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# NEWTON'S METHOD FOR STOCHASTIC DIFFERENTIAL EQUATIONS AND ITS PROBABILISTIC SECOND-ORDER ERROR ESTIMATE

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Abstract. Kawabata and Yamada [5] proposed an implicit Newton's method for nonlinear stochastic differential equations and proved its convergence. Later Amano [2] gave an explicit formulation of method and showed its direct error estimate. In this article, we prove a probabilistic second-order error estimate which has been an open problem since 1991.

## 1. Introduction

Let $a(t,x)$ and $b(t,x)$ be real-valued bounded $C^2$ smooth functions defined in the two dimensional Euclidean space $\mathbb{R}^2$. We assume that there exist nonnegative constants $A_1$, $A_2$, $B_1$ and $B_2$ satisfying

$$\big|\frac{\partial a}{\partial x}(t,x)\big| \le A_1\,, \quad \big|\frac{\partial^2 a}{\partial x^2}(t,x)\big| \le A_2$$

and

$$\big|\frac{\partial b}{\partial x}(t,x)\big| \le B_1\,, \quad \big|\frac{\partial^2 b}{\partial x^2}(t,x)\big| \le B_2$$

in $\mathbb{R}^2$.

Let $w(t)$, $t \ge 0$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{F}_t$, $t \ge 0$ be the natural filtration of $\mathcal{F}$. We assume that $\xi(t)$, $t \ge 0$ is a solution of the initial value problem for stochastic differential equation

$$d\xi(t) = a\big(t, \xi(t)\big)\,dt + b\big(t, \xi(t)\big)\,dw(t)\,, \quad \xi(0) = \xi_0\,, \tag{1.1}$$

where $\xi_0$ is a bounded random variable independent of $\mathcal{F}_t$, $t \ge 0$. Without loss of generality, we may assume that $\xi(t)$ is continuous with respect to $t \ge 0$.

For $T > 0$ and $1 \le p < \infty$, $L^p_w[0,T]$ stands for the class of all separable non-anticipative functions $f(t)$, $t \ge 0$ with respect to $\{\mathcal{F}_t\}$ satisfying

$$P\Big[\int_0^T |f(t)|^p\,dt < \infty\Big] = 1$$

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and $M_w^p[0,T]$ denotes the subset of $L_w^p[0,T]$ consisting of all functions $f(t)$ with

$$E\Big[\int_0^T |f(t)|^p\,dt\Big] < \infty\,.$$

It is well-known that $\xi(t) \in M_w^2[0,T]$ for any $T > 0$ (see, for example, [4]).

The explicit Newton's scheme for (1.1) is formulated as follows (see [2]): We define a sequence $\{\xi_n(t)\}$ by $\xi_0(t) = \xi_0$ and

$$\begin{aligned}
&\xi_{n+1}(t)\\
&= e^{\eta_n(t)}\Big(\xi_0 + \int_0^t \big(a_{0,n}(s) - b_{0,n}(s)b_{1,n}(s)\big)e^{-\eta_n(s)}ds + \int_0^t b_{0,n}(s)e^{-\eta_n(s)}dw(s)\Big)
\end{aligned}$$

for $n = 0,1,2,\dots$, where

$$\begin{gathered}
\eta_n(t) = \int_0^t \Big(a_{1,n}(s) - \frac{1}{2}\,(b_{1,n}(s))^2\Big)\,ds + \int_0^t b_{1,n}(s)\,dw(s)\,,\\
a_{0,n}(t) = a(t,\xi_n(t)) - \frac{\partial a}{\partial x}\big(t,\xi_n(t)\big)\,\xi_n(t)\,,\\
a_{1,n}(t) = \frac{\partial a}{\partial x}\big(t,\xi_n(t)\big)\,,\\
b_{0,n}(t) = b(t,\xi_n(t)) - \frac{\partial b}{\partial x}\big(t,\xi_n(t)\big)\,\xi_n(t)\,,\\
b_{1,n}(t) = \frac{\partial b}{\partial x}\big(t,\xi_n(t)\big)\,.
\end{gathered}$$

In this article, we shall estimate the approximation errors

$$\varepsilon_n(t) = \xi_n(t) - \xi(t), \quad n = 0,1,2,\dots\,. \tag{1.2}$$

**Theorem 1.1.** *For any $T > 0$, there exists a nonnegative constant $C$ depending only on $T$, $A_1$, $A_2$, $B_1$ and $B_2$ such that*

$$P\Big[\sup_{0\le t\le T}|\varepsilon_n(t)| \le \rho \text{ implies } \sup_{0\le t\le T}|\varepsilon_{n+1}(t)| \le R\rho^2\Big] \ge 1 - C\,R^{-1/2}$$

*for all $R \ge 1$, $0 < \rho \le 1$ and every $n = 0,1,2,\dots$.*

Our symbolic Newton's method may give a new possibility to the study of computer algebraic method in stochastic analysis (see, for example, [3]).

If the positive constants $T_0$, $A_1$, $A_2$, $B_1$ and $B_2$ are given, then a repeated use of Theorem 1.1 gives an approximate solution of (1.1) in terms of multiple stochastic integrals. For example, we first note that, seeing the proof of Theorem 1.1, we can choose $R \ge 1$ sufficiently large so that

$$CR^{-1/2} < \frac{1}{100}, \quad \text{where } C = C(A_1,A_2,B_1,B_2,T)$$

for all positive $T \le T_0$. Second, we take a small $\rho > 0$ so as to satisfy

$$R\rho < \frac{1}{10}\,.$$

Third, by using a martingale inequality (Lemma 2.8), we take a sufficiently small $T > 0$ such that

$$P\Big[\sup_{0\le t\le T}|\varepsilon_0(t)| \le \rho\Big] \ge 1 - \frac{1}{100}\,.$$

Now, a repeated use of Theorem 1.1 and

$$R^{-1}\big(R\rho\big)^{2^{10}} \le \frac{1}{10^{2^{10}}}$$

show that

$$P\Big[\sup_{0\le t\le T} |\varepsilon_{10}(t)| \le \frac{1}{10^{1024}}\Big] \ge 1 - \frac{1}{10}\,.$$

It is clear, by the definition of $\{\xi_n(t)\}$, that the approximate solution $\xi_{10}(t)$ has a multiple stochastic integral representation.

## 2. Preliminaries

The following two lemmas follow immediately from Itô's formula.

**Lemma 2.1.** *If $\alpha(t) \in L^1_w[0,T]$ and $\beta(t) \in L^2_w[0,T]$ for any $T > 0$ and if*

$$d\eta(t) = \alpha(t)\,dt + \beta(t)\,dw(t)\,,$$

*then*

$$de^{\eta(t)} = e^{\eta(t)}\,d\eta(t) + \frac{1}{2}\,\beta^2(t)e^{\eta(t)}\,dt\,.$$

*Proof.* For a function $f(x) = e^x$, Itô's formula gives

$$d\,f(\eta(t)) = \Big(f'(\eta(t))\alpha(t) + \frac{1}{2}\,f''(\eta(t))\beta^2(t)\Big)dt + f'(\eta(t))\beta(t)\,dw(t)\,;$$

this implies the desired formula. □

**Lemma 2.2.** *If $\alpha_1(t), \alpha_2(t) \in L^1_w[0,T]$ and $\beta_1(t), \beta_2(t) \in L^2_w[0,T]$ for any $T > 0$ and if*

$$d\xi_i(t) = \alpha_i(t)\,dt + \beta_i(t)\,dw(t)\,, \quad i = 1, 2\,,$$

*then*

$$d\big(\xi_1(t)\,\xi_2(t)\big) = \xi_2(t)\,d\xi_1(t) + \xi_1(t)\,d\xi_2(t) + \beta_1(t)\beta_2(t)\,dt\,.$$

*Proof.* Applying Itô's formula for a 2-dimensional diffusion process $(\xi_1(t), \xi_2(t))$ and a function $f(x_1, x_2) = x_1x_2$, we have

$$\begin{aligned}
&df\big(\xi_1(t), \xi_2(t)\big)\\
&= \Big(\sum_{i=1}^{2} \frac{\partial f}{\partial x_i}\big(\xi_1(t), \xi_2(t)\big)\alpha_i(t) + \frac{1}{2}\sum_{i,j=1}^{2} \frac{\partial^2 f}{\partial x_i \partial x_j}\big(\xi_1(t), \xi_2(t)\big)\beta_i(t)\beta_j(t)\Big)dt\\
&\quad + \sum_{i=1}^{2} \frac{\partial f}{\partial x_i}\big(\xi_1(t), \xi_2(t)\big)\beta_i(t)\,dw(t)\,;
\end{aligned}$$

this completes the proof. □

Lemmas 2.1 and 2.2 show the following three key lemmas.

**Lemma 2.3.** *For $n = 1, 2, 3, \dots$ and*

$$a_0(t) = a_{0,n}(t)\,, \quad a_1(t) = a_{1,n}(t)\,, \quad b_0(t) = b_{0,n}(t)\,, \quad b_1(t) = b_{1,n}(t)\,,$$

*the initial value problem for the linear stochastic differential equation*

$$d\xi(t) = \big(a_0(t) + a_1(t)\,\xi(t)\big)\,dt + \big(b_0(t) + b_1(t)\,\xi(t)\big)\,dw(t)\,, \quad \xi(0) = \xi_0$$

*has an explicit solution*

$$\zeta(t) = e^{\eta(t)}\Big(\xi_0 + \int_0^t \big(a_0(s) - b_0(s)b_1(s)\big)e^{-\eta(s)}\,ds + \int_0^t b_0(s)e^{-\eta(s)}\,dw(s)\Big)\,,$$

*where*

$$\eta(t) = \int_0^t \Big(a_1(s) - \frac{1}{2}b_1^2(s)\Big)\,ds + \int_0^t b_1(s)\,dw(s)\,.$$

*Proof.* Since Lemma 2.1 gives

$$de^{\eta(t)} = e^{\eta(t)}d\eta(t) + \frac{1}{2}b_1^2(t)e^{\eta(t)}\,dt = a_1(t)e^{\eta(t)}\,dt + b_1(t)e^{\eta(t)}\,dw(t)\,,$$

Lemma 2.2 shows

$$\begin{aligned}
d\zeta(t) &= d\left(e^{\eta(t)}\Big(\xi_0 + \int_0^t \big(a_0(s) - b_0(s)b_1(s)\big)e^{-\eta(s)}\,ds + \int_0^t b_0(s)e^{-\eta(s)}\,dw(s)\Big)\right)\\
&= \Big(\xi_0 + \int_0^t \big(a_0(s) - b_0(s)b_1(s)\big)e^{-\eta(s)}\,ds + \int_0^t b_0(s)e^{-\eta(s)}\,dw(s)\Big)\,de^{\eta(t)}\\
&\quad + e^{\eta(t)}\Big(\big(a_0(t) - b_0(t)b_1(t)\big)e^{-\eta(t)}\,dt + b_0(t)e^{-\eta(t)}\,dw(t)\Big)\\
&\quad + \big(b_1(t)e^{\eta(t)}\big)\big(b_0(t)e^{-\eta(t)}\big)\,dt\\
&= \zeta(t)\big(a_1(t)\,dt + b_1(t)\,dw(t)\big)\\
&\quad + \big(a_0(t) - b_0(t)b_1(t)\big)\,dt + b_0(t)\,dw(t) + b_1(t)b_0(t)\,dt\\
&= \big(a_0(t) + a_1(t)\,\zeta(t)\big)\,dt + \big(b_0(t) + b_1(t)\,\zeta(t)\big)\,dw(t)\,.
\end{aligned}$$

□

**Remark 2.4.** It follows immediately from the definition of $\xi_{n+1}(t)$ and Lemma 2.3 that $\xi_{n+1}(0) = \xi_0$ and

$$d\xi_{n+1}(t) = \big(a_{0,n}(t) + a_{1,n}(t)\,\xi_{n+1}(t)\big)\,dt + \big(b_{0,n}(t) + b_{1,n}(t)\,\xi_{n+1}(t)\big)\,dw(t)$$

for $n = 0, 1, 2, \dots$. Therefore, $\{\xi_n(t)\}$ is exactly the same sequence introduced by Kawabata and Yamada [5]; this implies the convergence

$$\lim_{n\to\infty} E\Big[\sup_{0\le t\le T} |\,\xi_n(t) - \xi(t)\,|^2\Big] = 0$$

for any $T > 0$. By their result, we have only to concentrate on the estimation of errors.

**Lemma 2.5.** *For $n = 0, 1, 2, \dots$, we have*

$$\varepsilon_{n+1}(t) = e^{\eta_n(t)}\Big(\int_0^t \big(\alpha_{0,n}(s) - \beta_{0,n}(s)b_{1,n}(s)\big)e^{-\eta_n(s)}ds + \int_0^t \beta_{0,n}(s)e^{-\eta_n(s)}dw(s)\Big)\,,$$

*where*

$$\begin{aligned}
\alpha_{0,n}(t) &= \varepsilon_n^2(t)\int_0^1 (\theta - 1)\,\frac{\partial^2 a}{\partial x^2}\big(t, \xi_n(t) - \theta\varepsilon_n(t)\big)\,d\theta\,,\\
\beta_{0,n}(t) &= \varepsilon_n^2(t)\int_0^1 (\theta - 1)\,\frac{\partial^2 b}{\partial x^2}\big(t, \xi_n(t) - \theta\varepsilon_n(t)\big)\,d\theta\,.
\end{aligned}$$

*Proof.* Since $\xi_{n+1}(t)$ is a solution of the linear stochastic differential equation in Remark 2.4, by (1.1) and (1.2), we have

$$\begin{aligned}
&d\varepsilon_{n+1}(t)\\
&= d\,\xi_{n+1}(t) - d\,\xi(t)\\
&= \Big(a(t,\xi_n(t)) - \frac{\partial a}{\partial x}\big(t,\xi_n(t)\big)\varepsilon_n(t) - a\big(t,\xi_n(t)-\varepsilon_n(t)\big) + a_{1,n}(t)\varepsilon_{n+1}(t)\Big)\,dt\\
&\quad + \Big(b(t,\xi_n(t)) - \frac{\partial b}{\partial x}\big(t,\xi_n(t)\big)\varepsilon_n(t) - b\big(t,\xi_n(t)-\varepsilon_n(t)\big) + b_{1,n}(t)\varepsilon_{n+1}(t)\Big)\,dw(t)\,.
\end{aligned}$$

Let us consider an auxiliary function

$$F(\theta) = a\big(t,\xi_n(t)-\theta\varepsilon_n(t)\big)\,,\quad 0\le\theta\le 1\,.$$

Then, integration by parts shows

$$F(1) = F(0) + F'(0) + \int_0^1 (1-\theta)\,F''(\theta)\,d\theta\,;$$

this gives

$$\alpha_{0,n}(t) = a(t,\xi_n(t)) - \frac{\partial a}{\partial x}\big(t,\xi_n(t)\big)\varepsilon_n(t) - a\big(t,\xi_n(t)-\varepsilon_n(t)\big)\,.$$

Similarly, we have

$$\beta_{0,n}(t) = b(t,\xi_n(t)) - \frac{\partial b}{\partial x}\big(t,\xi_n(t)\big)\varepsilon_n(t) - b\big(t,\xi_n(t)-\varepsilon_n(t)\big)\,.$$

Therefore, we obtain

$$d\varepsilon_{n+1}(t) = \big(\alpha_{0,n}(t) + a_{1,n}(t)\varepsilon_{n+1}(t)\big)\,dt + \big(\beta_{0,n}(t) + b_{1,n}(t)\varepsilon_{n+1}(t)\big)\,dw(t)\,;$$

seeing the proof of Lemma 2.3 for $a_0(t)=\alpha_{0,n}(t)$, $a_1(t)=a_{1,n}(t)$, $b_0(t)=\beta_{0,n}(t)$ and $b_1(t)=b_{1,n}(t)$, this completes the proof. □

**Lemma 2.6.** *For any $t>0$, we obtain*

$$\begin{gathered}
E\big[|e^{\eta_n(t)}-1|^2\big] \le 4t(A_1\sqrt{t}+B_1)^2 e^{4t(A_1\sqrt{t}+B_1)^2}\,,\\
E\big[|e^{-\eta_n(t)}-1|^2\big] \le 4t\big((A_1+B_1^2)\sqrt{t}+B_1\big)^2 e^{4t((A_1+B_1^2)\sqrt{t}+B_1)^2}\,.
\end{gathered}$$

*Proof.* Since Lemma 2.1 implies

$$de^{\eta_n(t)} = a_{1,n}(t)e^{\eta_n(t)}dt + b_{1,n}(t)e^{\eta_n(t)}dw(t)\,,$$

we easily have

$$\begin{gathered}
e^{\eta_n(t)}-1 = \int_0^t a_{1,n}(s)e^{\eta_n(s)}ds + \int_0^t b_{1,n}(s)e^{\eta_n(s)}dw(s)\,,\\
\big|\,a_{1,n}(s)e^{\eta_n(s)}\big| \le A_1 + A_1|e^{\eta_n(s)}-1|\,,\\
\big|b_{1,n}(s)e^{\eta_n(s)}\big| \le B_1 + B_1|e^{\eta_n(s)}-1|\,.
\end{gathered}$$

Hence, the stochastic Gronwall inequality [1] shows one of the desired estimates.

Similarly, by Lemma 2.1, we obtain

$$e^{-\eta_n(t)}-1 = -\int_0^t \big(a_{1,n}(s)-(b_{1,n}(s))^2\big)e^{-\eta_n(s)}ds - \int_0^t b_{1,n}(s)e^{-\eta_n(s)}dw(s)$$

and a simple calculation gives

$$\begin{gathered}\big|\big(a_{1,n}(s)-(b_{1,n}(s))^2\big)e^{-\eta_n(s)}\big| \le (A_1+B_1^2)+(A_1+B_1^2)|e^{-\eta_n(s)}-1|\,,\\ \big|b_{1,n}(s)e^{-\eta_n(s)}\big| \le B_1+B_1|e^{-\eta_n(s)}-1|\,.\end{gathered}$$

Therefore, by the stochastic Gronwall inequality [1], we obtain the remaining inequality. □

**Remark 2.7.** By Fubini's theorem and Lemma 2.6, we can show that

$$e^{\pm\eta_n(t)}=(e^{\pm\eta_n(t)}-1)+1\in M_w^2[0,T]$$

and

$$\begin{gathered}E\Big[\int_0^T e^{2\eta_n(t)}dt\Big]\le 2\int_0^T 4t(A_1\sqrt{t}+B_1)^2e^{4t(A_1\sqrt{t}+B_1)^2}dt+2T\,,\\ E\Big[\int_0^T e^{-2\eta_n(t)}dt\Big]\le 2\int_0^T 4t\big((A_1+B_1^2)\sqrt{t}+B_1\big)^2e^{4t((A_1+B_1^2)\sqrt{t}+B_1)^2}dt+2T\end{gathered}$$

for any $T>0$.

Martingale inequalities (see, for example, [4]) play important roles in the proof of our error estimate.

**Lemma 2.8.** *If $f(t)\in M_w^2[0,T]$, $T>0$, then*

$$P\Big[\sup_{0\le t\le T}\Big|\int_0^t f(s)\,dw(s)\Big|>\alpha\Big]\le\frac{1}{\alpha^2}\,E\Big[\int_0^T f^2(s)\,ds\Big]$$

*for any positive number $\alpha$.*

**Lemma 2.9.** *If $f(t)\in L_w^2[0,T]$, $T>0$, then*

$$P\Big[\sup_{0\le t\le T}\Big(\int_0^t f(s)\,dw(s)-\frac{\alpha}{2}\int_0^t f^2(s)\,ds\Big)>\beta\Big]\le e^{-\alpha\beta}$$

*for any positive numbers $\alpha$ and $\beta$.*

**Remark 2.10.** Since $b_{1,n}(t)\in L_w^2[0,T]$,

$$\begin{aligned}\eta_n(t)&=\int_0^t a_{1,n}(s)\,ds+\int_0^t b_{1,n}(s)\,dw(s)-\frac12\int_0^t\big(b_{1,n}(s)\big)^2ds\\ &\le A_1t+\int_0^t b_{1,n}(s)\,dw(s)-\frac12\int_0^t\big(b_{1,n}(s)\big)^2ds\end{aligned}$$

and

$$\begin{aligned}-\eta_n(t)&=\int_0^t(-a_{1,n}(s))\,ds+\int_0^t\big(b_{1,n}(s)\big)^2ds\\ &\quad+\int_0^t\big(-b_{1,n}(s)\big)\,dw(s)-\frac12\int_0^t\big(-b_{1,n}(s)\big)^2ds\\ &\le(A_1+B_1^2)\,t+\int_0^t\big(-b_{1,n}(s)\big)\,dw(s)-\frac12\int_0^t\big(-b_{1,n}(s)\big)^2ds\,,\end{aligned}$$

it follows from Lemma 2.9 that

$$\begin{aligned}
&P\big[\sup_{0\le s\le t} e^{\eta_n(s)} > R\big]\\
&\le P\Big[\sup_{0\le s\le t}\Big(\int_0^s b_{1,n}(u)\,dw(u) - \frac{1}{2}\int_0^s \big(b_{1,n}(u)\big)^2 du\Big) > -A_1 t + \log R\Big]\\
&\le e^{A_1 t} R^{-1}
\end{aligned}$$

and

$$\begin{aligned}
&P\big[\sup_{0\le s\le t} e^{-\eta_n(s)} > R\big]\\
&\le P\Big[\sup_{0\le s\le t}\Big(\int_0^s \big(-b_{1,n}(u)\big)\,dw(u) - \frac{1}{2}\int_0^s \big(-b_{1,n}(u)\big)^2 du\Big) > -(A_1+B_1^2)\,t + \log R\Big]\\
&\le e^{(A_1+B_1^2)\,t}\,R^{-1}
\end{aligned}$$

for all $R \ge 1$ and $0 \le t \le T$.

## 3. Proof of Theorem 1.1

*Proof.* Let us take real numbers $R \ge 1$ and $0 < \rho \le 1$ arbitrarily. Then, by Lemma 2.5, we can show that

$$\sup_{0\le t\le T} |\varepsilon_{n+1}(t)| > R\rho^2 \quad\text{and}\quad \sup_{0\le t\le T} |\varepsilon_n(t)| \le \rho$$

imply

$$\sup_{0\le t\le T} e^{\eta_n(t)} > \sqrt{R}$$

or

$$\sup_{0\le t\le T}\int_0^t \big|\alpha_{0,n}(s) - \beta_{0,n}(s)b_{1,n}(s)\big| e^{-\eta_n(s)} ds > \frac{\sqrt{R}}{2}\rho^2$$

or

$$\sup_{0\le t\le T}\Big|\int_0^t \beta_{0,n}(s)e^{-\eta_n(s)}dw(s)\Big| > \frac{\sqrt{R}}{2}\rho^2$$

for every $n = 0, 1, 2, \dots$. In fact, we have to use only an argument of contradiction.

By Remark 2.10, we easily have

$$P\big[\sup_{0\le t\le T} e^{\eta_n(t)} > \sqrt{R}\,\big] \le e^{TA_1} R^{-1/2}\,.$$

By

$$\big|\,\alpha_{0,n}(s) - \beta_{0,n}(s)b_{1,n}(s)\,\big| \le \frac{1}{2}(A_2 + B_1B_2)\varepsilon_n^2(s)\,,$$

Remark 2.10 and direct computation, it follows that

$$\begin{aligned}
&P\Big[\sup_{0\le t\le T}\int_0^t \big|\alpha_{0,n}(s) - \beta_{0,n}(s)b_{1,n}(s)\big| e^{-\eta_n(s)} ds > \frac{\sqrt{R}}{2}\rho^2 \text{ and } \sup_{0\le t\le T}|\varepsilon_n(t)| \le \rho\Big]\\
&\le T(A_2 + B_1B_2)e^{T(A_1+B_1^2)}\,R^{-1/2}\,.
\end{aligned}$$

Since

$$\beta_{0,n}(s) = \min\big(\varepsilon_n^2(s), \rho^2\big)\int_0^1 (\theta - 1)\,\frac{\partial^2 b}{\partial x^2}\big(s, \xi_n(s) - \theta\varepsilon_n(s)\big)\,d\theta$$

for $0 \le s \le T$ when $\sup_{0\le t\le T} |\varepsilon_n(t)| \le \rho$, Lemma 2.8 and Remark 2.7 show

$$\begin{aligned} &P\Big[\sup_{0\le t\le T}\Big|\int_0^t \beta_{0,n}(s)e^{-\eta_n(s)}dw(s)\Big| > \frac{\sqrt{R}}{2}\rho^2 \text{ and } \sup_{0\le t\le T}|\varepsilon_n(t)|\le\rho\Big]\\ &\le P\Big[\sup_{0\le t\le T}\Big|\int_0^t\Big(\min\big(\varepsilon_n^2(s),\rho^2\big)\int_0^1(\theta-1)\,\frac{\partial^2 b}{\partial x^2}\big(s,\xi_n(s)-\theta\varepsilon_n(s)\big)\,d\theta\Big)\\ &\quad\times e^{-\eta_n(s)}dw(s)\Big| > \frac{\sqrt{R}}{2}\rho^2\Big]\\ &\le 2B_2^2\Big(\int_0^T 4t\big((A_1+B_1^2)\sqrt{t}+B_1\big)^2 e^{4t((A_1+B_1^2)\sqrt{t}+B_1)^2}dt+T\Big)R^{-1}\,. \end{aligned}$$

Combining the above estimates, we can show that there exists a nonnegative constant $C = C(A_1, A_2, B_1, B_2, T)$ independent of $R$, $\rho$ and $n$ such that

$$P\Big[\sup_{0\le t\le T}|\varepsilon_{n+1}(t)| > R\rho^2 \text{ and } \sup_{0\le t\le T}|\varepsilon_n(t)|\le\rho\Big] \le C\,R^{-1/2}$$

for $n = 0, 1, 2, \dots$. Consequently, we have

$$P\Big[\sup_{0\le t\le T}|\varepsilon_n(t)|\le\rho \text{ implies } \sup_{0\le t\le T}|\varepsilon_{n+1}(t)|\le R\rho^2\Big] \ge 1 - C\,R^{-1/2}$$

for all $R \ge 1$, $0 < \rho \le 1$ and every $n = 0, 1, 2, \dots$. □

Finally, we give a slight improvement of Theorem 1.1. At the beginning of the proof of Theorem 1.1 where we have made a classification of an event, if we replace the lower bounds

$$\sqrt{R}\,, \quad \frac{\sqrt{R}}{2}\rho^2\,, \quad \frac{\sqrt{R}}{2}\rho^2$$

with

$$T^{-1/3}\sqrt{R}\,, \quad \frac{T^{1/3}\sqrt{R}}{2}\rho^2\,, \quad \frac{T^{1/3}\sqrt{R}}{2}\rho^2$$

respectively, then we can show that the above constant $C(A_1, A_2, B_1, B_2, T) = O(T^{1/3})$ as $T \to 0$. Therefore, our Newton's method may work better when it is used with a small time interval.

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