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REAL INTERPOLATION SPACES BETWEEN THE DOMAIN OF THE LAPLACE OPERATOR WITH TRANSMISSION CONDITIONS AND L^p ON A POLYGONAL DOMAIN

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Abstract. We provide a description of the real interpolation spaces between the domain of the Laplace operator (with transmission conditions in a polygonal domain Ω) and $L^p(\Omega)$ as interpolation spaces between $\mathcal{W}^{2,p}(\Omega)$ (possibly augmented with singular solutions) and $L^p(\Omega)$. This result relies essentially on estimates on the resolvent and the reiteration theorem.

1. Introduction

Let Ω be a polygonal domain of $\mathbb{R}^2$ divided into two polygons Ω_1 and Ω_2 separated by an interface Σ. Let the transmission conditions be defined as

$$u_1 = u_2 \quad \text{and} \quad \sum_{i=1}^{2} \alpha_i \frac{\partial u_i}{\partial \nu_i} = 0 \quad \text{on } \Sigma, \tag{1.1}$$

where ν_i denotes the unit normal vector to Σ directed outside Ω_i, u_i means the restriction of u to Ω_i, and α_1, α_2 are two positive real numbers such that $\alpha_1 \neq \alpha_2$.

Let A_p be the operator defined by

$$D_{A_p}(\Omega) = \{u \in H_0^1(\Omega) : \Delta u_i \in L^p(\Omega_i),\ i = 1, 2;\ (1.1) \text{ is satisfied } \},$$
$$A_p : u \mapsto \{-\Delta u_i\}_{i=1,2}\,.$$

Then A_p is the infinitesimal generator of an analytic semigroup on $L^p(\Omega)$ [3].

Let us define $\mathcal{W}^{s,p}(\Omega) := \{u \in H_0^1(\Omega); u_i \in W^{s,p}(\Omega_i),\ i = 1, 2 \text{ satisfying (1.1)}\}$ the space of piecewise $W^{s,p}$ functions on Ω which satisfies the transmission conditions (1.1). The space $\mathcal{W}^{s,p}(\Omega)$ will be equipped with the usual product norm of $\Pi_{i=1}^2 W^{s,p}(\Omega_i)$.

We know that $D_{A_p}(\Omega) = \operatorname{span}(\mathcal{W}^{2,p}(\Omega); S)$, the space spanned by $\mathcal{W}^{2,p}(\Omega)$ and S, where S is the finite set of singular solutions [6, 8].

By analogy with [2] who considered the Laplace operator subject to Dirichlet boundary conditions, we give a description of the real interpolation spaces related to the operator A_p. This result relies essentially on estimates on the resolvent and the reiteration theorem of real interpolation [7]. It is well known that information

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concerning real interpolation spaces between the basic space and the domain of the operator is crucial to get results of maximal regularity for parabolic problems.

This article is organized as follows: in Section 2, we recall the results concerning existence, uniqueness and regularity of the variational solution u of the following transmission problem with complex parameter λ

$$\begin{gathered} -\Delta u_i + \lambda u_i = f_i \quad \text{in } \Omega_i, \\ u_i = 0 \quad \text{on } \partial\Omega_i \setminus \Sigma, \\ u_1 = u_2 \quad \text{on } \Sigma, \\ \sum_{i=1}^{2} \alpha_i \frac{\partial u_i}{\partial \nu_i} = 0 \quad \text{on } \Sigma, \end{gathered} \tag{1.2}$$

where $f \in L^p(\Omega)$, $p > 1$.

The aim of Section 3 is to state the following estimates of problem (1.2) on the resolvent in an infinite sector G

$$\begin{gathered} \|u\|_{\mathcal{W}^{s,p}(G)} \le \frac{c}{\lambda^{1-\frac{s}{2}}} \|f\|_{0,p,G}, \quad s < \lambda_m + \frac{2}{p} \quad \text{for all } m, \\ \|u\|_{\operatorname{span}(\mathcal{W}^{s,p}(G);\mathcal{S})} \le \frac{c}{\lambda^{1-\frac{s}{2}}} \|f\|_{0,p,G}, \ s > \lambda_m + \frac{2}{p} \quad \text{for some } m, \end{gathered}$$

(see Section 3 for the definition of $\mathcal{S}$ and λ_m). For this purpose, we firstly establish the result for the case $\lambda = 1$. Applying the transformation $(x, y) \mapsto (tx, ty)$, $t = \lambda^{-1/2}$, problem (1.2) in G becomes

$$\begin{gathered} -\Delta u_i(tx, ty) + u_i(tx, ty) = t^2 f_i(tx, ty) \quad \text{in } G_i, \\ u_i(tx, ty) = 0 \quad \text{on } \partial G_i \setminus \Sigma, \\ u_1(tx, ty) = u_2(tx, ty) \quad \text{on } \Sigma, \\ \sum_{i=1}^{2} \alpha_i \frac{\partial u_i}{\partial \nu_i^t}(tx, ty) = 0 \quad \text{on } \Sigma, \end{gathered}$$

where ν_i^t is the normal vector with respect to the variables (tx, ty) directed outside G_i. This method of dilation relies on the invariance of the infinite sector G, under dilations and the homogeneity of the singular functions, therefore we come back to the previous case.

Section 4 is devoted to state such estimate in Ω polygonal. Via a partition of unity, problem (1.2) is locally reduced to a similar problem in an infinite sector, then we used the results of the previous section.

Thanks to the results of Section 4 and the reiteration theorem, we give in Section 5 a characterization of $D_{A_p}(\theta; p)$, $0 < \theta < 1$, as interpolation spaces between $\operatorname{span}(\mathcal{W}^{2,p}(\Omega); \mathcal{S})$ and $L^p(\Omega)$ or between $\mathcal{W}^{2,p}(\Omega)$ and $L^p(\Omega)$.

Let us finish this introduction with some notation used in the whole paper: if D is an open subset of $\mathbb{R}^2$, we denote by $L^p(D)$, $(p > 1)$ the Lebesgue spaces, and by $W^{s,p}(D)$, $s \ge 0$, the standard Sobolev spaces built on. The usual norm of $W^{s,p}(D)$ is denoted by $\| \cdot \|_{s,p,D}$. The space $H_0^1(D)$ is defined as usual by $H_0^1(D) := \{ v \in H^1(D); v = 0 \text{ on } \partial D \}$.

2. Regularity results of transmission problem in a polygonal domain

Let Ω be a bounded polygonal domain of $\mathbb{R}^2$ with a Lipschitz boundary Γ. We suppose that Ω is decomposed into two polygons Ω_1 and Ω_2 with an interface Σ satisfying

$$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2, \quad \Omega_1 \cap \Omega_2 = \emptyset, \quad \overline{\Omega}_1 \cap \overline{\Omega}_2 = \Sigma.$$

We assume that the boundaries $\partial\Omega_i$ of Ω_i $(i = 1, 2)$ is formed by open straight line segments $\Gamma_{i,j}, j = 1, \cdots, N_i$, with $N_i \in \mathbb{N}^\star$, enumerated clockwise such that

$$\Sigma = \Gamma_{1,1} = \Gamma_{2,1}, \quad \Gamma := \partial\Omega = \cup_{i=1,2} \cup_{j=2}^{N_i} \Gamma_{i,j}.$$

We denote by P_j, $j = 1, \dots, N_1 + N_2 - 2$ the vertices of Ω where

$$P_j = \overline{\Gamma_{1,j}} \cap \overline{\Gamma_{1,j+1}}, \quad j = 1, \dots, N_1 - 1$$
$$P_j = \overline{\Gamma_{2,j-N_1+1}} \cap \overline{\Gamma_{2,j-N_1+2}}, \quad j = N_1, \dots, N_1 + N_2 - 2.$$

At each point P_j, $(j \neq 1, j \neq N_1)$ we denote the measure of the angle P_j (measured from inside Ω) by ω_j. When $j = 1$ or $j = N_1$, the angle at P_j measured from inside Ω_i is denoted by ω_{ij}, $i = 1, 2$. See Figure 1 for an illustration.

For the transmission problem (1.2), the corresponding variational problem is

$$\int_\Omega \alpha(\nabla u \, \nabla \overline{v} + \lambda u \overline{v}) \, d\mathbf{x} = \int_\Omega \alpha f \overline{v} \, d\mathbf{x} \quad \forall v \in H_0^1(\Omega), \tag{2.1}$$

where $\alpha(\mathbf{x})$ is piecewise constant; i.e., $\alpha(\mathbf{x}) = \alpha_i > 0$ for $\mathbf{x} \in \Omega_i$, $i = 1, 2$.

For the rest of this article, $L^p(\Omega)$ will be equipped with the norm

$$\|u\|_{0,p} = \Big(\int_\Omega \alpha |u(\mathbf{x})|^p \; d\mathbf{x} \Big)^{1/p}.$$

First we recall the results concerning existence, uniqueness and regularity of the variational solution u of (2.1).

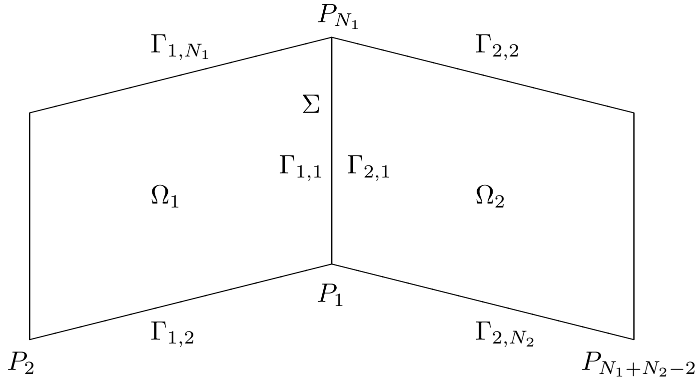

Figure 1. The domain Ω

Proposition 2.1. *For each $f \in L^p(\Omega)$, there exists a unique solution $u \in H_0^1(\Omega)$ of (2.1), for all $\lambda \in \mathbb{C} : \Re\lambda \geq 0$.*

For a proof of the above proposition, see [3, Lemma 3.1].

Let η_j be a cut-off function $\eta_j \equiv \eta_j(r) \in \mathcal{D}(\mathbb{R}^+)$ which is equal to 1 in a neighborhood of the vertex P_j, with compact support in an open set, which is disjoint to the other vertices of Ω, then the singularities of problem (1.2) take the form

$$S^{(jm)} = \eta_j r^{\lambda_{jm}} \sin\big(\lambda_{jm}\theta\big), \quad \lambda_{jm} = \frac{m\pi}{w_j}, \quad \text{when } j \neq 1,\ j \neq N_1,$$

and

$$S^{(jm)} = \eta_j r^{\lambda_{jm}} \varphi_{jm}(\theta), \quad \text{when } j = 1,\ j = N_1,$$

where λ_{jm} is a nonnegative real number and λ_{jm}^2, φ_{jm} are respectively the eigenvalues and eigenfunctions of the Sturm-Liouville problem:

$$\begin{gathered} \text{Find } \varphi \in H_0^1(]-\omega, \omega'[) \text{ such that} \\ -(\alpha(\theta)\varphi'(\theta))' = \lambda\alpha(\theta)\varphi(\theta), \end{gathered}$$

where $\omega = \omega_{12}$, $\omega' = \omega_{11}$, $\alpha(\theta) = \alpha_1$ for $\theta > 0$ and $\alpha(\theta) = \alpha_2$ for $\theta < 0$ if $j = 1$, while $\omega = \omega_{N_1 1}$, $\omega' = \omega_{2N_1}$, $\alpha(\theta) = \alpha_1$ for $\theta < 0$ and $\alpha(\theta) = \alpha_2$ for $\theta > 0$ if $j = N_1$.

The singular behavior of the solution of (2.1) is given by the following proposition (see [8, Theorem 2.27]).

Proposition 2.2. *If $\lambda_{jm} \neq \frac{2}{p'}$ for $1 \le j \le N_1 + N_2 - 2$ and for all $m \in \mathbb{N}^*$, then for each $f \in L^p(\Omega)$, there exist unique real numbers c_{jm} and a unique variational solution $u \in H_0^1(\Omega)$ of (1.2) which admits the decomposition*

$$u = u_R + \sum_{\lambda_{jm} \in]0, \frac{2}{p'}[,\ 1 \neq \lambda_{jm},\ 1 \le j \le N_1 + N_2 - 2} c_{jm} S^{(jm)}, \tag{2.2}$$

where $u_R \in \mathcal{W}^{2,p}(\Omega)$ is the regular part of u and the constants c_{jm} are the coefficients of the singular part.

3. L^p estimates in an infinite sector G

Let G be a plane sector consisting of two plane sectors G_1, G_2 with respective opening ω_1 and ω_2, separated by an interface Σ.

$$\begin{gathered} G_1 = \{(r\cos\theta, r\sin\theta);\ -\omega_1 < \theta < 0,\ r > 0\}, \\ G_2 = \{(r\cos\theta, r\sin\theta);\ 0 < \theta < \omega_2,\ r > 0\}, \\ \Sigma = \{(r, 0);\ r > 0\}. \end{gathered}$$

We consider the transmission problem (1.2) in the infinite sector G,

$$-\Delta u_i + \lambda u_i = f_i \quad \text{in } G_i, \tag{3.1}$$

$$u_i = 0 \quad \text{on } \partial G_i \setminus \Sigma, \tag{3.2}$$

$$u_1 = u_2 \quad \text{on } \Sigma, \tag{3.3}$$

$$\sum_{i=1}^{2} \alpha_i \frac{\partial u_i}{\partial \nu_i} = 0 \quad \text{on } \Sigma. \tag{3.4}$$

To obtain growth (with respect to λ) on $\|(-\Delta + \lambda)^{-1}\|$ in a given norm, we state the result on a finite sector denoted by $G_F := G \cap B(0, r')$, $r' > 0$. We shall obtain the same result for an infinite sector by taking limits, with respect to a sequence of cut-off functions.

Proposition 3.1. *If $\lambda_m \neq 2/p'$ for all $m \in \mathbb{N}^*$, then for each $f \in L^p(G_F)$, there exists a unique variational solution $u \in H_0^1(G_F)$ of (3.1)–(3.4) (with G_F instead of G) which admits the decomposition*

$$u = u_R + \sum_{\lambda_m \in]0, \frac{2}{p'}[,\ \lambda_m \neq 1} c_m S^{(m)}, \tag{3.5}$$

where $u_R \in \mathcal{W}^{2,p}(G_F)$ is the regular part of u, c_m are constants and $S^{(m)}$ are defined as in Section 2, the subscript j has been omitted since G contain only one vertex; furthermore u satisfies the estimates

$$\|u\|_{0,p,G_F} \leq \frac{1}{\Re\lambda} \|f\|_{0,p,G_F}, \quad \Re\lambda > 0, \tag{3.6}$$

$$\|u\|_{0,p,G_F} \leq \frac{p}{2|\Im\lambda|} \|f\|_{0,p,G_F}, \quad \Im\lambda \neq 0. \tag{3.7}$$

Consequently there exists a constant $c(p) > 0$ such that

$$\|u\|_{0,p,G_F} \leq \frac{c(p)}{|\lambda|} \|f\|_{0,p,G_F}, \quad \Re\lambda \geq 0,\ \lambda \neq 0, \tag{3.8}$$

and for the regular part we have

$$\|u_R\|_{\mathcal{W}^{2,p}(G_F)} \leq C\|(-\Delta + \lambda)u_R\|_{0,p,G_F}. \tag{3.9}$$

Proof. The decomposition of u into a regular part and a singular one is a direct consequence of Proposition 2.2. For (3.6), (3.7) and (3.8), see [3].

As in [3], we obtain (3.9) by applying [8, Theorem 2.27] and Peetre's lemma. Indeed:

$$\begin{aligned} \|u_R\|_{\mathcal{W}^{2,p}(G_F)} &\leq C\{\|\Delta u_R\|_{0,p,G_F} + \|u_R\|_{0,p,G_F}\} \\ &\leq C\{\|(-\Delta + \lambda)u_R\|_{0,p,G_F} + (1 + |\lambda|)\|u_R\|_{0,p,G_F}\}. \end{aligned} \tag{3.10}$$

Now it suffice to apply to u_R the estimate (3.8) to get (3.9). $\square$

As mentioned above, to obtain growth (with respect to λ) on $\|(-\Delta + \lambda)^{-1}\|$ in a given norm, we shall need a priori estimates when $\lambda = 1$ in that norm. Following up with dilations will give the required result.

Let us denote by $\mathcal{S}$ the set of singular functions; i.e.,

$$\mathcal{S} := \{S^{(m)}; \lambda_m \in]0, \frac{2}{p'}[\text{ with } \lambda_m \neq 1\}.$$

Define $D_{A_p}(G) := \operatorname{span}(\mathcal{W}^{2,p}(G); \mathcal{S})$.

3.1. L^p estimates for $\lambda = 1$.

Proposition 3.2. *Let us assume that $\lambda_m \neq 2/p'$ for all $m \in \mathbb{N}^*$. Let $u \in D_{A_p}(G)$ and let $0 \leq s \leq 2$, then there exists C such that*

$$\|u\|_{\mathcal{W}^{s,p}(G)} \leq C\|(A_p + 1)u\|_{0,p,G} \quad \text{if } s < \frac{2}{p} + \lambda_m, \text{ for all } m \tag{3.11}$$

and

$$\|u\|_{\operatorname{span}(\mathcal{W}^{s,p}(G);\mathcal{S})} \leq C\|(A_p + 1)u\|_{0,p,G} \quad \text{if } s \geq \frac{2}{p} + \lambda_m, \text{ for some } m. \tag{3.12}$$

Proof. Let $u \in \mathcal{W}^{2,p}(G)$ be a solution of (3.1)-(3.4) for $\lambda = 1$. Define the cut-off function $\eta \in C^2(\mathbb{R})$,

$$\eta(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1; \\ 0 & \text{for } x \ge 2, \end{cases}$$

and $0 \le \eta(x) \le 1$ for $1 \le x \le 2$.

Consider a sequence of such cut-off functions $\{\eta_n(r)\}$; $\eta_n(r) = \eta(r/n)$ where we choose (r,θ) as polar coordinates with origin at the vertex of the sector. For each n, let G^n be a finite sector which contains $\operatorname{supp}(\eta_n u)$, $G_i^n := G^n \cap G_i$ $(i = 1,2)$. Then $\eta_n u \in \mathcal{W}^{2,p}(G^n)$, and $\eta_n u$ is a solution of

$$(-\Delta + 1)(\eta_n u_i) = F_i,$$

where

$$F_i = \eta_n(-\Delta + 1)u_i - \Delta\eta_n u_i - 2\nabla\eta_n \nabla u_i \in L^p(G_i^n).$$

It follows from (3.9) that

$$\|\eta_n u\|_{\mathcal{W}^{2,p}(G^n)} \le C\|(A_p + 1)(\eta_n u)\|_{0,p,G^n}.$$

This implies

$$\begin{aligned} \|\eta_n u\|_{\mathcal{W}^{2,p}(G^n)} &= \Big(\sum_{|\beta| \le 2,\, 1 \le i \le 2} \big\|\partial^\beta(\eta_n u_i)\big\|^p_{0,G_i^n} \Big)^{1/p} \\ &\le C\Big\{ \|(A_p + 1)u\|_{0,p,G^n} + \frac{1}{n^2}\|u\|_{0,p,G^n} + \frac{1}{n}\|\nabla u\|_{0,p,G^n} \Big\}. \end{aligned} \tag{3.13}$$

We consider for example the term $\big\|\frac{\partial^2}{\partial x^2}(\eta_n u_i)\big\|^p_{0,G_i^n}$ in (3.13)

$$\big\|\frac{\partial^2}{\partial x^2}(\eta_n u_i)\big\|^p_{0,G_i^n} = \int_{G_i} \chi_n |f_i^n|^p \, r\, dr\, d\theta,$$

where

$$f_i^n(r,\theta) = \eta_n \frac{\partial^2 u_i}{\partial x^2} + \frac{2}{n}\eta'\Big(\frac{r}{n}\Big)\cos\theta \frac{\partial u_i}{\partial x} + \Big(\frac{1}{n}\frac{\sin^2\theta}{r}\eta'\Big(\frac{r}{n}\Big) + \frac{1}{n^2}\cos^2\theta\, \eta''\Big(\frac{r}{n}\Big)\Big)u_i,$$

and

$$\chi_n(r,\theta) = \begin{cases} 1 & \text{if } (r,\theta) \in G_i^n, \\ 0 & \text{otherwise}. \end{cases}$$

It is clear that

$$\lim_n f_i^n(r,\theta) = \frac{\partial^2 u_i}{\partial x^2},$$

and

$$|f_i^n|^p \le c\Big(\Big|\frac{\partial^2 u_i}{\partial x^2}\Big| + \Big|\frac{\partial u_i}{\partial x}\Big| + |u_i|\Big)^p \in L^1(G_i),$$

consequently, the dominated convergence theorem implies

$$\lim_n \big\|\frac{\partial^2}{\partial x^2}(\eta_n u_i)\big\|^p_{0,G_i^n} = \big\|\frac{\partial^2 u_i}{\partial x^2}\big\|^p_{0,G_i}.$$

Therefore, applying the same technique to the other terms in (3.13), we obtain

$$\|u\|_{\mathcal{W}^{2,p}(G)} \le C\|(A_p + 1)u\|_{0,p,G}, \tag{3.14}$$

hence the inequality (3.11) for $s = 2$.

To state (3.12) for $s = 2$, we apply Proposition 6.9 from the Appendix with $E = \mathcal{W}^{2,p}(G)$, $H = L^p(G)$, $F = \mathcal{S}$ and $A = -\Delta + 1$ subject to homogeneous Dirichlet boundary conditions and transmission conditions. Inequality (3.12) follows from (3.14), we obtain

$$\|u\|_{\mathrm{span}(\mathcal{W}^{2,p}(G);\mathcal{S})} \le C\|(A_p+1)u\|_{0,p,G}. \tag{3.15}$$

Since $D_{A_p}(G) = \mathrm{span}(\mathcal{W}^{2,p}(G);\mathcal{S})$, then $u \in D_{A_p}(G)$ can be written as

$$u = u_R + \sum_{\lambda_m \in]0,\, \frac{2}{p'}[,\ \lambda_m \neq 1} c_m S^{(m)}.$$

Therefore, if $s < \frac{2}{p} + \lambda_m$, for all m, $S^{(m)} \in \mathcal{W}^{s,p}(G)$ and we have

$$\begin{aligned}\|u\|_{\mathcal{W}^{s,p}(G)} &\le \|u_R\|_{\mathcal{W}^{s,p}(G)} + \sum_{\lambda_m \in]0,\, \frac{2}{p'}[,\ \lambda_m \neq 1} |c_m|\, \|S^{(m)}\|_{\mathcal{W}^{s,p}(G)},\\ &\le C\big\{\|u_R\|_{\mathcal{W}^{2,p}(G)} + \sum_{\lambda_m \in]0,\, \frac{2}{p'}[,\ \lambda_m \neq 1} |c_m|\big\},\end{aligned}$$

by Sobolev imbedding theorem. By the equivalence of norms on the space of finite dimension, the right-hand side is $\|u\|_{\mathrm{span}(\mathcal{W}^{2,p}(G);\mathcal{S})}$. Hence inequality (3.11) follows from (3.15).

If $s \ge \frac{2}{p} + \lambda_m$ for some m, we have

$$\|u\|_{\mathrm{span}(\mathcal{W}^{s,p}(G);\mathcal{S})} = \|u_R\|_{\mathcal{W}^{s,p}(G)} + \sum_{\lambda_m \in]0,\, \frac{2}{p'}[,\ \lambda_m \neq 1} |c_m|.$$

Inequality (3.12) follows from Sobolev imbedding theorem and inequality (3.15). $\square$

Now, using the results of Proposition 3.2 we shall state estimate on the resolvent of problem (3.1)-(3.4), this is the principle idea of the method of dilation which relies on applying the transformation $x \mapsto tx$. Taking advantage from the invariance of the infinite sector under dilation, problem (3.1)-(3.4) is transformed to similar problem with $\lambda = 1$.

3.2. **Estimates in dependence on λ.** In this section, we assume that $\lambda > 0$.

Proposition 3.3. *Let us assume that $\lambda_m \neq \frac{2}{p'}$ for all m and let $u \in D_{A_p}(G)$ be a solution of* (3.1) − (3.4)*. Then if $D_{A_p}(G) \subset \mathcal{W}^{s,p}(G)$, there exists a constant C such that*

$$\|u\|_{0,p,G} + \lambda^{-s/2}\|u\|_{\mathcal{W}^{s,p}(G)} \le \frac{C}{\lambda}\|(A_p+\lambda)u\|_{0,p,G}, \quad \text{for } 0 \le s < 1,\ (\lambda > 0), \tag{3.16}$$

and

$$\|u\|_{0,p,G} + \lambda^{-1/2}\|u\|_{1,p,G} + \lambda^{-s/2}\|u\|_{\mathcal{W}^{s,p}(G)} \le \frac{C}{\lambda}\|(A_p+\lambda)u\|_{0,p,G}, \tag{3.17}$$

for $1 \le s \le 2$, $(\lambda > 0)$. ($D_{A_p}(G) \subset \mathcal{W}^{s,p}(G)$ holds if $s < \lambda_m + \frac{2}{p}$ for all m).

Proof. Since the sectors G_i are invariant under positive dilations: $(x,y) \mapsto (tx,ty)$, $t > 0$, the solution $u \in D_{A_p}(G)$ of problem (3.1)-(3.4) satisfies

$$\begin{gathered} -\frac{\partial^2}{\partial (tx)^2}u_i(tx,ty) - \frac{\partial^2}{\partial (ty)^2}u_i(tx,ty) + \lambda u_i(tx,ty) = f_i(tx,ty) \quad \text{in } G_i,\\ u_i(tx,ty) = 0 \quad \text{on } \partial G_i \setminus \Sigma,\\ u_1(tx,ty) = u_2(tx,ty) \quad \text{on } \Sigma, \end{gathered}$$

$$\sum_{i=1}^{2} \alpha_i \frac{\partial u_i}{\partial \nu_i}(tx, ty) = 0 \quad \text{on } \Sigma.$$

Let $t = 1/\sqrt{\lambda}$, using the notation $u_i^t(x,y) = u_i(tx,ty)$, $f_i^t(x,y) = f_i(tx,ty)$, the above problem is equivalent to

$$\begin{gathered} -\Delta u_i^t + u_i^t = t^2 f_i^t \quad \text{in } G_i, \\ u_i^t = 0 \quad \text{on } \partial G_i \setminus \Sigma, \\ u_1^t = u_2^t \quad \text{on } \Sigma, \\ \sum_{i=1}^{2} \alpha_i \frac{\partial u_i^t}{\partial \nu_i^t} = 0 \quad \text{on } \Sigma, \end{gathered} \tag{3.18}$$

where ν_i^t is the normal vector with respect to the variables (tx, ty). By Proposition 3.2, u^t satisfies

$$\|u^t\|_{\mathcal{W}^{s,p}(G)} \le C\|t^2 f^t\|_{0,p,G}.$$

Using Proposition 6.10, we obtain

$$t^{-2/p}\big((1-t^s)\|u\|_{0,p,G} + t^s\|u\|_{\mathcal{W}^{s,p}(G)}\big) \le t^2 t^{-2/p}\|f\|_{0,p,G}, \quad \text{for } 0 \le s < 1$$

and

$$t^{-2/p}\big((1-t)\|u\|_{0,p,G} + t(1-t^{-1+s})\|u\|_{1,p,G} + t^s\|u\|_{\mathcal{W}^{s,p}(G)}\big) \le t^2 t^{-2/p}\|f\|_{0,p,G},$$

for $1 \le s \le 2$. This yields the required estimates for small t. □

Proposition 3.4. *Let $0 < s < 2$ and let $u \in D_{A_p}(G)$ be a solution of (3.1) − (3.4). Then if $D_{A_p}(G) \subset \operatorname{span}(\mathcal{W}^{s,p}(G); \mathcal{S})$, there exists a constant C such that*

$$\|u\|_{\operatorname{span}(\mathcal{W}^{s,p};\mathcal{S})(G)} \le \frac{C}{\lambda^{1-\frac{s}{2}}}\|(A_p + \lambda)u\|_{0,p,G}, \quad \lambda > 0. \tag{3.19}$$

Proof. We follow step by step the proof of [2, Theorem 3.10]. As in the proof of Proposition 3.3, problem (3.1)-(3.4) is transformed under the method of dilations to problem (3.18). In a neighborhood of the origin, the unique solutions of problems (3.1)-(3.4) and (3.18) may be written successively as

$$u(x,y) = u_R(x,y) + \sum_{\lambda_m \in]0, \frac{2}{p'}[, \ \lambda_m \ne 1} \gamma_m \eta(r) r^{\lambda_m} \varphi_m(\theta) \tag{3.20}$$

and

$$u^t(x,y) = v_R(x,y) + \sum_{\lambda_m \in]0, \frac{2}{p'}[, \ \lambda_m \ne 1} k_m \eta(r) r^{\lambda_m} \varphi_m(\theta),$$

where u_R and v_R are the regular parts, γ_m and k_m are the coefficients of the singular parts. Thanks to (3.12), we have

$$\|v_R\|_{\mathcal{W}^{s,p}(G)} \le C\|t^2 f^t\|_{0,p} = Ct^{2/p'}\|f\|_{0,p,G}, \tag{3.21}$$

$$|k_m| \le ct^{2/p'}\|f\|_{0,p,G}. \tag{3.22}$$

By the definition of u^t, we can write

$$u(x,y) = u^t(\frac{x}{t}, \frac{y}{t}) = v_R(\frac{x}{t}, \frac{y}{t}) + \sum_{\lambda_m \in]0, \frac{2}{p'}[, \ \lambda_m \ne 1} k_m \eta(\frac{r}{t}) \frac{1}{t^{\lambda_m}} r^{\lambda_m} \varphi_m(\theta). \tag{3.23}$$

Therefore, comparing (3.20) with (3.23) we obtain

$$u_R(x,y) = v_R(\frac{x}{t}, \frac{y}{t}) + \sum_{\lambda_m \in]0, \frac{2}{p'}[, \lambda_m \neq 1} r^{\lambda_m} \varphi_m(\theta)\big(k_m \eta(\frac{r}{t}) t^{-\lambda_m} - \gamma_m \eta(r)\big).$$

Since u_R and v_R have $\mathcal{W}^{2,p}(G)$ regularity, the term in brackets must vanishes in a neighborhood of the origin. Then $\gamma_m = k_m t^{-\lambda_m}$, and we have

$$u_R(x,y) = v_R(\frac{x}{t}, \frac{y}{t}) + \sum_{\lambda_m \in]0, \frac{2}{p'}[, \lambda_m \neq 1} r^{\lambda_m} \varphi_m(\theta)\gamma_m\big(\eta(\frac{r}{t}) - \eta(r)\big).$$

Consequently (3.22) leads to

$$|\gamma_m| \le C t^{\frac{2}{p'} - \lambda_m} \|f\|_{0,p,G}. \tag{3.24}$$

We shall now find a bound in $\|u_R\|_{\mathcal{W}^{s,p}(G)}$

$$\begin{aligned} \|u_R\|_{\mathcal{W}^{s,p}(G)} &\le \|v_R(\frac{\cdot}{t}, \frac{\cdot}{t})\|_{\mathcal{W}^{s,p}(G)} \\ &\quad + \sum_{\lambda_m \in]0, \frac{2}{p'}[, \lambda_m \neq 1} \|\gamma_m r^{\lambda_m} \varphi_m(\theta)[\eta(\frac{r}{t}) - \eta(r)]\|_{\mathcal{W}^{s,p}(G)}. \end{aligned} \tag{3.25}$$

Using (6.3) in Proposition 6.10 from the Appendix and (3.21), the first term in the right hand side in (3.25) is bounded by

$$t^{s-\frac{2}{p}} \|v_R(\frac{\cdot}{t}, \frac{\cdot}{t})\|_{\mathcal{W}^{s,p}(G)} \le \|v_R\|_{\mathcal{W}^{s,p}(G)} \le C t^{2-\frac{2}{p}} \|f\|_{0,p,G}.$$

The explicit form of the second term in (3.25) and the properties of the cut-off function η allows us to majorise it (see [2]):

$$\|\gamma_m r^{\lambda_m} t^m(\theta)(\eta(\frac{r}{t}) - \eta(r))\|_{\mathcal{W}^{s,p}(G)} \le C t^{2-s} \|f\|_{0,p,G}.$$

Summing up, we have the estimate

$$\|u_R\|_{\mathcal{W}^{s,p}(G)} \le c t^{2-s} \|f\|_{0,p,G}.$$

Owing to (3.20) and using (3.24), we obtain

$$\|u\|_{\mathrm{span}(\mathcal{W}^{s,p}(G);\mathcal{S})} \le C\Big(\lambda^{-1+\frac{s}{2}} + \sum_{\lambda_m \in]0, \frac{2}{p'}[, \lambda_m \neq 1} \lambda^{\frac{\lambda_m}{2} + \frac{1}{p} - 1}\Big) \|f\|_{0,p,G}.$$

As $s \ge \lambda_m + \frac{2}{p}$ implies $\frac{s}{2} - 1 \ge \frac{\lambda_m}{2} + \frac{1}{p} - 1$, we obtain the desired estimate for large λ as required. $\square$

Remark 3.5. Estimates (3.16), (3.17), (3.19) can be obtained with respect to $|\lambda|$ instead of λ for $\Re\lambda \ge 0$ and $\lambda \neq 0$. For this, we just replace in subsection 3.1 $(A_p + 1)u$ by $(A_p + \lambda)u$, $|\lambda| = 1$, and in the proof of Proposition 3.3, $t = 1/\sqrt{\lambda}$ by $t = 1/\sqrt{|\lambda|}$.

4. Resolvent estimate in dependence on λ in polygonal Ω

We consider in Ω (a polygon defined as in Section 2) the transmission problem (1.2). The results exposed in Section 2 ensures the existence of the resolvent $(A_p+\lambda)^{-1}$, where A_p is defined in the introduction, we recall that $D_{A_p}(\Omega) = \mathrm{span}(\mathcal{W}^{2,p}(\Omega);\mathcal{S})$, where $\mathcal{S}$ stands the set of singular functions

$$\mathcal{S} := \{S^{(jm)};\lambda_{jm}\in]0,\frac{2}{p'}[\text{ with } \lambda_{jm}\neq 1\}.$$

We shall now deduce the growth with respect to λ of $\|(A_p+\lambda)^{-1}f\|_{\mathcal{W}^{s,p}(\Omega)}$ $(\lambda>0)$ and $\|(A_p+\lambda)^{-1}f\|_{\mathrm{span}(\mathcal{W}^{s,p}(\Omega);\mathcal{S})}$.

Theorem 4.1. (i) *If $s<\lambda_{jm}+\frac{2}{p}$ for all j and m, then the unique solution $u\in D_{A_p}(\Omega)$ of (1.2) belongs to $\mathcal{W}^{s,p}(\Omega)$ and satisfies*

$$\|u\|_{\mathcal{W}^{s,p}(\Omega)} \le \frac{c}{\lambda^{1-\frac{s}{2}}}\|f\|_{0,p,\Omega}.$$

(ii) *If $s>\lambda_{jm}+\frac{2}{p}$ for some (j,m), then the unique solution $u\in D_{A_p}(\Omega)$ of (1.2) belongs to* $\mathrm{span}(\mathcal{W}^{s,p}(\Omega);\mathcal{S})$ *and satisfies*

$$\|u\|_{\mathrm{span}(\mathcal{W}^{s,p}(\Omega);\mathcal{S})} \le \frac{c}{\lambda^{1-\frac{s}{2}}}\|f\|_{0,p,\Omega}.$$

Proof. Let us cover Ω by a partition of unity φ_i, $i=1,2,\dots,N_1+N_2-2$. That is, $\Omega\subset\cup_{i=1}^{n}\theta_i$ and $\varphi_i\in D(\theta_i)$; $\sum_{i=1}^{N_1+N_2-2}\varphi_i=1$. We denote by $\widetilde{\varphi_i u}$ the extension of $\varphi_i u$ by zero outside of $\mathrm{supp}(\varphi_i u)$. There are two typical cases to consider:

- if $i=1$ or $i=N_1$, $\widetilde{\varphi_i u}$ is solution of the transmission problem (3.1)-(3.4) in an infinite sector G. Therefore it satisfies the estimates in Proposition 3.3 and Proposition 3.4.
- if $i\neq 1$ and $i\neq N_1$, $\widetilde{\varphi_i u}$ is solution of a Dirichlet problem for the Laplace operator in an infinite sector, consequently it also satisfies the estimates in Proposition 3.3 and Proposition 3.4 with $W^{s,p}$ instead of $\mathcal{W}^{s,p}$ (see [2, Proposition 3.8 and Theorem 3.10]).

We continue exactly as in the proof of [2, Theorems 4.1 and 4.3]. □

5. Characterization of $D_{A_p}(\theta;p)$

Theorem 5.1. *Suppose that $\lambda_{jm}\neq\frac{2}{p'}$, for all j and m, and set*

$$\mu := \min_{m\in\mathbb{N}^*,\ 1\le j\le N_1+N_2-2}\{\lambda_{jm};\lambda_{jm}\in]0,\frac{2}{p'}[,\ \lambda_{jm}\neq 1\},$$

then

(i) $(D_{A_p}(\Omega),L^p(\Omega))_{\beta,p}\subset(\mathcal{W}^{2,p}(\Omega),L^p(\Omega))_{\beta,p}$, *for* $1-\frac{\mu}{2}-\frac{1}{p}<\beta<1$,
(ii) $(D_{A_p}(\Omega),L^p(\Omega))_{\beta,p}\subset\mathrm{span}\big((\mathcal{W}^{2,p}(\Omega),L^p(\Omega))_{\beta,p};\mathcal{S}\big)$, *for* $0<\beta<1-\frac{\mu}{2}-\frac{1}{p}$,
(iii) $\mathrm{span}\big((\mathcal{W}^{2,p}(\Omega),L^p(\Omega))_{\theta,p};\mathcal{S}\big)\subset(D_{A_p}(\Omega),L^p(\Omega))_{\theta,p}$, *for* $0<\theta<1$.

Consequently

$$D_{A_p}(\theta;p)=\begin{cases}(\mathcal{W}^{2,p}(\Omega),L^p(\Omega))_{1-\theta,p}, & \text{if } 0<\theta<\frac{\mu}{2}+\frac{1}{p},\\ \mathrm{span}\big((\mathcal{W}^{2,p}(\Omega),L^p(\Omega))_{1-\theta,p};\mathcal{S}\big), & \text{if } \frac{\mu}{2}+\frac{1}{p}<\theta<1.\end{cases}$$

Proof. (i) Let $s < \mu + \frac{2}{p}$, then $s < \lambda_{jm} + \frac{2}{p}$, for all j and m, thus from Theorem 4.1, $D_{A_p}(\Omega) \subset \mathcal{W}^{s,p}(\Omega)$ and

$$\|(A_p + \lambda)^{-1}\|_{L^p(\Omega)\to\mathcal{W}^{s,p}(\Omega)} \le \frac{c}{\lambda^{1-\frac{s}{2}}}.$$

Now, applying Corollary 6.6 with $E = L^p(\Omega), F = \mathcal{W}^{s,p}(\Omega)$ and $\alpha = 1 - \frac{s}{2}$ we obtain

$$(D_{A_p}(\Omega), L^p(\Omega))_{(1-\theta)(1-\frac{s}{2})+\theta,p} \subset (\mathcal{W}^{s,p}(\Omega), L^p(\Omega))_{\theta,p}, \quad 0 < \theta < 1,$$

therefore, by simple substitution

$$\begin{aligned}(D_{A_p}(\Omega), L^p(\Omega))_{\beta,p} &\subset (\mathcal{W}^{s,p}(\Omega), L^p(\Omega))_{\frac{2}{s}(\beta-1+\frac{s}{2}),p}\\ &\subset \big(\Pi_{i=1}^2 W^{s,p}(\Omega_i), \Pi_{i=1}^2 L^p(\Omega_i)\big)_{\frac{2}{s}(\beta-1+\frac{s}{2}),p},\end{aligned}$$

consequently, thanks to Proposition 6.8, we obtain

$$\begin{aligned}(D_{A_p}(\Omega), L^p(\Omega))_{\beta,p} &\subset \Pi_{i=1}^2\big(W^{s,p}(\Omega_i), L^p(\Omega_i)\big)_{\frac{2}{s}(\beta-1+\frac{s}{2}),p}\\ &= \Pi_{i=1}^2\Big(\big(W^{2,p}(\Omega_i), L^p(\Omega_i)\big)_{1-\frac{s}{2},p}, L^p(\Omega_i)\Big)_{\frac{2}{s}(\beta-1+\frac{s}{2}),p}\\ &= \Pi_{i=1}^2\big(W^{2,p}(\Omega_i), L^p(\Omega_i)\big)_{\beta,p},\end{aligned}$$

the last step is by reiteration (see Corollary 6.4). A second application of Proposition 6.8 leads to

$$(D_{A_p}(\Omega), L^p(\Omega))_{\beta,p} \subset (\Pi_{i=1}^2 W^{2,p}(\Omega_i), L^p(\Omega))_{\beta,p}.$$

Finally, $0 < s < \mu + \frac{2}{p}$ implies that $1 - \frac{\mu}{2} - \frac{1}{p} < \beta < 1$, hence (i) is proved.

(ii) Let $s > \mu + \frac{2}{p}$, we have from Theorem 4.1,

$$\begin{gathered}D_{A_p}(\Omega) \subset \operatorname{span}(\mathcal{W}^{s,p}(\Omega); \mathcal{S}),\\ \|(A_p + \lambda)^{-1}\|_{L^p(\Omega)\to\operatorname{span}(\mathcal{W}^{s,p}(\Omega);\mathcal{S})} \le \frac{c}{\lambda^{1-\frac{s}{2}}}.\end{gathered}$$

Here, we apply Corollary 6.7 like in the first case but with $F = \operatorname{span}(\mathcal{W}^{s,p}(\Omega); \mathcal{S})$, we obtain

$$\begin{aligned}\big(D_{A_p}(\Omega), L^p(\Omega)\big)_{\theta(1-\frac{s}{2}),p} &\subset \big(\operatorname{span}(\mathcal{W}^{2,p}; \mathcal{S}), \operatorname{span}(\mathcal{W}^{s,p}; \mathcal{S})\big)_{\theta,p}\\ &= \operatorname{span}\Big(\big(\mathcal{W}^{2,p}(\Omega), \mathcal{W}^{s,p}(\Omega)\big)_{\theta,p}; \mathcal{S}\Big)\\ &\subset \operatorname{span}\Big(\big(\Pi_{i=1}^2 W^{2,p}(\Omega_i), L^p(\Omega)\big)_{\theta(1-\frac{s}{2}),p}; \mathcal{S}\Big).\end{aligned}$$

(iii) Clearly $\mathcal{S} \subset \big(D_{A_p}(\Omega), L^p(\Omega)\big)_{\theta,p}$ for all $0 < \theta < 1$. Further, $\mathcal{W}^{2,p}(\Omega) \subset D_{A_p}(\Omega)$. Hence

$$\big(\mathcal{W}^{2,p}(\Omega), L^p(\Omega)\big)_{\theta,p} \subset \big(D_{A_p}(\Omega), L^p(\Omega)\big)_{\theta,p}.$$

Therefore,

$$\operatorname{span}\big((\mathcal{W}^{2,p}(\Omega), L^p(\Omega))_{\theta,p}; \mathcal{S}\big) \subset \big(D_{A_p}(\Omega), L^p(\Omega)\big)_{\theta,p}.$$

□

6. Appendix

6.1. **The real interpolation spaces.** We recall here some basic material on the theory of real interpolation spaces and refer to [7].

Definition 6.1. Let A_0 and A_1 be two Banach spaces such that $A_0 \subset A_1$ with continuous injection. The space $(A_0, A_1)_{\theta,p}$ is the subspace of A_1 consisting of $x \in A_1$ such that there exists two functions u_0 and u_1 satisfying

$$\begin{gathered} x = u_0(t) + u_1(t), \quad t > 0, \\ t^{-\theta} u_0 \in L^p_*(A_0), \quad t^{1-\theta} u_1 \in L^p_*(A_1), \end{gathered} \tag{6.1}$$

where $L^p_*(A_0)$ and $L^p_*(A_1)$ are function spaces defined on $(0, +\infty)$ taking values in A_0 and A_1 respectively with the pth power integrable in the measure $\frac{dt}{t}, 1 \le p \le \infty$ and $0 < \theta < 1$. The norm of the space $(A_0, A_1)_{\theta,p}$ is

$$\|x\|_{(A_0,A_1)_{\theta,p}} = \inf\Big\{ \Big(\int_0^\infty \|t^{-\theta} u_0(t)\|^p_{A_0} \frac{dt}{t} \Big)^{1/p} + \Big(\int_0^\infty \|t^{1-\theta} u_1(t)\|^p_{A_1} \frac{dt}{t} \Big)^{1/p} \Big\},$$

the infimum is taken over all functions satisfying (6.1).

In the particular case, when A_0 is the domain D_A of a closed linear operator A in $E \equiv A_1$, equipped with the graph norm, we have another characterization which is very useful for identifying the spaces in concrete examples. Let $\rho(A) \supset \mathbb{R}_+$ and there exist C_A such that

$$\|(A + \lambda)^{-1}\|_{E \to E} \le \frac{C_A}{\lambda}, \quad \lambda > 0,$$

then $D_A(\theta; p)$ is the subspace of E consisting of x such that

$$t^\theta A(A + t)^{-1} x \in L^p_*(E).$$

The equivalence result is $D_A(\theta; p) \equiv (D_A, E)_{1-\theta,p}$.

Definition 6.2. A subspace X of $A_0 + A_1$ belongs to class $\underline{K}_\theta(A_0, A_1)$ if there exists a constant C such that

$$\|a\|_X \le C \|a\|^{1-\theta}_{A_0} \|a\|^\theta_{A_1}$$

for every $a \in A_0 \cap A_1$ assuming $0 \le \theta \le 1$. Equivalently, $(A_0, A_1)_{\theta,1} \subset X$. Thus $(A_0, A_1)_{\theta,p}$ is of class $\underline{K}_\theta(A_0, A_1)$.

We have the following reiteration theorem.

Theorem 6.3. *Let $X_i \in \underline{K}_{\theta_i}(A_0, A_1), i = 0, 1$, then*

$$(A_0, A_1)_{(1-\theta)\theta_0 + \theta\theta_1} \subset (X_0, X_1)_{\theta,p}, \ 0 < \theta < 1.$$

Corollary 6.4. *For $0 < \theta_0$, $\theta_1 < 1$, $1 \le p, q \le \infty$, we have*

$$\begin{gathered} ((X, Y)_{\theta_0,q}, Y)_{\theta,p} = (X, Y)_{(1-\theta)\theta_0 + \theta, p}, \\ (X, (X, Y)_{\theta_1,q})_{\theta,p} = (X, Y)_{\theta_1\theta, p}. \end{gathered}$$

The following result is due to Grisvard.

Theorem 6.5. *Let A be a closed operator with domain D_A in a Banach space E. Assume F is a Banach space such that $D_A \subset F \subset E$, with continuous injections (for the graph norm on D_A). Further, assume $(A + t)^{-1}$ exists for every $t \ge 0$ and there exists $\alpha \in (0, 1)$ such that*

$$\|(A + t)^{-1}\|_{E \to F} = O(t^{-\alpha}),$$

then $F \in \underline{K}_\alpha(D_A, E)$.

Corollary 6.6. *Under the assumption of Theorem 6.5,*

$$(D_A, E)_{(1-\theta)\alpha+\theta,p} \subset (F, E)_{\theta,p}.$$

The above corollary follows from Theorem 6.3 with $A_0 = D_A$, $A_1 = E$, $X_0 = F$, $X_1 = E$, $\theta_0 = \alpha$, $\theta_1 = 1$ and recall that trivially, $E \in \underline{K}_1(D_A, E)$.

Corollary 6.7. *Under the assumptions of Theorem 6.5*

$$(D_A, E)_{\alpha\theta,p} \subset (D_A, F)_{\theta,p}.$$

The above corollary follows from Theorem 6.3 with $A_0 = D_A$, $A_1 = E$, $X_0 = D_A$, $X_1 = F$, $\theta_0 = 0$, $\theta_1 = \alpha$ and recall that trivially, $D_A \in \underline{K}_0(D_A, E)$.

Proposition 6.8 ([4])**.** *Let A, B, C, D be Banach spaces such that C is continuously embedded into A; D is continuously embedded into B, then*

$$(A \times B, C \times D)_{\theta,p} = (A, C)_{\theta,p} \times (B, D)_{\theta,p}.$$

Proof. By using a.e. the equivalence theorem (see [9, p. 37]) and taking into account

$$(a + b)^p \le 2^{p-1}(a^p + b^p),\ a, b \ge 0,\ p \ge 1.$$

□

6.2. **Some basic tools.**

Proposition 6.9. *Let E, H be Banach spaces, $D = E \oplus F$ with $\dim F < \infty$. Assume that a continuous injective mapping A from D to H satisfies*

$$\|u\|_E \le c\|Au\|_H \tag{6.2}$$

for all $u \in E$ and some constant c. Then

$$\|u\|_D \le c'\|Au\|_H$$

for all $u \in D$ and some constant c'.

Proposition 6.10. *Let G be an infinite sector with vertex at the origin. Let v be a function in $W^{s,p}(G)$ with $0 \le s \le 2$. Since G is invariant under the transformation $(x, y) \mapsto (tx, ty)$, $t > 0$, $(x, y) \in G$, the function $v^t(x, y) = v(tx, ty)$ is well defined and $v^t \in W^{s,p}(G)$ with*

$$\|v^t\|_{s,p,G} = \begin{cases} t^{-2/p}\big((1 - t^s)\|v\|_{0,p,G} + t^s\|v\|_{s,p,G}\big) & \text{if } 0 \le s \le 1, \\ t^{-2/p}\big((1 - t)\|v\|_{0,p,G} + (t - t^s)\|v\|_{1,p,G} + t^s\|v\|_{s,p,G}\big) & \text{if } 1 \le s \le 2. \end{cases}$$

Consequently,

$$\|v^t\|_{s,p,G} \ge\ t^{s-\frac{2}{p}}\|v\|_{s,p,G}, \tag{6.3}$$

holds for small t.

Proof. Let $\mathbf{x} = (x, y)$, $\mathbf{x}' = (x', y')$.

$$\|v^t\|_{s,p,G} = \begin{cases} t^{-2/p}\Big(\|v\|_{0,p,G} + t^s\Big(\int\int_{G\times G} \frac{|v(\mathbf{x})-v(\mathbf{x}')|^p}{|\mathbf{x}-\mathbf{x}'|^{2+sp}}\, d\mathbf{x}\, d\mathbf{x}'\Big)^{1/p}\Big) \\ \quad \text{if } 0 < s < 1, \\ t^{-2/p}\big(\|v\|_{0,p,G} + t|v|_{1,p,G}\big) \quad \text{if } s = 1, \\ t^{-\frac{2}{p}+s}\Big(t^{-s}\|v\|_{0,p,G} + t^{1-s}|v|_{1,p,G} \\ +\Big(\int\int_{G\times G} \frac{|\frac{\partial v}{\partial x}(\mathbf{x})-\frac{\partial v}{\partial x}(\mathbf{x}')|^p}{|\mathbf{x}-\mathbf{x}'|^{2+\sigma p}}\, d\mathbf{x}\, d\mathbf{x}'\Big)^{1/p} \\ +\Big(\int\int_{G\times G} \frac{|\frac{\partial v}{\partial y}(\mathbf{x})-\frac{\partial v}{\partial y}(\mathbf{x}')|^p}{|\mathbf{x}-\mathbf{x}'|^{2+\sigma p}}\, d\mathbf{x}\, d\mathbf{x}'\Big)^{1/p}\Big), \\ \quad \text{if } 1 < s < 2, \text{ with } s = 1+\sigma, 0 < \sigma < 1, \\ t^{-2/p}\big(\|v\|_{0,p,G} + t|v|_{1,p,G} + t^2|v|_{2,p,G}\big) \quad \text{if } s = 2, \end{cases}$$

$$= \begin{cases} t^{-2/p}\big(\|v\|_{0,p,G} + t^s(\|v\|_{s,p,G} - \|v\|_{0,p,G})\big) \quad \text{if } 0 < s \le 1, \\ t^{-2/p}\big(\|v\|_{0,p,G} + t(\|v\|_{1,p,G} - \|v\|_{0,p,G}) + t^s(\|v\|_{s,p,G} - \|v\|_{1,p,G})\big) \\ \quad \text{if } 1 < s \le 2, \text{ with } s = 1+\sigma,\ 0 < \sigma \le 1. \end{cases}$$

□

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