY RCHID SIDI AMMI

FACULTY OF SCIENCES AND TECHNIQUES, MOULAY ISMAIL UNIVERSITY, AMNEA GROUP, ER-RACHIDIA, MOROCCO

E-mail address: `sidihammi@ua.pt`

EL HASSAN EL KINANI

FACULTY OF SCIENCES AND TECHNIQUES, MOULAY ISMAIL UNIVERSITY, GAA GROUP, ERRACHIDIA, MOROCCO

E-mail address: `elkinani.67@yahoo.com`

DELFIN F. M. TORRES

CENTER FOR RESEARCH AND DEVELOPMENT IN MATHEMATICS AND APPLICATIONS (CIDMA), DEPARTMENT OF MATHEMATICS, UNIVERSITY OF AVEIRO, CAMPUS UNIVERSITÁRIO DE SANTIAGO, 3810-193 AVEIRO, PORTUGAL

E-mail address: `delfim@ua.pt`