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FRACTIONAL-POWER APPROACH FOR SOLVING COMPLETE ELLIPTIC ABSTRACT DIFFERENTIAL EQUATIONS WITH VARIABLE-OPERATOR COEFFICIENTS

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Abstract. This work is devoted to the study of a complete abstract second-order differential equation of elliptic type with variable operators as coefficients. A similar equation was studied by Favini et al [6] using Green's kernels and Dunford functional calculus. Our approach is based on the semigroup theory, the fractional powers of linear operators, and the Dunford's functional calculus. We will prove the main result on the existence and uniqueness of a strict solutions using combining assumptions from Yagi [16], Da Prato-Grisvard [3], and Acquistapace-Terreni [1].

1. Introduction

In a complex Banach space X, we consider the complete abstract second-order differential equation with variable operators as coefficients

$$u''(x) + B(x)u'(x) + A(x)u(x) - \lambda u(x) = f(x), \quad x \in (0,1) \tag{1.1}$$

under the Dirichlet boundary conditions

$$u(0) = \varphi, \quad u(1) = \psi\,. \tag{1.2}$$

Here λ is a positive real number, $f \in C^\theta([0,1];X)$, $0 < \theta < 1$, φ and ψ are given elements in X, $(B(x))_{x\in[0,1]}$ is a family of bounded linear operators, and $(A(x))_{x\in[0,1]}$ is a family of closed linear operators whose domains $D(A(x))$ are not necessarily dense in X. Set

$$Q(x) = A(x) - \lambda I, \quad \lambda > 0,$$

and consider Problem (1.1)-(1.2) in an elliptic setting. We assume that the family of closed linear operators $(Q(x))_{x\in[0,1]}$ with domains $D(Q(x))$ satisfies the condition: There exists $C > 0$ such that for all $x \in [0,1]$ and all $z \ge 0$, exists $(Q(x) - zI)^{-1}$ in $L(X)$ and

$$\|(Q(x) - zI\ )^{-1}\|_{L(X)} \le C/(1+z), \tag{1.3}$$

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which holds in the sector (where θ_0 and r_0 are small positive numbers)

$$\Pi_{\theta_0,r_0} = \{z \in \mathbb{C}\backslash\{0\} : |\arg(z)| \le \theta_0\} \cup \{z \in \mathbb{C} : |z| \le r_0\}.$$

We assume that for the coefficients $B(x)$, there exists $C > 0$ such that for all $x \in [0,1]$,

$$\|B(x)\|_{L(X)} \le C. \tag{1.4}$$

The term $B(x)u'(x)$ in this work is considered as a "perturbation" in some sense.

Recently, Favini et al [6] studied this equation under some differentiability assumptions on the resolvents of operators $Q(x)$, and used Dunford's operational calculus to obtain some results concerning the solution.

In this article, we give an alternative approach: for each $x \in [0,1]$, Assumption (1.3) leads us to consider the square roots $\sqrt{-Q(x)}$. It is well known that $-\sqrt{-Q(x)}$ generates an analytic semigroup not necessarily strongly continuous at 0 (see Balakrishnan [2] for dense domains and Martinez-Sanz [13] for nondense domains). Therefore, we use this important property to analyze and improve the resutls cited above under some natural differentiability assumptions on the square roots $\sqrt{-Q(x)}$, extending the study of Acquistapace-Terreni [1] in the parabolic case. We also use real interpolation theory and Dunford's functional calculus. Here, we obtain necessary and sufficient conditions for the existence of the strict solution, by using the square roots $-\sqrt{-Q(x)}$, while in [6], the authors give only a sufficient conditions, by using the operators $Q(x)$. This equation was also studied by means of operator sums theory in Da Prato-Grisvard [3], while we opt for the technique of the fractional powers of linear operators.

This article is organized as follows: In Section 2, we specify the assumptions and we give the representation of the solution. Section 3 is devoted to the analysis of operators $e^{xK(x)}\varphi$, $\frac{d}{dx}(e^{xK(x)}\varphi)$ and $\frac{d^2}{dx^2}(e^{xK(x)}\varphi)$. In Section 4, we study the regularity of the functions written in the representation of the solution. In section 5, we will present an equation satisfied by the solution and then we will solve it. In Section 6, we will prove the main result on the existence and uniqueness of the strict solution of (1.1)-(1.2) (see Theorem 6.1). Finally, we will present an example of a partial differential equation where our abstract results apply.

2. Assumptions and representation of the solution

Assumptions. By (1.3), it is possible to define fractional powers of $(-Q(x))$. In particular, for every $x \in [0,1]$ and $\lambda > 0$, the square roots $-(-Q(x))^{1/2}$ are well defined and generate analytic semigroups

$$\left(e^{-(-Q(x))^{1/2}y}\right)_{y>0} = \left(e^{(-(-A(x)+\lambda)^{1/2})y}\right)_{y>0},$$

not necessarily strongly continuous at 0. It is well known that there exists a second sector

$$\Pi_{\theta_1+\pi/2,r_1} = \{z \in \mathbb{C}\backslash\{0\} : |\arg(z)| \le \theta_1 + \pi/2\} \cup \{z \in \mathbb{C} : |z| \le r_1\}$$

with small $\theta_1 > 0$ and $r_1 > 0$ such that $\rho(-(-Q(x))^{1/2}) \supset \Pi_{\theta_1+\pi/2,r_1}$ for every $x \in [0,1]$. Set

$$\Gamma_0 = \{z = \rho e^{\pm i\theta_0} : \rho \ge r_0\} \cup \{z \in \mathbb{C} : |z| = r_0 \text{ and } |\arg(z)| \ge \theta_0\}$$

oriented from $\infty e^{-i\theta_0}$ to $\infty e^{i\theta_0}$ and

$$\Gamma_1 = \{z = \rho e^{\pm i(\theta_1+\pi/2)} : \rho \ge r_1\} \cup \{z \in \mathbb{C} : |z| = r_1, |\arg(z)| \ge \theta_1 + \pi/2\},$$

$$\Gamma_2 = \{z = \rho e^{\pm i(\theta_1+\pi/2)} : \rho \geq r_1\} \cup \{z \in \mathbb{C} : |z| = r_1, |\arg(z)| \leq \theta_1 + \pi/2\}$$

oriented from $\infty e^{-i(\theta_1+\pi/2)}$ to $\infty e^{i(\theta_1+\pi/2)}$. Therefore, for all $x \in [0,1]$, $y > 0$ and all positive integer k, one has

$$\begin{gathered} e^{-(-Q(x))^{1/2}y} = -\frac{1}{2i\pi}\int_{\Gamma_1} e^{zy}\big(-(-Q(x))^{1/2} - zI\big)^{-1}dz, \\ [-(-Q(x))^{1/2}]^k e^{-(-Q(x))^{1/2}y} = -\frac{1}{2i\pi}\int_{\Gamma_1} z^k e^{zy}\big(-(-Q(x))^{1/2} - zI\big)^{-1}dz. \end{gathered} \tag{2.1}$$

The above equality has an analytic continuation (in z) in the sector $\Pi_{\theta_1+\pi/2,r_1}$ and for all $z \geq 0$, $x \in [0,1]$ (see Tanabe [15, p. 36, Formula (2.29)])

$$\Big(-(-Q(x))^{1/2} - zI\Big)^{-1} = -\frac{1}{2i\pi}\int_{\Gamma_0} \frac{(Q(x)-sI)^{-1}}{(-s)^{1/2}+z}ds. \tag{2.2}$$

The following formula also holds (see [15], p. 37, Relation (2.32))

$$\Big(-(-Q(x))^{1/2} - zI\Big)^{-1} = \frac{1}{\pi}\int_0^{\infty} \frac{\sqrt{s}(Q(x)-sI)^{-1}}{s+z^2}ds. \tag{2.3}$$

Remark 2.1. Also we have the representation of the semigroup

$$e^{-(-Q(x))^{1/2}y} = -\frac{1}{2i\pi}\int_{\Gamma_1} e^{zy}\Big(-(-Q(x))^{1/2} - zI\Big)^{-1}dz, \tag{2.4}$$

on Γ_2 and all the calculus with respect it can be done on Γ_1 or Γ_2.

Set

$$K(x) = -(-Q(x))^{1/2}.$$

Lemma 2.2. *Under Hypothesis* (1.3)*, there exists a constant $C > 0$ such that for all $z \in \Pi_{\theta_1+\pi/2,r_1}$,*

$$\|(K(x)-zI)^{-1}\|_{L(X)} \leq \frac{C}{|z|}. \tag{2.5}$$

In addition to Assumptions (1.3) and (1.4), we will assume that:

For all $z \in \Pi_{\theta_1+\pi/2,r_1}$, the mapping $x \mapsto (K(x)-zI)^{-1}$, defined on $[0,1]$, is in $C^2([0,1], L(X))$ and there exist $C > 0$, $\nu \in]1/2,1]$ and $\eta \in]0,1[$ such that for all $z \in \Pi_{\theta_1+\pi/2,r_1}$, and all $x, s \in [0,1]$,

$$\|\frac{\partial}{\partial x}(K(x)-zI)^{-1}\|_{L(X)} \leq \frac{C}{|z|^{\nu}}, \tag{2.6}$$

$$B(0)(X) \subset \overline{D(K(0))}, \quad B(1)(X) \subset \overline{D(K(1))}, \tag{2.7}$$

$$\begin{gathered} \frac{d}{dx}(K(x))^{-1}_{|x=0}(D(K(0))) \subset \overline{D(K(0))}, \\ \frac{d}{dx}(K(x))^{-1}_{|x=1}(D(K(1)) \subset \overline{D(K(1))}, \end{gathered} \tag{2.8}$$

$$\|\frac{\partial}{\partial x}(K(x)-zI)^{-1} - \frac{\partial}{\partial s}(K(s)-zI)^{-1}\|_{L(X)} \leq \frac{C|x-s|^{\eta}}{|z|^{\nu}}, \quad \eta+\nu-1>0, \tag{2.9}$$

$$\|\frac{\partial^2}{\partial x^2}(K(x)-zI)^{-1}\|_{L(X)} \leq C|z|^{1-\nu}, \tag{2.10}$$

$$\|\frac{d^2}{dx^2}(K(x))^{-1} - \frac{d^2}{ds^2}(K(s))^{-1}\|_{L(X)} \leq C|x-s|^{\eta}. \tag{2.11}$$

Remark 2.3. Observe that all the constants given above are independent of x and always, we have $\eta+\nu-1<\nu$ and $\eta+\nu-1<\eta$. Moreover, we can replace z by $\sqrt{\lambda}+z$ in (2.6), (2.9) and (2.10).

Comments on the Hypotheses. (1) We can also solve our problem using the following assumption of Yagi's type [17]:

$$\|K(x)(K(x)-zI)^{-1}\frac{d}{dx}(K(x))^{-1}\|_{L(X)} \leq \frac{C}{|z|^{\nu}}. \tag{2.12}$$

It is clear that (2.6) is slightly weaker in comparison with inequality (2.12). In fact (2.12) leads to ((2.6) by using the formula

$$\frac{\partial}{\partial x}(K(x)-zI)^{-1} = K(x)(K(x)-zI)^{-1}\frac{d}{dx}(K(x))^{-1}K(x)(K(x)-zI)^{-1}. \tag{2.13}$$

(2) It is easy to check that the inequality

$$\begin{aligned}
&\|K(x)(K(x)-zI)^{-1}\frac{d}{dx}(K(x))^{-1} - K(s)(K(s)-zI)^{-1}\frac{d}{ds}(K(s))^{-1}\|_{L(X)}\\
&\leq \frac{C|x-s|^{\eta}}{|z|^{\nu}}
\end{aligned}$$

implies (2.9) and since

$$\begin{aligned}
&K(x)(K(x)-zI)^{-1}\frac{d}{dx}(K(x))^{-1} - K(s)(K(s)-zI)^{-1}\frac{d}{ds}(K(s))^{-1}\\
&= K(x)(K(x)-zI)^{-1}\Big(\frac{d}{dx}(K(x))^{-1} - \frac{d}{ds}(K(s))^{-1}\Big)\\
&\quad + (K(x)(K(x)-zI)^{-1} - K(s)(K(s)-zI)^{-1})\frac{d}{ds}(K(s))^{-1}\\
&= K(x)(K(x)-zI)^{-1}\Big(\frac{d}{dx}(K(x))^{-1} - \frac{d}{ds}(K(s))^{-1}\Big)\\
&\quad + z\Big(\int_s^x \frac{\partial}{\partial r}(K(r)-zI)^{-1}dr\Big)\frac{d}{ds}(K(s))^{-1},
\end{aligned}$$

it follows that

$$\begin{aligned}
&\frac{d}{dx}(K(x))^{-1} - \frac{d}{ds}(K(s))^{-1}\\
&= (K(x)-zI)(K(x))^{-1}K(x)(K(x)-zI)^{-1}\frac{d}{dx}(K(x))^{-1}\\
&\quad - K(s)(K(s)-zI)^{-1}\frac{d}{ds}(K(s))^{-1}\\
&\quad - z(K(x)-zI)(K(x))^{-1}\Big(\int_s^x \frac{\partial}{\partial r}(K(r)-zI)^{-1}dr\Big)\frac{d}{ds}(K(s))^{-1}.
\end{aligned}$$

For $z=1$, one obtains

$$\begin{aligned}
&\frac{d}{dx}(K(x))^{-1} - \frac{d}{ds}(K(s))^{-1}\\
&= (K(x)-I)(K(x))^{-1}K(x)(K(x)-I)^{-1}\frac{d}{dx}(K(x))^{-1}\\
&\quad - K(s)(K(s)-I)^{-1}\frac{d}{ds}(K(s))^{-1}\\
&\quad - (K(x)-I)(K(x))^{-1}\Big(\int_s^x e^{xz}\frac{\partial}{\partial r}(K(r)-I)^{-1}dr\Big)\frac{d}{ds}(K(s))^{-1},
\end{aligned}$$

which leads to

$$\|\frac{d}{dx}(K(x))^{-1}-\frac{d}{ds}(K(s))^{-1}\|_{L(X)}\leq C(|x-s|^{\eta}+|x-s|)\leq C|x-s|^{\eta}.$$

(3) From (2.13), we also derive the formula

$$\begin{aligned}&\frac{\partial^2}{\partial x^2}(K(x)-zI)^{-1}\\&=K(x)(K(x)-zI)^{-1}\frac{d^2(K(x))^{-1}}{dx^2}K(x)(K(x)-zI)^{-1}\\&\quad+2zK(x)(K(x)-zI)^{-1}\Big(\frac{d(K(x))^{-1}}{dx}K(x)(K(x)-zI)^{-1}\Big)^2.\end{aligned}\tag{2.14}$$

(4) Observe that, for a large enough $|z|$, Formula (2.14) leads to (2.10), since

$$\|\frac{\partial^2}{\partial x^2}(K(x)-zI)^{-1}\|_{L(X)}=O(1)+O(|z|^{1-\nu})=O(|z|^{1-\nu}).$$

Also, one deduces from (2.11) that (see [10, p.113])

$$\begin{aligned}&\|\frac{d^2}{dx^2}(K(x)-z)^{-1}-\frac{d^2}{ds^2}(K(s)-z)^{-1}\|_{L(X)}\\&\leq C(|x-s|^{\eta}+(|x-s|+|x-s|^{\eta})|z|^{1-\nu}+|x-s||z|^{2-2\nu}).\end{aligned}\tag{2.15}$$

Representation of the solution. Let us recall briefly the case when $B(x)\equiv 0$ and $Q(x)=Q$ is a constant operator satisfying the natural ellipticity hypothesis mentioned above (we will take $K=-(-Q)^{1/2}$). In this case, the representation of the solution u is given by the formula (see Krein [9] or Favini et al [5])

$$u(x)=e^{xK}\xi_0+e^{(1-x)K}\xi_1+\frac{1}{2}\int_0^x e^{(x-s)K}K^{-1}f(s)ds+\frac{1}{2}\int_x^1 e^{(s-x)K}K^{-1}f(s)ds,$$

where

$$\begin{aligned}\xi_0&=(I-Z)^{-1}(\varphi-e^{K}\psi)-\frac{(I-Z)^{-1}}{2}\int_0^1 e^{sK}K^{-1}f(s)ds\\&\quad+\frac{(I-Z)^{-1}}{2}\int_0^1 e^{(2-s)K}K^{-1}f(s)ds,\\ \xi_1&=(I-Z)^{-1}(\psi-e^{K}\varphi)-\frac{(I-Z)^{-1}}{2}\int_0^1 e^{(1-s)K}K^{-1}f(s)ds\\&\quad+\frac{(I-Z)^{-1}}{2}\int_0^1 e^{(1+s)K}K^{-1}f(s)ds,\end{aligned}$$

and

$$Z=e^{2K},\quad (I-Z)^{-1}=\frac{1}{2i\pi}\int_{\Gamma_1}\frac{e^{2z}}{1-e^{2z}}(K-zI)^{-1}dz+I.$$

(see Lunardi [12, p. 60] for the invertibility of $I-Z$). In our situation, we will seek a solution of (1.1)-(1.2) in the form

$$\begin{aligned}u(x)&=e^{xK(x)}\xi_0^*(x)+e^{(1-x)K(x)}\xi_1^*(x)+\frac{1}{2}\int_0^x e^{(x-s)K(x)}(K(x))^{-1}f^*(s)ds\\&\quad+\frac{1}{2}\int_x^1 e^{(s-x)K(x)}(K(x))^{-1}f^*(s)ds,\end{aligned}\tag{2.16}$$

where

$$\begin{aligned}\xi_0^*(x) &= (I-Z(x))^{-1}(\varphi^* - e^{K(x)}\psi^*) - \frac{(I-Z(x))^{-1}}{2}\int_0^1 e^{sK(x)}(K(x))^{-1}f^*(s)ds\\ &\quad + \frac{(I-Z(x))^{-1}}{2}\int_0^1 e^{(2-s)K(x)}(K(x))^{-1}f^*(s)ds,\end{aligned}$$

$$\begin{aligned}&\xi_1^*(x)\\ &= (I-Z(x))^{-1}(\psi^* - e^{K(x)}\varphi^*) - \frac{(I-Z(x))^{-1}}{2}\int_0^1 e^{(1-s)K(x)}(K(x))^{-1}f^*(s)ds\\ &\quad + \frac{(I-Z(x))^{-1}}{2}\int_0^1 e^{(1+s)K(x)}(K(x))^{-1}f^*(s)ds,\end{aligned}$$

and

$$Z(x) = e^{2K(x)}, \quad (I-Z(x))^{-1} = \frac{1}{2i\pi}\int_{\Gamma_1}\frac{e^{2z}}{1-e^{2z}}(K(x)-zI)^{-1}dz + I.$$

Here φ^*, ψ^* and f^* are unknown elements to be determined in an adequate space ($f^* \in C^\beta([0,1];X)$, $(0<\beta<1)$), to obtain a strict solution u of (1.1)-(1.2). Recall that a strict solution is a function u such that

$$\begin{gathered}u \in C^2([0,1],X),\ u(x) \in D(Q(x)) \text{ for every } x \in [0,1],\\ x \mapsto Q(x)u(x) \in C([0,1],X),\end{gathered}$$

and u satisfies (1.1)-(1.2). Formal calculus gives

$$u(0) = \varphi^* = \varphi, \quad u(1) = \psi^* = \psi.$$

Therefore, it suffices to seek f^* in an appropriate space with the representation

$$\begin{aligned}u(x) &= Y(x)\Big((e^{xK(x)} - e^{(2-x)K(x)})\varphi + (e^{(1-x)K(x)} - e^{(1+x)K(x)})\psi\Big)\\ &\quad - \frac{Y(x)}{2}\int_0^1 e^{(x+s)K(x)}(K(x))^{-1}f^*(s)ds\\ &\quad + \frac{Y(x)}{2}\int_0^1 e^{(2+x-s)K(x)}(K(x))^{-1}f^*(s)ds\\ &\quad - \frac{Y(x)}{2}\int_0^1 e^{(2-x-s)K(x)}(K(x))^{-1}f^*(s)ds\\ &\quad + \frac{Y(x)}{2}\int_0^1 e^{(2-x+s)K(x)}(K(x))^{-1}f^*(s)ds\\ &\quad + \frac{1}{2}\Big(\int_0^x e^{(x-s)K(x)}(K(x))^{-1}f^*(s)ds\\ &\quad + \int_x^1 e^{(s-x)K(x)}(K(x))^{-1}f^*(s)ds\Big)\\ &= d_0(x)\varphi + d_1(x)\psi + m(x,f^*) + v(x,f^*),\end{aligned} \tag{2.17}$$

where $Y(x) = (I-Z(x))^{-1}$ and $v(x,f^*)$ is defined by the last two integrals, gives a strict solution to Problem (1.1)-(1.2).

3. Some basic results

To study existence, uniqueness and regularity of the strict solution u we need to show some basic results. To this end, we will study the behavior of operators $e^{xK(x)}\varphi$, $\frac{d}{dx}\big(e^{xK(x)}\varphi\big)$ and $\frac{d^2}{dx^2}\big(e^{xK(x)}\varphi\big)$ near 0, knowing that similar results can be obtained, near 1, for operators $e^{(1-x)K(x)}\psi$, $\frac{d}{dx}\big(e^{(1-x)K(x)}\psi\big)$ and $\frac{d^2}{dx^2}\big(e^{(1-x)K(x)}\psi\big)$, where $\psi \in D(\big(K(1)\big)^2 = D\big(A(1)\big))$.

3.1. **Analysis of operators $e^{xK(x)}\varphi$ and $\frac{d}{dx}(e^{xK(x)}\varphi)$.** Let $\xi > 0$ and $x \in [0,1]$. From the representation

$$e^{\xi K(x)} = -\frac{1}{2i\pi}\int_{\Gamma_1} e^{\xi z}(K(x)-z)^{-1}dz,$$

the operators

$$\begin{aligned}
\frac{\partial}{\partial x}e^{\xi K(x)} &= -\frac{1}{2i\pi}\int_{\Gamma_1} e^{\xi z}\frac{\partial}{\partial x}(K(x)-zI)^{-1}dz,\\
\frac{\partial^2}{\partial x^2}e^{\xi K(x)} &= -\frac{1}{2i\pi}\int_{\Gamma_1} e^{\xi z}\frac{\partial^2}{\partial x^2}(K(x)-z)^{-1}dz,
\end{aligned}$$

are well defined. Moreover

$$\|\frac{\partial}{\partial x}e^{\xi K(x)}\| \le \frac{C}{\xi^{1-\nu}}, \tag{3.1}$$

$$\|\frac{\partial^2}{\partial x^2}e^{\xi K(x)}\| \le \frac{C}{\xi^{2-\nu}}. \tag{3.2}$$

In the sequel, we need to use the following formula, as in [6, p. 126]: for all $\varphi \in D(K(0))$, one has

$$\begin{aligned}
&(K(x)-z)^{-1}\varphi\\
&= \frac{K(x)(K(x)-z)^{-1}\varphi}{z} - \frac{\varphi}{z}\\
&= \frac{1}{z}\Big((K(0))^{-1} - (K(x))^{-1} + x\frac{d}{dx}(K(x))^{-1}\Big)K(0)\varphi - \frac{\varphi}{z}\\
&\quad + (K(x)-z)^{-1}\Big((K(0))^{-1} - (K(x))^{-1} + x\frac{d}{dx}(K(x))^{-1}\Big)K(0)\varphi\\
&\quad + \frac{(K(x)-z)^{-1}}{z}K(0)\varphi - \frac{x}{z}\frac{d}{dx}(K(x))^{-1}K(0)\varphi\\
&\quad - x(K(x)-z)^{-1}\frac{d}{dx}(K(x))^{-1}K(0)\varphi,
\end{aligned}$$

which leads to

$$\begin{aligned}
&\frac{\partial}{\partial x}(K(x)-z)^{-1}\varphi\\
&= \frac{\partial}{\partial x}(K(x)-z)^{-1}((K(0))^{-1} - (K(x))^{-1} + x\frac{d}{dx}(K(x))^{-1})K(0)\varphi\\
&\quad - (K(x)-z)^{-1}\frac{d}{dx}(K(x))^{-1}K(0)\varphi + \frac{1}{z}\frac{\partial}{\partial x}(K(x)-z)^{-1}K(0)\varphi\\
&\quad - \frac{1}{z}\frac{d}{dx}(K(x))^{-1}K(0)\varphi - x\frac{\partial}{\partial x}(K(x)-z)^{-1}\frac{d}{dx}(K(x))^{-1}K(0)\varphi,
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}(K(x)-z)^{-1}\varphi \\
&= \frac{\partial^2}{\partial x^2}(K(x)-z)^{-1}((K(0))^{-1}-(K(x))^{-1}+x\frac{d}{dx}(K(x))^{-1})K(0)\varphi \\
&\quad -(K(x)-z)^{-1}\frac{d^2(K(x))^{-1}}{dx^2}K(0)\varphi - 2\frac{\partial}{\partial x}(K(x)-z)^{-1}\frac{d(K(x))^{-1}}{dx}K(0)\varphi \\
&\quad +\frac{1}{z}\frac{\partial^2}{\partial x^2}(K(x)-z)^{-1}K(0)\varphi - x\frac{\partial^2}{\partial x^2}(K(x)-z)^{-1}\frac{d}{dx}(K(x))^{-1}K(0)\varphi.
\end{aligned} \tag{3.4}$$

Lemma 3.1. *There exists $C>0$ such that*

1. *for all $x>0$, $\varphi\in D(K(0))$, $\|\varphi-e^{xK(0)}\varphi\|_X\le Cx\|K(0)\varphi\|_X$,*
2. *for all $x>0$, $\varphi\in D((K(0))^2)$, $\|\varphi-e^{xK(0)}\varphi\|_X\le Cx^2\|(K(0))^2\varphi\|_X$.*

Proof. Let $x>0$. Given some $\varepsilon>0$ with $\varepsilon<x$, one has

$$e^{\varepsilon K(0)}\varphi-e^{xK(0)}\varphi=\frac{1}{2i\pi}\int_{\Gamma_1}(-e^{\varepsilon z}+e^{xz})\frac{K(0)(K(0)-zI)^{-1}}{z}\varphi dz.$$

(1) If $\varphi\in D(K(0))$, then

$$\begin{aligned}
e^{\varepsilon K(0)}\varphi-e^{xK(0)}\varphi &= \int_\varepsilon^x\Big(\frac{1}{2i\pi}\int_{\Gamma_1}e^{\xi z}(K(0)-zI)^{-1}K(0)\varphi dz\Big)d\xi \\
&= -\int_\varepsilon^x e^{\xi K(0)}K(0)\varphi d\xi
\end{aligned}$$

and

$$\|e^{\varepsilon K(0)}\varphi-e^{xK(0)}\varphi\|_X\le C(x-\varepsilon)\|K(0)\varphi\|_X.$$

(2) If $\varphi\in D((K(0))^2)$, then

$$\|e^{\varepsilon K(0)}\varphi-e^{xK(0)}\varphi\|_X\le C(x^2-\varepsilon^2)\|(K(0))^2\varphi\|_X.$$

We obtain the two estimates when ε approaches 0. □

Lemma 3.2. *Under Hypotheses* (1.3), (1.4), (2.6)–(2.9), *there exists a constant $C>0$ such that*

1. *For all $\varphi\in X$, $x>0$, $\|e^{xK(x)}\varphi\|_X\le C\|\varphi\|_X$.*
2. *For all $\varphi\in X$, $x>0$, $s\ge 0$, $\|K(s)e^{xK(s)}\varphi\|_X\le (C/x)\|\varphi\|_X$.*
3. *In particular, if $\varphi\in D(K(0))$, $x>0$, $s\in[0,x]$, then*
$$\|K(s)e^{xK(s)}\varphi\|_X\le C\|K(0)\varphi\|_X.$$
4. *The map $x\mapsto e^{xK(x)}\varphi$ belongs to the space $C([0,1];X)$ if and only if $\varphi\in\overline{D(K(0))}=\overline{D(A(0))}$, in this case $\lim_{x\to0}e^{xK(x)}\varphi=\varphi$.*
5. *The map $x\mapsto e^{xK(0)}\varphi$ belongs to the space $C^\theta([0,1];X)$ if and only if $\varphi\in D_{K(0)}(\theta;+\infty)$.*

Proof. We recall that $D_{K(0)}(\theta;+\infty)$ is the known real interpolation space defined, for instance, in [7]. The first two statements are well known. Let $\varphi\in D(K(0))$, $x>0$, $s\in[0,x]$, then

$$K(s)e^{xK(s)}\varphi=K(s)e^{xK(s)}\Big((K(0))^{-1}-(K(s))^{-1}\Big)K(0)\varphi+e^{xK(s)}K(0)\varphi$$

$$
\begin{aligned}
&= K(s)e^{xK(s)}\Big((K(0))^{-1} - (K(s))^{-1} + s\frac{d}{ds}(K(s))|_{s=0}^{-1}\Big)K(0)\varphi \\
&\quad - sK(s)e^{xK(s)}\Big(\frac{d}{ds}(K(s))|_{s=0}^{-1} - \frac{d}{ds}(K(s))^{-1}\Big)K(0)\varphi \\
&\quad - sK(s)e^{xK(s)}\frac{d}{ds}(K(s))^{-1}K(0)\varphi + e^{xK(s)}K(0)\varphi \\
&= a_1 + a_2 + a_3 + a_4.
\end{aligned}
$$

Writing the first term as

$$
a_1 = -K(s)e^{xK(s)}\int_0^s \Big(\frac{d}{d\xi}(K(\xi))^{-1} - \frac{d}{ds}(K(s))|_{s=0}^{-1}\Big)d\xi K(0)\varphi.
$$

Using the fact that

$$
\begin{aligned}
&\|\int_0^s (\frac{d}{d\xi}(K(\xi))^{-1} - \frac{d}{ds}(K(s))|_{s=0}^{-1})d\xi\| \\
&\le \int_0^s \xi \sup_{0<\theta<1}\|\frac{d^2}{d\xi^2}(K(\xi))|_{\xi=\theta s}^{-1}\|_{L(X)}d\xi \\
&\le Cs^2 \le Cx^2,
\end{aligned}
$$

we obtain

$$
\|a_1\|_X \le Cx\|K(0)\varphi\|_X, \quad \|a_2\|_X \le Cx\|K(0)\varphi\|_X.
$$

Concerning the two terms above, it is easy to see that

$$
\begin{gathered}
\|sK(s)e^{xK(s)}\frac{d}{ds}(K(s))^{-1}K(0)\varphi\|_X \le C(\frac{s}{x})\|K(0)\varphi\|_X, \\
\|e^{xK(s)}K(0)\varphi\|_X \le C\|K(0)\varphi\|_X.
\end{gathered}
$$

The fourth statement can be treated as follows: we write

$$
e^{xK(x)}\varphi = e^{xK(0)}\varphi + \Big(e^{xK(x)} - e^{xK(0)}\Big)\varphi.
$$

According to Sinestrari [14] and Haase [8], we know that

$$
x \mapsto e^{xK(0)}\varphi \in C([0,1];X) \ \text{ if and only if } \ \varphi \in \overline{D(K(0))} = \overline{D(A(0))}.
$$

Moreover, we have (by (3.1))

$$
\|(e^{xK(x)} - e^{xK(0)})\varphi\|_X = \|\int_0^x \frac{\partial}{\partial\xi}e^{xK(\xi)}\varphi d\xi\|_X \le Cx^\nu\|\varphi\|_X.
$$

The last statement can be proved similarly. □

Corollary 3.3. *Under Hypotheses* (1.3), (1.4) *and* (2.6)–(2.9), *there exists a constant $C > 0$ such that*

1. *For all $\psi \in X$, $0 \le x < 1$, $\|e^{(1-x)K(x)}\psi\|_X \le C\|\psi\|_X$.*
2. *For all $\psi \in X$, $0 \le x < 1$, $s \ge 0$, $\|K(s)e^{(1-x)K(s)}\psi\|_X \le C/(1-x)\|\psi\|_X$.*
3. *In particular, if $\psi \in D(K(1))$, $0 \le x < 1$, $s \in [x,1]$, then*
$$
\|K(s)e^{(1-x)K(s)}\psi\|_X \le C\|K(1)\psi\|_X.
$$
4. *The map $x \mapsto e^{(1-x)K(x)}\psi$ belongs to the space $C([0,1];X)$ if and only if $\psi \in \overline{D(K(1))} = \overline{D(A(1))}$, in this case $\lim_{x\to 1} e^{(1-x)K(x)}\psi = \psi$.*
5. *The map $x \mapsto e^{(1-x)K(1)}\psi$ belongs to the space $C^\theta([0,1];X\ )$ if and only if $\psi \in D_{K(1)}(\theta;+\infty)$.*

Remark 3.4. Observe that, by the well known reiteration property, one has
$$D_{K(0)}(\theta;+\infty)=D_{A(0)}(\theta/2;+\infty);\ D_{K(1)}(\theta;+\infty)=D_{A(1)}(\theta/2;+\infty).$$

Lemma 3.5. *Let $\varphi\in D((K(0))^2)=D(A(0))$. Then, under Hypotheses* (1.3) *and* (2.6)–(2.9)*, the function $x\mapsto \frac{d}{dx}(e^{xK(x)}\varphi)$ belongs to the space $C^{\min(\eta,\nu)}([0,1];X)$ and $\frac{d}{dx}(e^{xK(x)}\varphi)\to K(0)\varphi$, as $x\to 0$.*

Proof. It is sufficient to prove the result near 0. Let $x>0$, $\varphi\in D((K(0))^2)$, one has
$$e^{xK(x)}\varphi=-\frac{1}{2i\pi}\int_{\Gamma_1}e^{xz}(K(x)-zI)^{-1}\varphi dz,$$
and
$$\begin{aligned}
&\frac{d}{dx}(e^{xK(x)}\varphi)\\
&=-\frac{1}{2i\pi}\int_{\Gamma_1}ze^{xz}(K(x)-zI)^{-1}\varphi dz-\frac{1}{2i\pi}\int_{\Gamma_1}e^{xz}\frac{\partial}{\partial x}(K(x)-zI)^{-1}\varphi dz\\
&=-U(x)\varphi-V(x)\varphi.
\end{aligned}$$
For $U(x)\varphi$, we use remark 2.1 and the following decomposition (as in [1, p. 25])
$$\begin{aligned}
U(x)\varphi&=U(x)[\varphi-e^{xK(0)}\varphi]+U(x)((K(0))^{-1}-(K(x))^{-1})K(0)e^{xK(0)}\varphi\\
&\quad+\frac{1}{2i\pi}\int_{\Gamma_2}e^{xz}\frac{\partial}{\partial x}(K(x)-zI)^{-1}(K(x))^{-1}K(0)e^{xK(0)}\varphi dz.
\end{aligned}$$
Computing
$$\begin{aligned}
&\frac{1}{2i\pi}\int_{\Gamma_2}e^{xz}\frac{\partial}{\partial x}(K(x)-zI)^{-1}(K(x))^{-1}K(0)e^{xK(0)}\varphi dz\\
&-\frac{1}{2i\pi}\int_{\Gamma_2}\frac{e^{xz}}{z}\frac{\partial}{\partial x}(K(x)-zI)^{-1}K(0)e^{xK(0)}\varphi dz,
\end{aligned}$$
we obtain
$$\begin{aligned}
&\frac{\partial}{\partial x}(K(x)-zI)^{-1}(K(x))^{-1}-\frac{1}{z}\frac{\partial}{\partial x}(K(x)-zI)^{-1}\\
&=K(x)(K(x)-zI)^{-1}\frac{d}{dx}(K(x))^{-1}(K(x)-zI)^{-1}\\
&\quad-\frac{1}{z}K(x)(K(x)-zI)^{-1}\frac{d}{dx}(K(x))^{-1}K(x)(K(x)-zI)^{-1}\\
&=\frac{d}{dx}(K(x))^{-1}(K(x)-zI)^{-1}\\
&\quad+z(K(x)-zI)^{-1}\frac{d}{dx}(K(x))^{-1}(K(x)-zI)^{-1}\\
&\quad-\frac{1}{z}\frac{d}{dx}(K(x))^{-1}K(x)(K(x)-zI)^{-1}\\
&\quad-(K(x)-zI)^{-1}\frac{d}{dx}(K(x))^{-1}K(x)(K(x)-zI)^{-1}\\
&=\frac{d}{dx}(K(x))^{-1}(K(x)-zI)^{-1}-\frac{1}{z}\frac{d}{dx}(K(x))^{-1}K(x)(K(x)-zI)^{-1}\\
&\quad-(K(x)-zI)^{-1}\frac{d}{dx}(K(x))^{-1}
\end{aligned}$$

$$= -\frac{1}{z}\frac{d}{dx}(K(x))^{-1} - (K(x) - zI)^{-1}\frac{d}{dx}(K(x))^{-1},$$

so

$$\begin{aligned}
&\frac{1}{2i\pi}\int_{\Gamma_2} e^{xz}\frac{\partial}{\partial x}(K(x) - zI)^{-1}(K(x))^{-1}K(0)e^{xK(0)}\varphi dz \\
&= \frac{1}{2i\pi}\int_{\Gamma_2}\frac{e^{xz}}{z}\frac{\partial}{\partial x}(K(x) - zI)^{-1}K(0)e^{xK(0)}\varphi dz \\
&\quad - \frac{d}{dx}(K(x))^{-1}K(0)e^{xK(0)}\varphi + e^{xK(x)}\frac{d}{dx}(K(x))^{-1}K(0)e^{xK(0)}\varphi.
\end{aligned}$$

This yields

$$\begin{aligned}
U(x)\varphi &= U(x)(\varphi - e^{xK(0)}\varphi) \\
&\quad + U(x)((K(0))^{-1} - (K(x))^{-1})K(0)e^{xK(0)}\varphi \\
&\quad + \frac{1}{2i\pi}\int_{\Gamma_2}\frac{e^{xz}}{z}\frac{\partial}{\partial x}(K(x) - zI)^{-1}K(0)e^{xK(0)}\varphi dz \\
&\quad + (e^{xK(x)} - I)\frac{d}{dx}(K(x))^{-1}K(0)e^{xK(0)}\varphi \\
&= U_1(x)\varphi + U_2(x)\varphi + U_3(x)\varphi + U_4(x)\varphi.
\end{aligned}$$

Consequently,

$$\begin{gathered}
\|U_1(x)\varphi\|_X \le Cx^{\nu+1}\|(K(0))^2\varphi\|_X, \quad \|U_2(x)\varphi\|_X \le Cx^{\nu+1}\|(K(0))^2\varphi\|_X, \\
\|U_3(x)\varphi\|_X \le Cx^{\nu+1}\|(K(0))^2\varphi\|_X.
\end{gathered}$$

To prove that $U_4(x)\varphi \to 0$, as $x \to 0$, we write, for all $x > 0$,

$$\begin{aligned}
U_4(x)\varphi &= (e^{xK(x)} - I)\frac{d}{dx}(K(x))^{-1}K(0)e^{xK(0)}\varphi \\
&= (e^{xK(x)} - I)[\frac{d}{dx}(K(x))^{-1} - \frac{d}{dx}(K(x))^{-1}_{|x=0}]e^{xK(0)}K(0)\varphi \\
&\quad + (e^{xK(x)} - I)\frac{d}{dx}(K(x))^{-1}_{|x=0}[e^{xK(0)}K(0)\varphi - K(0)\varphi] \\
&\quad + (e^{xK(x)} - I)\frac{d}{dx}(K(x))^{-1}_{|x=0}K(0)\varphi \\
&= U_{41}(x)\varphi + U_{42}(x)\varphi + U_{43}(x)\varphi;
\end{aligned}$$

then

$$\begin{gathered}
\|U_{41}(x)\varphi\|_X \le Cx^{\eta}\|K(0)\varphi\|_X \to 0, \quad \text{as } x \to 0, \\
\|U_{42}(x)\varphi\|_X \le C\|[e^{xK(0)}K(0)\varphi - K(0)\varphi]\|_X \to 0, \quad \text{as } x \to 0,
\end{gathered}$$

since $K(0)\varphi \in D(K(0)) \subset \overline{D(K(0))}$. Finally, in virtue of Lemma 3.2, statement 4 and Assumption (2.8), one obtains $U_{43}(x)\varphi \to 0$, as $x \to 0$ if and only if $\frac{d}{dx}(K(x))|^{-1}_{x=0}K(0)\varphi \in \overline{D(K(0))}$. Thus $U(x)\varphi \to 0$, as $x \to 0$. For $V(x)\varphi$, one writes

$$\begin{aligned}
V(x)\varphi &= \frac{1}{2i\pi}\int_{\Gamma_1} ze^{xz}((K(x) - zI)^{-1} - (K(0) - zI)^{-1})\varphi dz \\
&\quad + \frac{1}{2i\pi}\int_{\Gamma_1} e^{xz}\frac{(K(0) - zI)^{-1}}{z}(K(0))^2\varphi dz \\
&= V_1(x)\varphi + V_2(x)\varphi.
\end{aligned}$$

It is clear that $V_2(x)\varphi \to -K(0)\varphi$, as $x \to 0$. Concerning $V_1(x)\varphi$, one has

$$\begin{aligned} V_1(x)\varphi &= \frac{1}{2i\pi}\int_{\Gamma_1} ze^{xz}\Big(\int_0^x \frac{\partial}{\partial\sigma}(K(\sigma)-zI)^{-1} - \frac{\partial}{\partial\sigma}(K(\sigma)-zI)^{-1}_{|\sigma=0}d\sigma\Big)\varphi dz \\ &\quad + \frac{x}{2i\pi}\int_{\Gamma_1} ze^{xz}\frac{\partial}{\partial\sigma}(K(\sigma)-zI)^{-1}_{|\sigma=0}\varphi dz \\ &= (b_1)+(b_2). \end{aligned}$$

Since $\varphi = (I - e^{xK(0)})\varphi + e^{xK(0)}\varphi$ and from Lemma 3.1, it follows that

$$\|(b_1)\|_X \le Cx^{\eta+\nu}\|(K(0))^2\varphi\|_X, \quad \|(b_2)\|_X \le Cx^{\nu+1}\|(K(0))^2\varphi\|_X.$$

Hence $V_2(x)\varphi \to 0$, as $x \to 0$. □

3.2. **Analysis of operator $\frac{d^2}{dx^2}(e^{xK(x)}\varphi)$.** One writes

$$\begin{aligned} \frac{d^2}{dx^2}(e^{xK(x)}\varphi) &= -\frac{1}{2i\pi}\int_{\Gamma_1} z^2e^{xz}(K(x)-zI)^{-1}\varphi dz \\ &\quad - \frac{1}{i\pi}\int_{\Gamma_1} ze^{xz}\frac{\partial}{\partial x}(K(x)-zI)^{-1}\varphi dz \\ &\quad - \frac{1}{2i\pi}\int_{\Gamma_1} e^{xz}\frac{\partial^2}{\partial x^2}(K(x)-zI)^{-1}\varphi dz \\ &= S_1(x)\varphi + S_2(x)\varphi + S_3(x)\varphi. \end{aligned} \tag{3.5}$$

Next, we study the behavior of operators S_1, S_2 and S_3.

Lemma 3.6. *Let $\varphi \in D((K(0))^2) = D(A(0))$ and assume (1.3) and (2.6)–(2.9). Then, for all $x \in]0,1]$, one has $S_1(x)\varphi = e^{xK(0)}(K(0))^2\varphi + \mathcal{F}_1(x)\varphi$, where the function $x \mapsto \mathcal{F}_1(x)\varphi$ belongs to the space $C^\nu([0,1];X)$. Moreover $S_1(x)\varphi \to (K(0))^2\varphi$, as $x \to 0$, if and only if $(K(0))^2\varphi \in \overline{D(K(0))} = D(A(0))$.*

Proof. It is sufficient to prove the behavior of S_1 near 0. Set

$$\begin{aligned} S_1(x) &= -\frac{1}{2i\pi}\int_{\Gamma_1} z^2e^{xz}((K(x)-zI)^{-1} - (K(0)-zI)^{-1})dz \\ &\quad - \frac{1}{2i\pi}\int_{\Gamma_1} z^2e^{xz}(K(0)-zI)^{-1}dz \\ &= S_{11}(x) + S_{12}(x). \end{aligned}$$

For $x > 0$, we have $\|S_{11}(x)\| \le Cx^{\nu-2}$. Now, write

$$S_{11}(x)\varphi = S_{11}(x)(\varphi - e^{xK(0)}\varphi) + S_{11}(x)e^{xK(0)}\varphi.$$

Due to Lemma 3.1, we obtain

$$\|S_{11}(x)(\varphi - e^{xK(0)}\varphi)\|_X \le Cx^\nu\|(K(0))^2\varphi\|_X,$$

which implies that $S_{11}(x)(\varphi - e^{xK(0)}\varphi) \to 0$, as $x \to 0$. From the estimate

$$\|e^{xK(0)}\varphi\|_X \le Cx^2\|(K(0))^2\varphi\|_X,$$

it follows that $S_{11}(x)e^{xK(0)}\varphi \to 0$, as $x \to 0$. For the term $S_{12}(x)\varphi$, we write

$$S_{12}(x)\varphi = e^{xK(0)}(K(0))^2\varphi,$$

and we know that $e^{xK(0)}(K(0))^2\varphi \to (K(0))^2\varphi$, as $x \to 0$, if and only if $(K(0))^2\varphi \in \overline{D(K(0))} = D(A(0))$. □

Lemma 3.7. *Let $\varphi \in D((K(0))^2) = D(A(0))$ and assume* (1.3) *and* (2.6)–(2.9). *Then, the function $x \mapsto S_2(x)\varphi$ belongs to the space $C^{\min(\eta,\nu)}([0,1];X)$ and $S_2(x)\varphi \to 0$, as $x \to 0$.*

Proof. It is sufficient to prove the Hölderianity near 0. For $x > 0$, $\varphi \in X$, it is not difficult to see that $\|S_2(x)\varphi\|_X \le Cx^{\nu-2}\|\varphi\|_X$. So, we can write

$$\begin{aligned}
S_2(x)\varphi &= S_2(x)(\varphi - e^{xK(0)}\varphi) + S_2(x)((K(0))^{-1} - (K(x))^{-1})K(0)e^{xK(0)}\varphi \\
&\quad - \frac{1}{i\pi}\int_{\Gamma_1} e^{xz}\frac{\partial}{\partial x}(K(x)-zI)^{-1}K(0)e^{xK(0)}\varphi dz \\
&\quad + \frac{1}{i\pi}\int_{\Gamma_1} ze^{xz}(K(x)-zI)^{-1}\frac{d}{dx}(K(x))^{-1}K(0)e^{xK(0)}\varphi dz \\
&= S_{21}(x)\varphi + S_{22}(x)\varphi + S_{23}(x)\varphi + S_{24}(x)\varphi.
\end{aligned}$$

In fact, it suffices to prove that

$$\begin{aligned}
&-\frac{1}{i\pi}\int_{\Gamma_1} ze^{xz}\frac{\partial}{\partial x}(K(x)-zI)^{-1}(K(x))^{-1}dz \\
&+\frac{1}{i\pi}\int_{\Gamma_1} e^{xz}\frac{\partial}{\partial x}(K(x)-zI)^{-1}dz \\
&-\frac{1}{i\pi}\int_{\Gamma_1} ze^{xz}(K(x)-zI)^{-1}\frac{d}{dx}(K(x))^{-1}dz = 0.
\end{aligned}$$

From the calculus done in the proof of Lemma 3.5, one obtains

$$\begin{aligned}
&-z\frac{\partial}{\partial x}(K(x)-zI)^{-1}(K(x))^{-1} + \frac{\partial}{\partial x}(K(x)-zI)^{-1} - z(K(x)-zI)^{-1}\frac{d}{dx}(K(x))^{-1} \\
&= \frac{d}{dx}(K(x))^{-1},
\end{aligned}$$

and by integrating at the left of Γ_1, we obtain $\frac{1}{i\pi}\int_{\Gamma_1} e^{xz}\frac{d}{dx}(K(x))^{-1}dz = 0$. Now, in virtue of Lemma 3.1, for all $\varphi \in D((K(0))^2)$,

$$\begin{gathered}
\|S_{21}(x)\varphi\|_X \le Cx^\nu\|(K(0))^2\varphi\|_X, \quad \|S_{22}(x)\varphi\|_X \le Cx^\nu\|(K(0))^2\varphi\|_X, \\
\|S_{23}(x)\varphi\|_X \le Cx^\nu\|(K(0))^2\varphi\|_X.
\end{gathered}$$

Finally,

$$\begin{aligned}
&S_{24}(x)\varphi \\
&= \frac{1}{2i\pi}\int_{\Gamma_1} ze^{xz}\Big((K(x)-zI)^{-1} - (K(0)-zI)^{-1}\Big)\frac{d}{dx}(K(x))^{-1}K(0)e^{xK(0)}\varphi dz \\
&\quad + \frac{1}{2i\pi}\int_{\Gamma_1} ze^{xz}(K(0)-zI)^{-1}\Big(\frac{d(K(x))^{-1}}{dx} - \frac{d(K(x))^{-1}}{dx}|_{x=0}\Big)K(0)e^{xK(0)}\varphi dz \\
&\quad + \frac{1}{2i\pi}\int_{\Gamma_1} ze^{xz}(K(0)-zI)^{-1}\frac{d(K(x))^{-1}}{dx}|_{x=0}K(0)e^{xK(0)}\varphi dz \\
&= (a) + (b) + (c),
\end{aligned}$$

and thus

$$\begin{gathered}
\|(a)\|_X \le Cx^\nu\|(K(0))^2\varphi\|_X, \quad \|(b)\|_X \le Cx^\eta\|(K(0))^2\varphi\|_X, \\
\|(c)\|_X \le Cx^\nu\|(K(0))^2\varphi\|_X.
\end{gathered}$$

Summarizing, we obtain $S_2(x)\varphi \to 0$, as $x \to 0$. $\square$

Lemma 3.8. *Let $\varphi \in D((K(0))^2) = D(A(0))$ and assume (1.3) and (2.6)–(2.11). Then, for all $x \in]0,1]$*

$$S_3(x)\varphi = e^{xK(0)}(-\frac{d^2}{dx^2}(K(x))|_{x=0}^{-1}(K(0)\varphi)) + \mathcal{F}_3(x)\varphi$$

where the function $x \mapsto \mathcal{F}_3(x)\varphi$ belongs to the space $C^{\min(\eta,\nu)}([0,1];X)$. Moreover

$$S_3(x)\varphi \to -\frac{d^2}{dx^2}(K(x))|_{x=0}^{-1}(K(0))\varphi, \quad \text{as } x \to 0$$

if and only if

$$-\frac{d^2}{dx^2}(K(x))|_{x=0}^{-1}(K(0))\varphi \in \overline{D(K(0))} = \overline{D(A(0))}.$$

Proof. Using Formula (3.4), one writes

$$\begin{aligned}
S_3(x)\varphi &= -\frac{1}{2i\pi}\int_{\Gamma_1} e^{xz}\frac{\partial^2}{\partial x^2}(K(x)-z)^{-1} \\
&\quad \times ((K(0))^{-1} - (K(x))^{-1} + x\frac{d}{dx}(K(x))^{-1})K(0)\varphi dz \\
&\quad + \frac{1}{2i\pi}\int_{\Gamma_1} e^{xz}(K(x)-z)^{-1}\frac{d^2}{dx^2}(K(x))^{-1}K(0)\varphi dz \\
&\quad + \frac{1}{i\pi}\int_{\Gamma_1} e^{xz}\frac{\partial}{\partial x}(K(x)-z)^{-1}\frac{d}{dx}(K(x))^{-1}K(0)\varphi dz \\
&\quad - \frac{1}{2i\pi}\int_{\Gamma_1} \frac{e^{xz}}{z}\frac{\partial^2}{\partial x^2}(K(x)-z)^{-1}K(0)\varphi dz \\
&\quad + \frac{1}{2i\pi}\int_{\Gamma_1} e^{xz}.x.\frac{\partial^2}{\partial x^2}(K(x)-z)^{-1}\frac{d}{dx}(K(x))^{-1}K(0)\varphi dz \\
&= S_{31}(x)\varphi + S_{32}(x)\varphi + S_{33}(x)\varphi + S_{34}(x)\varphi + S_{35}(x)\varphi.
\end{aligned}$$

We need the behavior of operators $S_{31}, S_{32}, S_{33}, S_{34}$ and S_{35} near 0. To this end, we use the formula

$$\|(I - e^{xK(0)})K(0)\varphi\|_X \le Cx\|(K(0))^2\varphi\|_X. \tag{3.6}$$

For the terms $S_{31}(x)\varphi$, $S_{33}(x)\varphi$, $S_{34}(x)\varphi$ and $S_{35}(x)\varphi$, write

$$K(0)\varphi = (K(0)\varphi - e^{xK(0)}K(0)\varphi) + e^{xK(0)}K(0)\varphi,$$

and use (3.6); then one obtains, as $x \to 0$,

$$\begin{gathered}
\|S_{31}(x)\varphi\|_X \le Cx^{\eta+\nu}\|(K(0))^2\varphi\|_X \to 0, \|S_{33}(x)\varphi\|_X \le Cx^{\nu}\|(K(0))^2\varphi\|_X \to 0, \\
\|S_{34}(x)\varphi\|_X \le Cx^{\nu}\|(K(0))^2\varphi\|_X \to 0, \|S_{35}(x)\varphi\|_X \le Cx^{\nu}\|(K(0))^2\varphi\|_X \to 0.
\end{gathered}$$

For the term $S_{32}(x)\varphi$, we have

$$\begin{aligned}
&S_{32}(x)\varphi \\
&= \frac{1}{2i\pi}\int_{\Gamma_1} e^{xz}(K(x)-z)^{-1}\Big(\frac{d^2}{dx^2}(K(x))^{-1} - \frac{d^2}{dx^2}(K(x))|_{x=0}^{-1}\Big)K(0)\varphi dz \\
&\quad + \frac{1}{2i\pi}\int_{\Gamma_1} e^{xz}\Big((K(x)-z)^{-1} - (K(0)-z)^{-1}\Big)\frac{d^2}{dx^2}(K(x))|_{x=0}^{-1}K(0)\varphi dz
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2i\pi}\int_{\Gamma_1} e^{xz}(K(0)-z)^{-1}\frac{d^2}{dx^2}(K(x))|_{x=0}^{-1}K(0)\varphi dz\\
&=(a_1)+(a_2)+(a_3),
\end{aligned}
$$

and clearly, as $x\to 0$,

$$
\|(a_1)\|_X\le Cx^{\eta}\|K(0)\varphi\|_X\to 0, \|(a_2)\|_X\le Cx^{\nu}\|K(0)\varphi\|_X\to 0,
$$

and

$$
(a_3)=e^{xK(0)}\Big(-\frac{d^2}{dx^2}(K(x))|_{x=0}^{-1}K(0)\varphi\Big)\to\Big(-\frac{d^2}{dx^2}(K(x))|_{x=0}^{-1}K(0)\varphi\Big)
$$

if and only if

$$
\Big(-\frac{d^2}{dx^2}(K(x))|_{x=0}^{-1}K(0)\varphi\Big)\in\overline{D(K(0))}=\overline{D(A(0))}.
$$

□

4. Study of regularity of the solution

Taking into account representation (2.17), we compute the term

$$
\mathrm{Op}(u)(x)=u''(x)+B(x)u'(x)+Q(x)u(x), \tag{4.1}
$$

for all $x\in\,]0,1[$ and we will analyze its behavior near 0 and 1.

Regularity of $\mathrm{Op}(d_0)$ **and** $\mathrm{Op}(d_1)$**.** Recall that $x\mapsto Y(x)=(I-e^{2K(x)})^{-1}\in C^2([0,1],L(X))$ and let

$$
d_0(x)\varphi=(I-e^{(2-2x)K(x)})Y(x)e^{xK(x)}\varphi:=U_0(x)Y(x)e^{xK(x)}\varphi.
$$

Proposition 4.1. *Let $\varphi\in D((K(0))^2)$. Assume* (1.3), (1.4), (2.6)–(2.11). *Then, the function $x\mapsto\mathrm{Op}(d_0(\cdot)\varphi)(x)$ belongs to the space $C^{\min(\eta,\nu)}([0,1];X)$.*

Proof. For all $x\in]0,1]$, one has

$$
Q(x)d_0(x)\varphi=-(K(x))^2U_0(x)Y(x)e^{xK(x)}\varphi,
$$

$$
d_0'(x)\varphi=[U_0(x)Y'(x)+U_0'(x)Y(x)]e^{xK(x)}\varphi+U_0(x)Y(x)\frac{d}{dx}e^{xK(x)}\varphi.
$$

So

$$
\begin{aligned}
&B(x)d_0'(x)\varphi\\
&=B(x)[U_0(x)Y'(x)+U_0'(x)Y(x)]e^{xK(x)}\varphi+B(x)U_0(x)Y(x)\frac{d}{dx}e^{xK(x)}\varphi\\
&:=G_\lambda(x)\varphi=\sum_{i=1}^{3}G_\lambda^i(x)\varphi.
\end{aligned} \tag{4.2}
$$

Also, one obtains

$$
\begin{aligned}
d_0''(x)\varphi&=[U_0(x)Y''(x)+2U_0'(x)Y'(x)+U_0''(x)Y(x)]e^{xK(x)}\varphi\\
&\quad+[2U_0(x)Y'(x)+2U_0'(x)Y(x)]\frac{d}{dx}e^{xK(x)}\varphi+U_0(x)Y(x)\frac{d^2}{dx^2}(e^{xK(x)}\varphi)\\
&=[U_0(x)Y''(x)+2U_0'(x)Y'(x)+U_0''(x)Y(x)]e^{xK(x)}\varphi\\
&\quad+[2U_0(x)Y'(x)+2U_0'(x)Y(x)]\frac{d}{dx}e^{xK(x)}\varphi\\
&\quad+U_0(x)Y(x)[S_1(x)\varphi+S_2(x)\varphi+S_3(x)\varphi],
\end{aligned}
$$

which leads to

$$\mathrm{Op}(d_0(\cdot)\varphi)(x) = F_\lambda(x)\varphi + G_\lambda(x)\varphi, \tag{4.3}$$

where

$$\begin{aligned} F_\lambda(x)\varphi &= [U_0(x)Y''(x) + 2U_0'(x)Y'(x) + U_0''(x)Y(x)]e^{xK(x)}\varphi \\ &\quad + [2U_0(x)Y'(x) + 2U_0'(x)Y(x)]\frac{d}{dx}e^{xK(x)}\varphi \\ &\quad + U_0(x)Y(x)[S_2(x)\varphi + S_3(x)\varphi] \\ &:= \sum_{i=1}^{3} F_\lambda^i(x)\varphi. \end{aligned} \tag{4.4}$$

Since

$$U_0'(x) = -\frac{1}{i\pi}\int_{\Gamma_1} ze^{(2-2x)z}(K(x)-zI)^{-1}dz + \frac{1}{2i\pi}\int_{\Gamma_1} e^{(2-2x)z}\frac{\partial}{\partial x}(K(x)-zI)^{-1}dz,$$

and

$$\begin{aligned} U_0''(x) &= \frac{2}{i\pi}\int_{\Gamma_1} z^2e^{(2-2x)z}(K(x)-zI)^{-1}dz - \frac{2}{i\pi}\int_{\Gamma_1} ze^{(2-2x)z}\frac{\partial}{\partial x}(K(x)-zI)^{-1}dz \\ &\quad + \frac{1}{2i\pi}\int_{\Gamma_1} e^{(2-2x)z}\frac{\partial^2}{\partial x^2}(K(x)-zI)^{-1}dz, \end{aligned}$$

it is easy to see that

$$[0,1[\ni x \mapsto U_0'(x) \in C([0,1[;L(X)); \quad [0,1[\ni x \mapsto U_0''(x) \in C([0,1[;L(X)).$$

Let us compute the limit of the following term, as $x \to 0$,

$$[U_0(x)Y''(x) + 2U_0'(x)Y'(x) + U_0''(x)Y(x)]e^{xK(x)}\varphi,$$

which will be important in the sequel. Recall that $(e^{xK(x)}\varphi)_{x=0} = \varphi$. For all $x \in [0,1]$, one has

$$Y'(x) = Y(x)(\frac{d}{dx}e^{2K(x)})Y(x),$$

$$Y''(x) = 2Y(x)(\frac{d}{dx}e^{2K(x)})Y(x)(\frac{d}{dx}e^{2K(x)})Y(x) + Y(x)(\frac{d^2}{dx^2}e^{2K(x)})Y(x).$$

Using Lemmas 3.2 and 3.5, one obtains

$$\begin{aligned} &[U_0(0)Y''(0) + 2U_0'(0)Y'(0) + U_0''(0)Y(0)]\varphi \\ &+ [2U_0(0)Y'(0) + 2U_0'(0)Y(0)]\big(\frac{d}{dx}e^{xK(x)}\varphi\big)\big|_{x=0} \\ &= 4K(0)e^{2K(0)}Y(0)(\frac{d}{dx}e^{2K(x)})\big|_{x=0}Y(0)\varphi \\ &\quad - \frac{4}{2i\pi}\int_{\Gamma_1} ze^{2z}\frac{\partial}{\partial x}(K(x)-zI)\big|_{x=0}^{-1}Y(0)\varphi dz \\ &= 4((a)+(b)). \end{aligned}$$

It is clear that (a) is in $D((K(0))^n)$ for any $n \in \mathbb{N}^*$. For (b), one has

$$\begin{aligned} &\frac{-1}{2i\pi}\int_{\Gamma_1} ze^{2z}\frac{\partial}{\partial x}(K(x)-zI)|_{x=0}^{-1}Y(0)\varphi dz \\ &= \frac{-1}{2i\pi}\int_{\Gamma_1} ze^{2z}\frac{d}{dx}(K(x))|_{x=0}^{-1}K(0)(K(0)-zI)^{-1}Y(0)\varphi dz \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2i\pi}\int_{\Gamma_1} z^2 e^{2z}(K(0)-zI)^{-1}\frac{d}{dx}(K(x))|_{x=0}^{-1}K(0)(K(0)-zI)^{-1}Y(0)\varphi dz\\
&= (b_1)+(b_2),
\end{aligned}
$$

where, clearly $(b_2)\in D(K(0))$. For (b_1), write

$$
\begin{aligned}
(b_1) &= \frac{d}{dx}(K(x))|_{x=0}^{-1}\Big(\frac{-1}{2i\pi}\int_{\Gamma_1} ze^{2z}K(0)(K(0)-zI)^{-1}Y(0)\varphi dz\Big)\\
&= \frac{d}{dx}(K(x))|_{x=0}^{-1}\Big(\frac{-1}{2i\pi}\int_{\Gamma_1} ze^{2z}(K(0)-zI)^{-1}Y(0)K(0)\varphi dz\Big)\\
&= \frac{d}{dx}(K(x))|_{x=0}^{-1}(K(0)e^{2K(0)}Y(0)K(0)\varphi),
\end{aligned}
$$

obviously $K(0)e^{2K(0)}Y(0)K(0)\varphi\in D(K(0))$. Due to Assumption (2.8), one obtains $(b_1)\in\overline{D(K(0))}$. On the other hand, by Lemmas 3.7 and 3.8, since $U_0(0)Y(0)=I$, it follows that

$$
U_0(x)Y(x)(S_2(x)\varphi+S_3(x)\varphi)\to(-\frac{d^2}{dx^2}(K(x))|_{x=0}^{-1}K(0)\varphi)
$$

if and only if $(-\frac{d^2}{dx^2}(K(x))|_{x=0}^{-1}K(0)\varphi\big)\in\overline{D(K(0))}=\overline{D(A(0))}$. Therefore $F_\lambda(0)\varphi=\Psi_0(\varphi)-\frac{d^2}{dx^2}(K(x))|_{x=0}^{-1}K(0)\varphi$, where $\Psi_0(\varphi)\in\overline{D(K(0))}$. Thanks to (4.2), one obtains

$$
G_\lambda(0)\varphi=B(0)(2e^{2K(0)}Y(0)K(0)\varphi+(K(0))\varphi)=\Psi_0^*(\varphi),
$$

with $\Psi_0^*(\varphi)\in\overline{D(K(0))}$ (due to (2.7). Summarizing, one obtains

$$
F_\lambda(0)\varphi+G_\lambda(0)\varphi=\Phi_0^*(\varphi)-\frac{d^2}{dx^2}(K(x))|_{x=0}^{-1}(K(0))^{-1}(K(0))^2\varphi \tag{4.5}
$$

where $\Phi_0^*(\varphi)=\Psi_0(\varphi)+\Psi_0^*(\varphi)\in\overline{D(K(0))}$.

Concerning the operator d_1, for all $x\in[0,1[$, one has

$$
d_1(x)\psi=(I-e^{2xK(x)})Y(x)e^{(1-x)K(x)}\psi:=U_1(x)Y(x)e^{(1-x)K(x)}\psi.
$$

Let $\psi\in D((K(1))^2)$. We will treat its regularity near 1. By using similar arguments as those done for d_0, one has

$$
Q(x)d_1(x)\psi=-(K(x))^2U_1(x)Y(x)e^{(1-x)K(x)}\psi,
$$

and

$$
d_1'(x)\psi=[U_1(x)Y'(x)+U_1'(x)Y(x)]e^{(1-x)K(x)}\psi+U_1(x)Y(x)\frac{d}{dx}e^{(1-x)K(x)}\psi.
$$

Hence

$$
\begin{aligned}
B(x)d_1'(x)\psi &= B(x)[U_1(x)Y'(x)+U_1'(x)Y(x)]e^{(1-x)K(x)}\psi\\
&\quad + B(x)U_1(x)Y(x)\frac{d}{dx}e^{(1-x)K(x)}\psi\\
&:= T_\lambda(x)\psi,
\end{aligned}
$$

and

$$
\begin{aligned}
&d_1''(x)\psi\\
&= [U_1(x)Y''(x)+2U_1'(x)Y'(x)+U_1''(x)Y(x)]e^{(1-x)K(x)}\psi\\
&\quad + [2U_1(x)Y'(x)+2U_1'(x)Y(x)]\frac{d}{dx}e^{(1-x)K(x)}\psi+U_1(x)Y(x)\frac{d^2}{dx^2}e^{(1-x)K(x)}\psi
\end{aligned}
$$

$$= U_1(x)Y(x)(K(x))^2 e^{(1-x)K(x)}\psi + S_\lambda(x)\psi,$$

with

$$S_\lambda(1)\psi := \Psi_1(\psi) - \frac{d^2}{dx^2}(K(x))|_{x=1}^{-1} K(1)\psi,$$

where $\Psi_1(\psi) \in \overline{D(K(1))}$. Also, $T_\lambda(1)\psi = \Psi_1^*(\psi)$, with $\Psi_1^*(\psi) \in D(K(1)) \subset \overline{D(K(1))}$ (due to (2.7)). Summarizing, one obtains

$$S_\lambda(1)\psi + T_\lambda(1)\psi = \Phi_1^*(\psi) - Q(1)\psi - \frac{d^2}{dx^2}(K(x))|_{x=1}^{-1} K(1)\psi \tag{4.6}$$

where $\Phi_1^*(\psi) = \Psi_1(\psi) + \Psi_1^*(\psi) \in \overline{D(K(1))}$. Here

$$\mathrm{Op}(d_1(\cdot)\psi)(x) = S_\lambda(x)\psi + T_\lambda(x)\psi. \tag{4.7}$$

□

Then, as in the previous proposition, we obtain the following result.

Proposition 4.2. *Let $\psi \in D((K(1))^2)$ and assume (1.3), (1.4) and (2.6)–(2.11). Then, the function $x \mapsto \mathrm{Op}(d_1(\cdot)\psi)(x)$ belongs to the space $C^{\min(\eta,\nu)}([0,1];X)$.*

Regularity of $\mathrm{Op}(m)$. Recall that, for all $x \in]0,1[$, one has

$$\begin{aligned} m(x,f^*) = &-\frac{Y(x)}{2}\int_0^1 e^{(x+s)K(x)}(K(x))^{-1}f^*(s)ds \\ &+ \frac{Y(x)}{2}\int_0^1 e^{(2+x-s)K(x)}(K(x))^{-1}f^*(s)ds \\ &- \frac{Y(x)}{2}\int_0^1 e^{(2-x-s)K(x)}(K(x))^{-1}f^*(s)ds \\ &+ \frac{Y(x)}{2}\int_0^1 e^{(2-x+s)K(x)}(K(x))^{-1}f^*(s)ds \\ := &\sum_{i=1}^4 m_i(x,f^*). \end{aligned}$$

Since, for x, $s \in]0,1[$, one has

$$e^{(x+s)K(x)} - e^{(2-x+s)K(x)} = \Big(I - e^{(2-2x)K(x)}\Big)e^{(x+s)K(x)} = U_0(x)e^{(x+s)K(x)},$$

$$\begin{aligned} e^{(2-x-s)K(x)} - e^{(2+x-s)K(x)} &= \Big(I - e^{2xK(x)}\Big)e^{(1-x)K(x)}e^{(1-s)K(x)} \\ &= U_1(x)e^{(2-(x+s))K(x)}, \end{aligned}$$

then, one can write

$$\begin{aligned} m(x,f^*) &= (m_1(x,f^*) + m_4(x,f^*)) + (m_2(x,f^*) + m_3(x,f^*)) \\ &= -\frac{Y(x)}{2}\int_0^1 (e^{(x+s)K(x)} - e^{(2-x+s)K(x)})(K(x))^{-1}f^*(s)ds \\ &\quad + \frac{Y(x)}{2}\int_0^1 (e^{(2+x-s)K(x)} - e^{(2-x-s)K(x)})(K(x))^{-1}f^*(s)ds \\ &= -\frac{U_0(x)Y(x)}{2}\int_0^1 e^{(x+s)K(x)}(K(x))^{-1}f^*(s)ds \end{aligned}$$

$$- \frac{U_1(x)Y(x)}{2} \int_0^1 e^{(2-(x+s))K(x)}(K(x))^{-1} f^*(s)ds.$$

Proposition 4.3. *Assume* (1.3), (1.4) *and* (2.6)–(2.11). *Then the function* $x \mapsto \operatorname{Op}(m(.,f^*))(x)$ *belongs to the space* $C^{\min(\eta,\nu)}([0,1];X)$.

Proof. By using Dunford calculus and the formula

$$(K(x)-z)^{-1}(K(x))^{-1} = \frac{1}{z}[(K(x)-z)^{-1} - (K(x))^{-1}],$$

we obtain, for all $x \in]0,1[$ and $f^* \in C^\beta([0,1];X)$,

$$\begin{aligned}
m'(x,f^*) = &-\frac{1}{2}[U_0'(x)Y(x)+U_0(x)Y'(x)] \int_0^1 e^{(x+s)K(x)}(K(x))^{-1} f^*(s)ds \\
&- \frac{U_0(x)Y(x)}{2} \int_0^1 e^{(x+s)K(x)} f^*(s)ds \\
&+ \frac{U_0(x)Y(x)}{4i\pi} \int_0^1 \int_{\Gamma_1} \frac{e^{(x+s)z}}{z} \frac{\partial}{\partial x}(K(x)-zI)^{-1} f^*(s)\,dz\,ds \\
&- \frac{1}{2}[U_1'(x)Y(x)+U_1(x)Y'(x)] \int_0^1 e^{(2-(x+s))K(x)}(K(x))^{-1} f^*(s)ds \\
&+ \frac{U_1(x)Y(x)}{2} \int_0^1 e^{(2-(x+s))K(x)} f^*(s)ds \\
&+ \frac{U_1(x)Y(x)}{4i\pi} \int_0^1 \int_{\Gamma_1} \frac{e^{(2-(x+s))z}}{z} \frac{\partial}{\partial x}(K(x)-zI)^{-1} f^*(s)\,dz\,ds.
\end{aligned}$$

Also,

$$\begin{aligned}
&B(x)m'(x,f^*) \\
&= -\frac{B(x)}{2}[U_0'(x)Y(x)+U_0(x)Y'(x)] \int_0^1 e^{(x+s)K(x)}(K(x))^{-1} f^*(s)ds \\
&\quad - \frac{B(x)U_0(x)Y(x)}{2} \int_0^1 e^{(x+s)K(x)} f^*(s)ds \\
&\quad + \frac{B(x)U_0(x)Y(x)}{4i\pi} \int_0^1 \int_{\Gamma_1} \frac{e^{(x+s)z}}{z} \frac{\partial}{\partial x}(K(x)-zI)^{-1} f^*(s)\,dz\,ds \\
&\quad - \frac{B(x)}{2}[U_1'(x)Y(x)+U_1(x)Y'(x)] \int_0^1 e^{(2-(x+s))K(x)}(K(x))^{-1} f^*(s)ds \\
&\quad + \frac{B(x)U_1(x)Y(x)}{2} \int_0^1 e^{(2-(x+s))K(x)} f^*(s)ds \\
&\quad + \frac{B(x)U_1(x)Y(x)}{4i\pi} \int_0^1 \int_{\Gamma_1} \frac{e^{(2-(x+s))z}}{z} \frac{\partial}{\partial x}(K(x)-zI)^{-1} f^*(s)\,dz\,ds \\
&:= N_\lambda(f^*)(x) = \sum_{i=1}^6 N_{\lambda i}(f^*)(x).
\end{aligned}$$

Thanks to Hypotheses (1.4) and (2.6), all the previous integrals are absolutely convergent. For instance, concerning the term $N_{\lambda 1}(f^*)(x)$, by (2.1), we obtain

$$\|N_{\lambda 1}(f^*)(x)\|_X \le Cx\|f^*\|_{C(X)}.$$

The other terms are regular near 0 and they will be treated similarly near 1. On the other hand, let us calculate $Q(x)m(x,f^*)$ for $x\in]0,1[$. One has

$$\begin{aligned} Q(x)m(x,f^*) &= (K(x))^2(\frac{U_0(x)Y(x)}{2}\int_0^1 e^{(x+s)K(x)}(K(x))^{-1}f^*(s)ds) \\ &\quad + (K(x))^2(\frac{U_1(x)Y(x)}{2}\int_0^1 e^{(2-(x+s))K(x)}(K(x))^{-1}f^*(s)ds). \end{aligned} \tag{4.8}$$

These two integrals are absolutely convergent by semigroup properties and the Hölderianity of f^*. For $x\in]0,1[$, one has

$$m''(x,f^*) = (m_1''(x,f^*)+m_4''(x,f^*)) + (m_2''(x,f^*)+m_3''(x,f^*)),$$

where

$$\begin{aligned} &m_1''(x,f^*)+m_4''(x,f^*) \\ &= -\frac{1}{2}[U_0(x)Y''(x)+2U_0'(x)Y'(x)+U_0''(x)Y(x)]\int_0^1 e^{(x+s)K(x)}(K(x))^{-1}f^*(s)ds \\ &\quad -\frac{1}{2}[2U_0'(x)Y(x)+2U_0(x)Y'(x)]\int_0^1 \frac{\partial}{\partial x}\{e^{(x+s)K(x)}(K(x))^{-1}\}f^*(s)ds \\ &\quad -\frac{1}{2}U_0(x)Y(x)\int_0^1 \frac{\partial^2}{\partial x^2}\{e^{(x+s)K(x)}(K(x))^{-1}\}f^*(s)ds, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &m_2''(x,f^*)+m_3''(x,f^*) \\ &= -\frac{1}{2}[U_1(x)Y''(x)+2U_1'(x)Y'(x)+U_1''(x)Y(x)]\int_0^1 e^{(2-x-s)K(x)}(K(x))^{-1}f^*(s)ds \\ &\quad -\frac{1}{2}[2U_1'(x)Y(x)+2U_1(x)Y'(x)]\int_0^1 \frac{\partial}{\partial x}\{e^{(2-x-s)K(x)}(K(x))^{-1}\}f^*(s)ds \\ &\quad -\frac{1}{2}U_1(x)Y(x)\int_0^1 \frac{\partial^2}{\partial x^2}\{e^{(2-x-s)K(x)}(K(x))^{-1}\}f^*(s)ds. \end{aligned} \tag{4.10}$$

We will treat the regularity of $m''(x,f^*)$ near 0 and 1. To this end, near 0, it suffices to study the regularity of $m_1''(x,f^*)+m_4''(x,f^*)$ because $m_2''(x,f^*)+m_3''(x,f^*)$ is continuous near 0. Similarly, we treat the term $m_2''(x,f^*)+m_3''(x,f^*)$ near 1. We have to study the regularity, near 0, of the terms

$$\begin{gathered} M_1(x,f^*) = \int_0^1 e^{(x+s)K(x)}(K(x))^{-1}f^*(s)ds, \\ M_2(x,f^*) = \int_0^1 \frac{\partial}{\partial x}\{e^{(x+s)K(x)}(K(x))^{-1}\}f^*(s)ds, \\ M_3(x,f^*) = \int_0^1 \frac{\partial^2}{\partial x^2}\{e^{(x+s)K(x)}(K(x))^{-1}\}f^*(s)ds. \end{gathered}$$

The term $M_1(x,f^*)$ can be treated similarly to $N_{\lambda 1}(f^*)(x)$ and the term $M_2(x,f^*)$ as $(N_{\lambda i}(f^*)(x))_{i=2,3}$. For $M_3(x)$ one obtains

$$M_3(x) = -\frac{1}{2i\pi}\int_0^1\int_{\Gamma_1} ze^{(x+s)z}(K(x)-zI)^{-1}f^*(s)\,dz\,ds$$

$$
\begin{aligned}
& - \frac{1}{i\pi} \int_0^1 \int_{\Gamma_1} e^{(x+s)z} \frac{\partial}{\partial x} (K(x) - zI)^{-1} f^*(s) \, dz \, ds \\
& - \frac{1}{2i\pi} \int_0^1 \int_{\Gamma_1} \frac{e^{(x+s)z}}{z} \frac{\partial^2}{\partial x^2} (K(x) - zI)^{-1} f^*(s) \, dz \, ds \\
:= & \sum_{i=1}^{3} M_{3i}(x).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& - \frac{U_0(x) Y(x)}{2} M_3(x) + Q(x)(m_1(x, f^*) + m_4(x, f^*)) \\
& = - \frac{U_0(x) Y(x)}{i\pi} \int_0^1 \int_{\Gamma_1} e^{(x+s)z} \frac{\partial}{\partial x} (K(x) - zI)^{-1} f^*(s) \, dz \, ds \\
& \quad - \frac{U_0(x) Y(x)}{2i\pi} \int_0^1 \int_{\Gamma_1} \frac{e^{(x+s)z}}{z} \frac{\partial^2}{\partial x^2} (K(x) - zI)^{-1} f^*(s) \, dz \, ds
\end{aligned}
$$

and from the formula (2.14), it follows that

$$
\begin{aligned}
& - \frac{U_0(x) Y(x)}{2} M_3(x) + Q(x)(m_1(x, f^*) + m_4(x, f^*)) \\
& = - \frac{U_0(x) Y(x)}{i\pi} \int_0^1 \int_{\Gamma_1} e^{(x+s)z} \frac{\partial}{\partial x} (K(x) - zI)^{-1} f^*(s) \, dz \, ds \\
& \quad - \frac{U_0(x) Y(x)}{2i\pi} \int_0^1 \int_{\Gamma_1} e^{(x+s)z} \frac{K(x)(K(x) - zI)^{-1}}{z} \frac{d^2 (K(x))^{-1}}{dx^2} \\
& \quad \times K(x)(K(x) - zI)^{-1} f^*(s) \, dz \, ds \\
& \quad - \frac{U_0(x) Y(x)}{i\pi} \int_0^1 \int_{\Gamma_1} e^{(x+s)z} K(x)(K(x) - zI)^{-1} \\
& \quad \times \Big(\frac{d (K(x))^{-1}}{dx} K(x)(K(x) - zI)^{-1} \Big)^2 f^*(s) \, dz \, ds \\
& = - \frac{U_0(x) Y(x)}{i\pi} [I_1(x) + I_2(x) + I_3(x)].
\end{aligned}
$$

Concerning the term $I_1(x)$ (the term $I_3(x)$ will be treated similarly), one has

$$
\begin{aligned}
I_1(x) & = \int_0^1 \int_{\Gamma_1} e^{(x+s)z} \frac{\partial}{\partial x} (K(x) - zI)^{-1} f^*(s) \, dz \, ds \\
& = \int_0^1 \int_{\Gamma_1} e^{(x+s)z} [\frac{\partial}{\partial x} (K(x) - zI)^{-1} - \frac{\partial}{\partial x} (K(x) - zI)|_{x=0}^{-1}] f^*(s) \, dz \, ds \\
& \quad + \int_0^1 \int_{\Gamma_1} e^{(x+s)z} \frac{\partial}{\partial x} (K(x) - zI)|_{x=0}^{-1} f^*(s) \, dz \, ds \\
& = (I_{11}) + (I_{12}).
\end{aligned}
$$

Hence, as $x \to 0$,

$$
\|(I_{11})\|_X \le C x^{\eta} \|f^*\|_{C(X)} \to 0, \quad \|(I_{12})\|_X \le C x^{\nu} \|f^*\|_{C(X)} \to 0.
$$

Concerning the term $I_2(x)$, one writes

$$\begin{aligned}
I_2(x) &= \int_0^1 \int_{\Gamma_1} e^{(x+s)z} \frac{K(x)(K(x)-zI)^{-1}}{z} \frac{d^2(K(x))^{-1}}{dx^2} \\
&\quad \times [K(x)(K(x)-zI)^{-1} - K(0)(K(0)-zI)^{-1}] f^*(s)\, dz\, ds \\
&\quad + \int_0^1 \int_{\Gamma_1} e^{(x+s)z} \frac{K(x)(K(x)-zI)^{-1}}{z} \Big[\frac{d^2(K(x))^{-1}}{dx^2} - \frac{d^2(K(x))}{dx^2}\Big|_{x=0}^{-1} \Big] \\
&\quad \times K(0)(K(0)-zI)^{-1} f^*(s)\, dz\, ds \\
&\quad + \int_0^1 \int_{\Gamma_1} e^{(x+s)z} \frac{[K(x)(K(x)-zI)^{-1} - K(0)(K(0)-zI)^{-1}]}{z} \\
&\quad \times \frac{d^2(K(x))^{-1}}{dx^2}\Big|_{x=0} K(0)(K(0)-zI)^{-1} f^*(s)\, dz\, ds \\
&= (I_{21}) + (I_{22}) + (I_{23}).
\end{aligned}$$

Then, as $x \to 0$, one obtains

$$\|(I_{21})\|_X \le Cx\|f^*\|_{C(X)} \to 0, \quad \|(I_{22})\|_X \le Cx^\eta \|f^*\|_{C(X)} \to 0,$$
$$\|(I_{23})\|_X \le Cx\|f^*\|_{C(X)} \to 0.$$

We apply a similar treatment for the term (4.10) near 1. Summing up, one obtains

$$\begin{aligned}
&m_1''(x,f^*) + m_4''(x,f^*) + Q(x)(m_1(x,f^*) + m_4(x,f^*)) \\
&= \frac{1}{2}[U_0(x)Y''(x) + 2U_0'(x)Y'(x) + U_0''(x)Y(x)] \int_0^1 e^{(x+s)K(x)}(K(x))^{-1} f^*(s) ds \\
&\quad - \frac{1}{2}[2U_0'(x)Y(x) + 2U_0(x)Y'(x)] \int_0^1 \frac{\partial}{\partial x}\{e^{(x+s)K(x)}(K(x))^{-1}\} f^*(s) ds \\
&\quad + \frac{U_0(x)Y(x)}{2i\pi} \int_0^1 \int_{\Gamma_1} e^{(x+s)z} \frac{\partial}{\partial x}(K(x)-zI)^{-1} f^*(s)\, dz\, ds \\
&\quad + \frac{U_0(x)Y(x)}{4i\pi} \int_0^1 \int_{\Gamma_1} \frac{e^{(x+s)z}}{z} \frac{\partial^2}{\partial x^2}(K(x)-zI)^{-1} f^*(s)\, dz\, ds.
\end{aligned}$$

Similarly, one obtains

$$\begin{aligned}
&m_2''(x,f^*) + m_3''(x,f^*) + Q(x)(m_2(x,f^*) + m_3(x,f^*)) \\
&= -\frac{1}{2}[U_1(x)Y''(x) + 2U_1'(x)Y'(x) + U_1''(x)Y(x)] \\
&\quad \times \int_0^1 e^{(2-x-s)K(x)}(K(x))^{-1} f^*(s) ds \\
&\quad - \frac{1}{2}[2U_1'(x)Y(x) + 2U_1(x)Y'(x)] \int_0^1 \frac{\partial}{\partial x}\{e^{(2-x-s)K(x)}(K(x))^{-1}\} f^*(s) ds \\
&\quad - \frac{U_1(x)Y(x)}{2i\pi} \int_0^1 \int_{\Gamma_1} e^{(2-x-s)z} \frac{\partial}{\partial x}(K(x)-zI)^{-1} f^*(s)\, dz\, ds \\
&\quad + \frac{U_1(x)Y(x)}{4i\pi} \int_0^1 \int_{\Gamma_1} \frac{e^{(2-x-s)z}}{z} \frac{\partial^2}{\partial x^2}(K(x)-zI)^{-1} f^*(s)\, dz\, ds.
\end{aligned} \tag{4.11}$$

Therefore,

$$\mathrm{Op}(m(.,f^*))(x) = M_\lambda(f^*)(x) + N_\lambda(f^*)(x), \tag{4.12}$$

where

$$\begin{aligned} M_\lambda(f^*)(x) &= m_1''(x,f^*) + m_4''(x,f^*) + Q(x)(m_1(x,f^*) + m_4(x,f^*)) \\ &\quad + m_2''(x,f^*) + m_3''(x,f^*) + Q(x)(m_2(x,f^*) + m_3(x,f^*)). \end{aligned} \tag{4.13}$$

□

Regularity of $\mathrm{Op}(v)$**.**

Proposition 4.4. *Assume* (1.3)*,* (1.4)*,* (2.6)*–*(2.11)*. Then, the function* $x \mapsto \mathrm{Op}(v(.,f^*))(x)$ *belongs to the space* $C^{\min(\beta,\eta+\nu-1)}([0,1];X)$*.*

Proof. Recall that, for $x \in]0,1[$, one has

$$v(x,f^*) = \frac{1}{2}\int_0^x e^{(x-s)K(x)}(K(x))^{-1}f^*(s)ds + \frac{1}{2}\int_x^1 e^{(s-x)K(x)}(K(x))^{-1}f^*(s)ds,$$

where $f^* \in C^\beta([0,1];X)$ (β will be specified, $0 < \beta < 1$). So

$$\begin{aligned} Q(x)v(x,f^*) &= -\frac{1}{2}\int_0^x K(x)e^{(x-s)K(x)}(f^*(s) - f^*(x))ds \\ &\quad - \frac{1}{2}\int_x^1 K(x)e^{(s-x)K(x)}(f^*(s) - f^*(x))ds \\ &\quad + f^*(x) - \frac{1}{2}e^{xK(x)}f^*(x) - \frac{1}{2}e^{(1-x)K(x)}f^*(x). \end{aligned} \tag{4.14}$$

Due to the properties of analytic semigroups and the Hölderianity of f^*, the first two integrals are absolutely convergent. On the other hand,

$$\begin{aligned} v'(x,f^*) &= \frac{1}{2}\int_0^x e^{(x-s)K(x)}f^*(s)ds - \frac{1}{2}\int_x^1 e^{(s-x)K(x)}f^*(s)ds \\ &\quad + \frac{1}{2}\int_0^x e^{(x-s)K(x)}\frac{d}{dx}(K(x))^{-1}f^*(s)ds \\ &\quad + \frac{1}{2}\int_x^1 e^{(s-x)K(x)}\frac{d}{dx}(K(x))^{-1}f^*(s)ds \\ &\quad + \frac{1}{2}\int_0^x (\frac{\partial}{\partial x}e^{\eta K(x)})|_{\eta=(x-s)}(K(x))^{-1}f^*(s)ds \\ &\quad + \frac{1}{2}\int_x^1 (\frac{\partial}{\partial x}e^{\eta K(x)})|_{\eta=(s-x)}(K(x))^{-1}f^*(s)ds \\ &:= \sum_{i=1}^6 w_i(x,f^*). \end{aligned}$$

Let us simplify the terms $w_3(x,f^*) + w_5(x,f^*)$ and $w_4(x,f^*) + w_6(x,f^*)$. By using Dunford calculus, one writes

$$\begin{aligned} & w_3(x,f^*) + w_5(x,f^*) \\ &= -\frac{1}{4i\pi}\int_0^x \int_{\Gamma_1} e^{(x-s)z}(K(x) - zI)^{-1}\frac{d}{dx}(K(x))^{-1}f^*(s)\,dz\,ds \\ &\quad - \frac{1}{4i\pi}\int_0^x \int_{\Gamma_1} e^{(x-s)z}\Big(\frac{\partial}{\partial x}(K(x) - zI)^{-1}(K(x))^{-1}\Big)f^*(s)\,dz\,ds \\ &= \beta_1(x) + \beta_2(x). \end{aligned}$$

From the formula (see the proof of Lemma 3.5)

$$\begin{aligned}&\frac{\partial}{\partial x}(K(x)-zI)^{-1}(K(x))^{-1}-\frac{1}{z}\frac{\partial}{\partial x}(K(x)-zI)^{-1}\\&=-\frac{1}{z}\frac{d}{dx}(K(x))^{-1}-(K(x)-zI)^{-1}\frac{d}{dx}(K(x))^{-1},\end{aligned}$$

we deduce

$$\begin{aligned}\beta_2(x)=&\frac{1}{4i\pi}\int_0^x\int_{\Gamma_1}\frac{e^{(x-s)z}}{z}\frac{d}{dx}(K(x))^{-1}f^*(s)\,dz\,ds\\&-\frac{1}{4i\pi}\int_0^x\int_{\Gamma_1}\frac{e^{(x-s)z}}{z}\frac{\partial}{\partial x}(K(x)-zI)^{-1}f^*(s)\,dz\,ds\\&+\frac{1}{4i\pi}\int_0^x\int_{\Gamma_1}e^{(x-s)z}(K(x)-zI)^{-1}\frac{d}{dx}(K(x))^{-1}f^*(s)\,dz\,ds\\=&\ \beta_{21}(x)+\beta_{22}(x)+\beta_{23}(x),\end{aligned}$$

hence $\beta_1(x)+\beta_{23}(x)=0$ and by integrating at the left hand side of Γ_1, it results that $\beta_{21}(x)=0$. Then

$$w_3(x,f^*)+w_5(x,f^*)=-\frac{1}{4i\pi}\int_0^x\int_{\Gamma_1}\frac{e^{(x-s)z}}{z}\frac{\partial}{\partial x}(K(x)-zI)^{-1}f^*(s)\,dz\,ds.$$

Similarly, one obtains

$$w_4(x,f^*)+w_6(x,f^*)=-\frac{1}{4i\pi}\int_x^1\int_{\Gamma_1}\frac{e^{(s-x)z}}{z}\frac{\partial}{\partial x}(K(x)-zI)^{-1}f^*(s)\,dz\,ds.$$

Thus

$$\begin{aligned}&w_3(x,f^*)+w_4(x,f^*)+w_5(x,f^*)+w_6(x,f^*)\\&=-\frac{1}{4i\pi}\int_0^x\int_{\Gamma_1}\frac{e^{(x-s)z}}{z}\frac{\partial}{\partial x}(K(x)-zI)^{-1}f^*(s)\,dz\,ds\\&\quad-\frac{1}{4i\pi}\int_x^1\int_{\Gamma_1}\frac{e^{(s-x)z}}{z}\frac{\partial}{\partial x}(K(x)-zI)^{-1}f^*(s)\,dz\,ds.\end{aligned}$$

Consequently,

$$\begin{aligned}&v'(x,f^*)\\&=w_1(x,f^*)+w_2(x,f^*)-\frac{1}{4i\pi}\int_0^x\int_{\Gamma_1}\frac{e^{(x-s)z}}{z}\frac{\partial}{\partial x}(K(x)-zI)^{-1}f^*(s)\,dz\,ds\\&\quad-\frac{1}{4i\pi}\int_x^1\int_{\Gamma_1}\frac{e^{(s-x)z}}{z}\frac{\partial}{\partial x}(K(x)-zI)^{-1}f^*(s)\,dz\,ds.\end{aligned}$$

Due to (2.6) and the semigroups properties, all these terms are well defined and it is the same for $B(x)v'(x,f^*)$, where

$$\begin{aligned} B(x)v'(x,f^*) &= B(x)w_1(x,f^*) + B(x)w_2(x,f^*) \\ &\quad - \frac{B(x)}{4i\pi}\int_0^x\int_{\Gamma_1}\frac{e^{(x-s)z}}{z}\frac{\partial}{\partial x}(K(x)-zI)^{-1}f^*(s)\,dz\,ds \\ &\quad - \frac{B(x)}{4i\pi}\int_x^1\int_{\Gamma_1}\frac{e^{(s-x)z}}{z}\frac{\partial}{\partial x}(K(x)-zI)^{-1}f^*(s)\,dz\,ds \\ &:= W_\lambda(f^*)(x) = \sum_{i=1}^4 W_{\lambda i}(f^*)(x), \end{aligned} \tag{4.15}$$

from which we have $B(0)v'(0,f^*) \in \overline{D(K(0))}$. Now, to differentiate $w_1(.,f^*) + w_2(.,f^*)$, one uses the method presented, for instance, in Tanabe [15, Theorem 3.3.4, p. 70.], For $0<\varepsilon\le x<1$, let us put

$$\frac{1}{2}\int_0^{x-\varepsilon} e^{(x-s)K(x)}f^*(s)ds - \frac{1}{2}\int_{x+\varepsilon}^1 e^{(s-x)K(x)}f^*(s)ds := w_1^\varepsilon(x,f^*) + w_2^\varepsilon(x,f^*).$$

Hence

$$\begin{aligned} &(w_1^\varepsilon)'(x,f^*) + (w_2^\varepsilon)'(x,f^*) \\ &= \frac{1}{2}e^{\varepsilon K(x)}(f^*(x-\varepsilon)+f^*(x+\varepsilon)) - e^{\varepsilon K(x)}f^*(x) + \frac{1}{2}e^{xK(x)}f^*(x) \\ &\quad + \frac{1}{2}e^{(1-x)K(x)}f^*(x) + \frac{1}{2}\int_0^{x-\varepsilon}K(x)e^{(x-s)K(x)}(f^*(s)-f^*(x))ds \\ &\quad + \frac{1}{2}\int_{x+\varepsilon}^1 K(x)e^{(s-x)K(x)}(f^*(s)-f^*(x))ds \\ &\quad + \frac{1}{2}\int_0^{x-\varepsilon}\Big(\frac{\partial}{\partial x}e^{\eta K(x)}\Big)|_{\eta=(x-s)}f^*(s)ds - \frac{1}{2}\int_{x+\varepsilon}^1\Big(\frac{\partial}{\partial x}e^{\eta K(x)}\Big)|_{\eta=(s-x)}f^*(s)ds, \end{aligned}$$

and as $\varepsilon\to 0$, we obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0}[(w_1^\varepsilon)'(x,f^*) + (w_2^\varepsilon)'(x,f^*)] \\ &= \frac{1}{2}e^{xK(x)}f^*(x) + \frac{1}{2}e^{(1-x)K(x)}f^*(x) \\ &\quad + \frac{1}{2}\int_0^x K(x)e^{(x-s)K(x)}(f^*(s)-f^*(x))ds \\ &\quad + \frac{1}{2}\int_x^1 K(x)e^{(s-x)K(x)}(f^*(s)-f^*(x))ds \\ &\quad + \frac{1}{2}\int_0^x\Big(\frac{\partial}{\partial x}e^{\eta K(x)}\Big)|_{\eta=(x-s)}f^*(s)ds - \frac{1}{2}\int_x^1\Big(\frac{\partial}{\partial x}e^{\eta K(x)}\Big)|_{\eta=(s-x)}f^*(s)ds. \end{aligned}$$

All these integrals are absolutely convergent. To calculate $v''(x,f^*)+Q(x)v(x,f^*)$, we need to compute $w_3'(x,f^*)+w_4'(x,f^*)+w_5'(x,f^*)+w_6'(x,f^*)$. One has

$$\begin{aligned} &w_3'(x,f^*)+w_4'(x,f^*)+w_5'(x,f^*)+w_6'(x,f^*) \\ &= -\frac{1}{4i\pi}\int_0^x\int_{\Gamma_1}e^{(x-s)z}\frac{\partial}{\partial x}(K(x)-zI)^{-1}f^*(s)\,dz\,ds \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{4i\pi}\int_0^x \int_{\Gamma_1} \frac{e^{(x-s)z}}{z}\frac{\partial^2}{\partial x^2}(K(x)-zI)^{-1}f^*(s)\,dz\,ds \\
&+ \frac{1}{4i\pi}\int_x^1 \int_{\Gamma_1} e^{(s-x)z}\frac{\partial}{\partial x}(K(x)-zI)^{-1}f^*(s)\,dz\,ds \\
&- \frac{1}{4i\pi}\int_x^1 \int_{\Gamma_1} \frac{e^{(s-x)z}}{z}\frac{\partial^2}{\partial x^2}(K(x)-zI)^{-1}f^*(s)\,dz\,ds.
\end{aligned}
$$

Due to (2.6) and (2.10), the previous integrals are convergent. For instance,

$$
\| - \frac{1}{4i\pi}\int_0^x \int_{\Gamma_1} \frac{e^{(x-s)z}}{z}\frac{\partial^2}{\partial x^2}(K(x)-zI)^{-1}f^*(s)\,dz\,ds\| \le Cx^{\nu}\|f^*\|_{C(X)}.
$$

The other integrals are similarly convergent. Therefore,

$$
v''(x,f^*) + Q(x)v(x,f^*) = f^*(x) + V_\lambda(f^*)(x),
$$

where

$$
\begin{aligned}
&V_\lambda(f^*)(x) \\
&= \int_0^x \Big(\frac{\partial}{\partial x}e^{\eta K(x)}\Big)|_{\eta=(x-s)}f^*(s)ds - \int_x^1 \Big(\frac{\partial}{\partial x}e^{\eta K(x)}\Big)|_{\eta=(s-x)}f^*(s)ds \\
&\quad - \frac{1}{4i\pi}\int_0^x \int_{\Gamma_1} \frac{e^{(x-s)z}}{z}\frac{\partial^2}{\partial x^2}(K(x)-zI)^{-1}f^*(s)\,dz\,ds \\
&\quad - \frac{1}{4i\pi}\int_x^1 \int_{\Gamma_1} \frac{e^{(s-x)z}}{z}\frac{\partial^2}{\partial x^2}(K(x)-zI)^{-1}f^*(s)\,dz\,ds.
\end{aligned} \tag{4.16}
$$

On the other hand, by a simple computation and due to (2.8), we find $V_\lambda(f^*)(0) \in \overline{D(K(0))}$, $V_\lambda(f^*)(1) \in \overline{D(K(1))}$. Summarizing the above calculations, one gets

$$
\mathrm{Op}(v(.,f^*))(x) = f^*(x) + V_\lambda(f^*)(x) + W_\lambda(f^*)(x). \tag{4.17}
$$

Since $V_\lambda(f^*)+W_\lambda(f^*) \in C^{\eta+\nu-1}([0,1];X)$ and $f^* \in C^\beta([0,1];X)$, then the function $x \mapsto \mathrm{Op}(v(.,f^*))(x)$ belongs to $C^{\min(\beta,\eta+\nu-1)}([0,1];X)$. □

Remark 4.5. Observe that all the Hölderianities studied above were done near 0 or 1 and from this study we deduce the Hölderianity in $[0,1]$.

5. Equation satisfied by the solution and its resolution

The previous computations prove that the representation given in (2.17) satisfies the abstract equation

$$
\begin{aligned}
&u''(x) + B(x)u'(x) + Q(x)u(x) \\
&= (F_\lambda(x)\varphi + G_\lambda(x)\varphi) + (S_\lambda(x)\psi + T_\lambda(x)\psi) \\
&\quad + (M_\lambda(f^*)(x) + N_\lambda(f^*)(x)) + (V_\lambda(f^*)(x) + W_\lambda(f^*)(x) + f^*(x)) \\
&= f(x), \quad \text{for } x \in]0,1[,
\end{aligned} \tag{5.1}
$$

To determine the unknown function f^*, we need to use the following result.

Lemma 5.1. *Under hypothesis* (1.3), *there exists a constant $C > 0$ such that*

$$
\|(K(x)-zI)^{-1}\|_{L(X)} \le \frac{C}{\sqrt{\lambda}}, \tag{5.2}
$$

for all $\lambda > 0$, $z \in \Pi_{\theta_1+\pi/2,r_1}$, $x \in [0,1]$ *and*

$$\|(-A(x)+\lambda)^{\kappa} e^{y(-(-A(x)+\lambda)^{1/2})}\|_{L(X)} \le Ce^{-\omega y\lambda^{1/2}}, \tag{5.3}$$

for some $\omega > 0$ *and all* $\kappa \in \mathbb{R}$, $y > 0$, $\lambda > 0$.

Proof. For the first estimate, by (2.3), using the change of variable $s \leftrightarrow \lambda s$ and the well known formula $\int_0^\infty \frac{s^{-\kappa}}{(s+1)}ds = \frac{\pi}{\sin(\kappa\pi)}$, $\kappa > 0$, we obtain, for $z > 0$ (for instance)

$$\|(K(x)-zI)^{-1}\|_{L(X)} \le C\int_0^\infty \frac{\sqrt{s}}{(\lambda+s)(s+z^2)}ds \le C\int_0^\infty \frac{\sqrt{s}}{(\lambda+s)s}ds \le \frac{C}{\sqrt{\lambda}}.$$

For the second estimate, the complete proof is given in [4, Lemma 2.6, statement b, p. 103]. □

Now, we prove the following result.

Proposition 5.2. *Let* $\varphi \in D((K(0))^2)$, $\psi \in D((K(1))^2)$ *and* $f \in C^\theta([0,1];X)$, $0 < \theta < 1$. *Assume* (1.3), (1.4), (2.6)–(2.11), *and that* u *given in* (2.17) *is a strict solution of* (1.1)-(1.2). *Then the function* f^*, *in the space* $C([0,1];X)$, *satisfies the equation*

$$(I+R_\lambda)(f^*)(\cdot) = f(\cdot) - F_\lambda(\cdot)\varphi - G_\lambda(\cdot)\varphi - S_\lambda(\cdot)\psi - T_\lambda(\cdot)\psi, \tag{5.4}$$

where

$$R_\lambda(f^*)(\cdot) = M_\lambda(f^*)(\cdot) + N_\lambda(f^*)(\cdot) + V_\lambda(f^*)(\cdot) + W_\lambda(f^*)(\cdot). \tag{5.5}$$

Moreover, there exists $\lambda^* > 0$ *such that for all* $\lambda \ge \lambda^*$, *operator* $I + R_\lambda$ *is invertible in* $C([0,1];X)$ *and*

$$(f^*)(\cdot) = (I+R_\lambda)^{-1}[f(\cdot) - F_\lambda(\cdot)\varphi - G_\lambda(\cdot)\varphi - S_\lambda(\cdot)\psi - T_\lambda(\cdot)\psi]. \tag{5.6}$$

Proof. To solve (5.4) in the space $C([0,1];X)$, we have to estimate $\|R_\lambda\|_{L(C([0,1];X))}$ (see (5.5)), for a large $\lambda > 0$. Let us, for instance, estimate some terms contained in $V_\lambda(f^*)(x)$; that is, (see (4.16))

$$\begin{aligned}
&V_\lambda(f^*)(x)\\
&= \int_0^x \big(\frac{\partial}{\partial x}e^{\xi K(x)}\big)|_{\xi=(x-s)} f^*(s)ds - \int_x^1 \big(\frac{\partial}{\partial x}e^{\xi K(x)}\big)|_{\xi=(s-x)} f^*(s)ds\\
&\quad - \frac{1}{4i\pi}\int_0^x \int_{\Gamma_1} \frac{e^{(x-s)z}}{z}\frac{\partial^2}{\partial x^2}(K(x)-zI)^{-1} f^*(s)\,dz\,ds\\
&\quad - \frac{1}{4i\pi}\int_x^1 \int_{\Gamma_1} \frac{e^{(s-x)z}}{z}\frac{\partial^2}{\partial x^2}(K(x)-zI)^{-1} f^*(s)\,dz\,ds\\
&= (R_1) + (R_2) + (R_3) + (R_4).
\end{aligned}$$

Due to Remark 2.3, we have the following estimate with respect to parameter λ,

$$\|\frac{\partial}{\partial x}e^{\xi K(x)}\| \le \frac{C}{\lambda^{(1-\eta)/2}\xi^{1-(\eta+\nu-1)}}. \tag{5.7}$$

Then, for the term (R_1) ((R_2) will be treated similarly), one obtains

$$\|(R_1)\|_X \le \frac{C}{\lambda^{(1-\eta)/2}}\int_0^x (x-s)^{(\eta+\nu-1)-1}ds \le \frac{C}{\lambda^{(1-\eta)/2}}\|f^*\|_{C(X)}.$$

For the term (R_3) (the term (R_4) will be treated similarly), from (2.14) and using
$$\frac{K(x)(K(x)-zI)^{-1}}{z}=(K(x)-zI)^{-1}+\frac{I}{z},$$
it follows that
$$\begin{aligned}
&(R_3)\\
&=-\frac{1}{4i\pi}\int_0^x\int_{\Gamma_1}e^{(x-s)z}(K(x)-zI)^{-1}\frac{d^2(K(x))^{-1}}{dx^2}K(x)(K(x)-zI)^{-1}f^*(s)\,dz\,ds\\
&\quad-\frac{1}{4i\pi}\int_0^x\int_{\Gamma_1}e^{(x-s)z}\frac{d^2(K(x))^{-1}}{dx^2}(K(x)-zI)^{-1}f^*(s)\,dz\,ds\\
&\quad-\frac{1}{4i\pi}\int_0^x\int_{\Gamma_1}\frac{e^{(x-s)z}}{z}\frac{d^2(K(x))^{-1}}{dx^2}f^*(s)\,dz\,ds\\
&\quad-\frac{1}{2i\pi}\int_0^x\int_{\Gamma_1}e^{(x-s)z}\frac{\partial}{\partial x}(K(x)-zI)^{-1}\frac{d(K(x))^{-1}}{dx}K(x)(K(x)-zI)^{-1}f^*(s)\,dz\,ds\\
&=(R_{31})+(R_{32})+(R_{33})+(R_{34}).
\end{aligned}$$
One observes that $(R_{33})=0$. Concerning the term (R_{31}) (similarly for (R_{32})), by the use of the estimates (see [11])
$$\exists C>0:\forall\lambda>0,\ \forall z\in\Gamma_1,\quad |z+\sqrt{\lambda}|\geq C|z| \text{ and } |z+\sqrt{\lambda}|\geq C\sqrt{\lambda},$$
and Remark 2.3, one obtains
$$\begin{aligned}
\|(R_{31})\|_X&\leq C\|f^*\|_{C(X)}\int_0^x\int_{\Gamma_1}\frac{|e^{(x-s)z}|}{|z+\sqrt{\lambda}|^{1/2}|z+\sqrt{\lambda}|^{1/2}}|dz|ds\\
&\leq C\|f^*\|_{C(X)}\int_0^x\int_{\Gamma_1}\frac{1}{\lambda^{1/4}}\frac{e^{Re(\sigma)}d|\sigma|}{\big(\frac{|\sigma|}{x-s}\big)^{1/2}(x-s)}ds\\
&\leq\frac{C}{\lambda^{1/4}}\|f^*\|_{C(X)}.
\end{aligned}$$
Similarly for the term (R_{34}), it results that
$$\begin{aligned}
\|(R_{34})\|_X&\leq C\|f^*\|_{C(X)}\int_0^x\int_{\Gamma_1}\frac{|e^{(x-s)z}||dz|ds}{|z+\sqrt{\lambda}|^{\nu+\eta-1}|z+\sqrt{\lambda}|^{1-\eta}}\\
&\leq C\|f^*\|_{C(X)}\int_0^x\int_{\Gamma_1}\frac{1}{\lambda^{(\nu+\eta-1)/2}}\frac{e^{Re(\sigma)}d|\sigma|}{\big(\frac{|\sigma|}{x-s}\big)^{1-\eta}(x-s)}ds\\
&\leq\frac{C}{\lambda^{(\nu+\eta-1)/2}}\|f^*\|_{C(X)}.
\end{aligned}$$
A similar analysis for the other terms prove the existence of some λ^* such that for all $\lambda\geq\lambda^*$, one has $\|R_\lambda\|_{L(C([0,1];X))}<1$. Therefore, $(I+R_\lambda)$ is invertible for $\lambda\geq\lambda^*$ in $C([0,1];X)$ and we obtain (5.6). □

In the sequel, we need the following result concerning $f^*(0)$ and $f^*(1)$.

Proposition 5.3. *Let $\varphi\in D((K(0))^2)$, $\psi\in D((K(1))^2)$ and $f\in C^\theta([0,1];X)$, $0<\theta<1$. Assume* (1.3), (1.4), (2.6)–(2.11), *and that u given in* (2.17) *is a strict solution of* (1.1)-(1.2). *Then*
$$f^*(0)=f(0)+\frac{d^2}{dx^2}(K(x))|_{x=0}^{-1}K(0)\varphi+\Phi_0^*(\varphi)+r_0(f^*,\psi),$$

$$f^*(1) = f(1) + \frac{d^2}{dx^2}(K(x))|_{x=1}^{-1} K(1)\psi + \Phi_1^*(\psi) + r_1(f^*,\varphi),$$

where $\Phi_0^*(\varphi), r_0(f^*,\psi) \in \overline{D(K(0))}$ *and* $\Phi_1^*(\psi), r_1(f^*,\varphi) \in \overline{D(K(1))}$. *Moreover* $f^* \in C^{\beta}([0,1];X)$, *where* $\beta = \min(\eta+\nu-1,\theta)$.

Proof. We have

$$\begin{aligned} f^*(0) &= f(0) - F_\lambda(0)\varphi - G_\lambda(0)\varphi - S_\lambda(0)\psi - T_\lambda(0)\psi \\ &\quad - M_\lambda(f^*)(0) - N_\lambda(f^*)(0) - V_\lambda(f^*)(0) - W_\lambda(f^*)(0). \end{aligned}$$

The term

$$-S_\lambda(0)\psi - T_\lambda(0)\psi - M_\lambda(f^*)(0) - N_\lambda(f^*)(0) - V_\lambda(f^*)(0) - W_\lambda(f^*)(0) := r_0(f^*,\psi)$$

is in $\overline{D(K(0))}$, and

$$-F_\lambda(1)\varphi - G_\lambda(1)\varphi - M_\lambda(f^*)(1) - N_\lambda(f^*)(1) - V_\lambda(f^*)(1) - W_\lambda(f^*)(1) := r_1(f^*,\varphi)$$

is in $\overline{D(K(1))}$. From (4.5), we have seen that

$$F_\lambda(0)\varphi + G_\lambda(0)\varphi = \Phi_0^*(\varphi) - \frac{d^2}{dx^2}(K(x))|_{x=0}^{-1} K(0)\varphi$$

where $\Phi_0^*(\varphi) \in \overline{D(K(0))}$. Therefore,

$$f^*(0) = f(0) + \frac{d^2}{dx^2}(K(x))|_{x=0}^{-1} K(0)\varphi + \Phi_0^*(\varphi) + r_0(f^*,\psi),$$

where $\Phi_0^*(\varphi) \in \overline{D(K(0))}$ and $r_0(f^*,\psi) \in \overline{D(K(0))}$.

Similarly, from (4.6), we obtain

$$f^*(1) = f(1) + \frac{d^2}{dx^2}(K(x))|_{x=1}^{-1} K(1)\psi + \Phi_1^*(\psi) + r_1(f^*,\varphi),$$

where $\Phi_1^*(\psi) \in \overline{D(K(1))}$ and $r_1(f^*,\varphi) \in \overline{D(K(1))}$. We know that, under our Hypotheses and the fact that $\varphi \in D((K(0))^2)$ and $\psi \in D((K(1))^2)$, the function

$$\begin{aligned} x \mapsto\ & F_\lambda(x)\varphi + G_\lambda(x)\varphi + S_\lambda(x)\psi + T_\lambda(x)\psi + M_\lambda(f^*)(x) \\ &+ N_\lambda(f^*)(x) + V_\lambda(f^*)(x) + W_\lambda(f^*)(x) \end{aligned}$$

belongs to $C^{\eta+\nu-1}([0,1];X)$. Then we deduce that if f^* exists, it belongs necessarily to $C^{\min(\eta+\nu-1,\theta)}([0,1];X)$; therefore $\beta = \min(\eta+\nu-1,\theta)$. $\square$

6. Main result

Now, we present our main result which is concerns the existence and uniqueness of the strict solution of Problem (1.1)-(1.2).

Theorem 6.1. *Let* $\varphi \in D((K(0))^2)$, $\psi \in D((K(1))^2)$ *and* $f \in C^\theta([0,1];X)$, $0 < \theta < 1$. *Then, under Hypotheses* (1.3), (1.4), (2.6)–(2.11), *there exists* $\lambda^* > 0$ *such that for all* $\lambda \geq \lambda^*$, *the function* u *given in the representation* (2.16) *is the unique strict solution of Problem* (1.1)-(1.2) *if and only if*

$$\begin{aligned} f(0) + (K(0))^2\varphi + \frac{d^2}{dx^2}(K(x))|_{x=0}^{-1} K(0)\varphi \in \overline{D(K(0))} = \overline{D(Q(0))}, \\ f(1) + (K(1))^2\psi + \frac{d^2}{dx^2}(K(x))|_{x=1}^{-1} K(1)\psi \in \overline{D(K(1))} = \overline{D(Q(1))}. \end{aligned} \tag{6.1}$$

This theorem, can be stated as follows.

Theorem 6.2. *Let $\varphi \in D(A(0))$, $\psi \in D(A(1))$ and $f \in C^\theta([0,1];X)$, $0<\theta<1$. Then, under Hypotheses* (1.3), (1.4), (2.6)–(2.11)*, there exists $\lambda^*>0$ such that for all $\lambda \geq \lambda^*$, the function u given in the representation* (2.16) *is the unique strict solution of Problem* (1.1)-(1.2) *if and only if*

$$
\begin{gathered}
f(0)-A(0)\varphi+\frac{d^2}{dx^2}(\lambda-A(x))|_{x=0}^{-1/2}(\lambda-A(0))^{1/2}\varphi \in \overline{D(A(0))}, \\
f(1)-A(1)\psi+\frac{d^2}{dx^2}(\lambda-A(x))|_{x=1}^{-1/2}(\lambda-A(1))^{1/2}\psi \in \overline{D(A(1))}.
\end{gathered}
$$

Proof. It is sufficient to consider the case $\psi=0$ and prove that $x \mapsto Q(x)u(x)=-((K(x))^2)u(x) \in C([0,1];X)$. One has

$$
\begin{aligned}
(K(x))^2u(x) &= Y(x)(I-e^{(2-2x)K(x)})(K(x))^2e^{xK(x)}\varphi+(K(x))^2m(x,f^*) \\
&\quad+\frac{1}{2}\int_0^x K(x)e^{(x-s)K(x)}(f^*(s)-f^*(x))ds \\
&\quad+\frac{1}{2}\int_x^1 K(x)e^{(s-x)K(x)}(f^*(s)-f^*(x))ds \\
&\quad+\frac{1}{2}e^{xK(x)}f^*(x)+\frac{1}{2}e^{(1-x)K(x)}f^*(x)-f^*(x).
\end{aligned}
$$

On the other hand, for $m(x,f^*)$, we will consider only the first term (which may exhibit a singularity near 0; the other terms are regular when we apply $(K(x))^2$). We have

$$
\begin{aligned}
(K(x))^2m(x,f^*) &= -\frac{Y(x)K(x)}{2}\int_0^1 e^{(x+s)K(x)}f^*(s)ds+R_m(x,f^*) \\
&= -\frac{Y(x)K(x)}{2}\int_0^1 e^{(x+s)K(x)}(f^*(s)-f^*(0))ds \\
&\quad-\frac{Y(x)}{2}(e^{K(x)}-I)e^{xK(x)}f^*(0)+R_m(x,f^*).
\end{aligned}
$$

Summing up, we obtain

$$
\begin{aligned}
&(K(x))^2u(x) \\
&= Y(x)\big(I-e^{(2-2x)K(x)}\big)(K(x))^2e^{xK(x)}\varphi \\
&\quad+\frac{1}{2}e^{xK(x)}f^*(x)-\frac{Y(x)}{2}\big(e^{K(x)}-I\big)e^{xK(x)}f^*(0)+\frac{1}{2}e^{(1-x)K(x)}f^*(x)-f^*(x) \\
&\quad-\frac{Y(x)K(x)}{2}\int_0^1 e^{(x+s)K(x)}(f^*(s)-f^*(0))ds+R_m(x,f^*) \\
&\quad+\frac{1}{2}\int_0^x K(x)e^{(x-s)K(x)}(f^*(s)-f^*(x))ds \\
&\quad+\frac{1}{2}\int_x^1 K(x)e^{(s-x)K(x)}(f^*(s)-f^*(x))ds.
\end{aligned}
$$

It is well known that the last four terms are continuous on $[0,1]$. The term $\frac{1}{2}e^{(1-x)K(x)}f^*(x)$ is easy to treat. Let us handle the first three terms

$$
\begin{aligned}
&Y(x)(I-e^{(2-2x)K(x)})(K(x))^2e^{xK(x)}\varphi+\frac{1}{2}e^{xK(x)}f^*(x) \\
&-\frac{Y(x)}{2}(e^{K(x)}-I)e^{xK(x)}f^*(0)
\end{aligned}
$$

$$\begin{aligned}
&= [Y(x)\big(I - e^{(2-2x)K(x)}\big) - I](K(x))^2 e^{xK(x)}\varphi + (K(x))^2 e^{xK(x)}\varphi \\
&\quad - \frac{[Y(x) - I]}{2}\big(e^{K(x)} - I\big)e^{xK(x)} f^*(0) - \frac{1}{2}e^{(1+x)K(x)} f^*(0) \\
&\quad + \frac{1}{2}e^{xK(x)}(f^*(x) - f^*(0)) + e^{xK(x)} f^*(0).
\end{aligned}$$

It remains to study $(K(x))^2 e^{xK(x)}\varphi + e^{xK(x)} f^*(0)$. As previously seen, write

$$\begin{aligned}
&(K(x))^2 e^{xK(x)}\varphi + e^{xK(x)} f^*(0) \\
&= ((K(x))^2 - (K(0))^2)e^{xK(x)}\varphi + (K(0))^2 e^{xK(x)}\varphi + e^{xK(x)} f^*(0) \\
&= ((K(x))^2 - (K(0))^2)e^{xK(x)}\varphi + (K(0))^2\big(e^{xK(x)} - e^{xK(0)}\big)\varphi \\
&\quad + [e^{xK(x)} - e^{xK(0)}]f^*(0) + (K(0))^2 e^{xK(0)}\varphi + e^{xK(0)} f^*(0) \\
&= (a) + (b) + (c) + (d) + (e).
\end{aligned}$$

One obtains (a), (b) tend to 0, as $x \to 0$; $(c) \in C([0,1];X)$ and due to Lemma 3.5, the term

$$\begin{aligned}
&(d) + (e) \\
&= e^{xK(0)}((K(0))^2\varphi + f(0) + \frac{d^2}{dx^2}(K(x))|_{x=0}^{-1} K(0)\varphi) + e^{xK(0)}(\Phi_0^*(\varphi) + r_0(f^*, \psi))
\end{aligned}$$

is in $C([0,1];X)$ if and only if

$$f(0) + (K(0))^2\varphi + \frac{d^2}{dx^2}(K(x))|_{x=0}^{-1} K(0)\varphi \in \overline{D(K(0))} = \overline{D(Q(0))} = \overline{D(A(0))}.$$

Similar arguments give us the compatibility condition in 1. □

Remark 6.3. Note that our main result improves the results concerning the study of (1.1) done in [6]. Indeed, it gives necessarily and sufficient conditions for the existence of strict solution by using the square roots $-\sqrt{-Q(x)}$, while in [6], the authors give, only, a sufficient conditions for the existence of strict solution, using the operators $Q(x)$.

7. Example

Consider the complex Banach space $X = C([0,1])$ with its usual sup-norm and define the family of closed linear operators $(-A(x))^{1/2}$ for all $x \in [0,1]$ by

$$\begin{gathered}
D(-(-A(x))^{1/2}) = \{\varphi \in C^2([0,1]) : a(x)\varphi(0) - b(x)\varphi'(0) = 0;\ \varphi(1) = 0\} \\
(-(-A(x))^{1/2}\varphi)(y) = \varphi''(y),\ y \in [0,1].
\end{gathered}$$

We are inspired by an analogous example in [1, p. 52] from which it is easy to deduce that

$$\begin{gathered}
D(A(x)) = \{\varphi \in C^4([0,1]) : a(x)\varphi(0) - b(x)\varphi'(0) = 0, \varphi(1) = 0; \\
a(x)\varphi''(0) - b(x)\varphi'''(0) = 0,\ \varphi''(1) = 0\} \\
(A(x)\varphi)(y) = -\varphi^{(iv)}(y), \quad y \in [0,1].
\end{gathered}$$

We assume that $a, b \in C^{2,\kappa}[0,1]$, $a(x) \geq 0$ and $\min_{x\in[0,1]} b(x) > 0$. Let us define the family of bounded linear operators $(B(x))_{x\in[0,1]}$ by

$$D(B(x)) = X; \quad (B(x)\varphi)(y) = c(x)\varphi(y).$$

where c is an arbitrary function in $C^{\kappa}([0,1])$. Then, all our results apply to the following concrete quasi-elliptic boundary value problem for a large $\lambda > 0$,

$$\begin{gathered}
\frac{\partial^2 u}{\partial x^2}(x,y) + c(x)\frac{\partial u}{\partial x}(x,y) - \frac{\partial^4 u}{\partial y^4}(x,y) - \lambda u(x,y) = f(x,y), \quad x,y \in (0,1), \\
a(x)u(x,0) - b(x)\frac{\partial u}{\partial y}(x,0) = 0, \quad x \in (0,1), \\
a(x)\frac{\partial^2 u}{\partial y^2}(x,0) - b(x)\frac{\partial^3 u}{\partial y^3}(x,0) = 0, \quad x \in (0,1), \\
u(x,1) = 0 = \frac{\partial^2 u}{\partial y^2}(x,1), \quad x \in (0,1), \\
u(0,y) = \varphi(y), \quad u(1,y) = \psi(y), \quad y \in (0,1),
\end{gathered}$$

provided that $f \in C^{\theta}([0,1];X)$.

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