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# PICONE-TYPE INEQUALITY AND STURMIAN COMPARISON THEOREMS FOR QUASILINEAR ELLIPTIC OPERATORS WITH $p(x)$-LAPLACIANS

NORIO YOSHIDA

Abstract. A Picone-type inequality for quasilinear elliptic operators with mixed nonlinearities and with $p(x)$-Laplacian is established, and Sturmian comparison theorems are derived on the basis of the Picone-type inequality. Generalizations to more general quasilinear elliptic operators with $p(x)$-Laplacians and specializations to quasilinear ordinary differential operators with $p(t)$-Laplacians are shown.

## 1. Introduction

Recently Picone identities and Sturmian comparison theorems for $p$-Laplacian equations have been developed; see for example, Allegretto [1], Allegretto and Huang [3, 4], Bognár and Došlý [5], Došlý and Řehák [6], Dunninger [7], Kusano, Jaroš and Yoshida [10], Yoshida [12, 13]. Picone identities or inequalities play an important role in establishing Sturmian comparison theorems and oscillation results.

Much current interest has been focused on various mathematical problems with variable exponent growth condition (see [8]). The study of such problems arise from nonlinear elasticity theory, electrorheological fluids (cf. [11, 17]).

The operator $\nabla \cdot \big(|\nabla u|^{p(x)-2}\nabla u\big)$ is said to be $p(x)$-Laplacian ($p(x) > 1$), and becomes $p$-Laplacian $\nabla \cdot \big(|\nabla u|^{p-2}\nabla u\big)$ if $p(x) = p$ (constant), where the central dot denotes the scalar product, $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ and $|x|$ denotes the Euclidean length of $x \in \mathbb{R}^n$.

Since the pioneering work of Zhang [16], there has been an increasing interest in studying oscillation problems for $p(x)$-Laplacian equations (cf. Yoshida [14, 15]). The $p(t)$-Laplacian ($p(t) > 1$) equation

$$\big(|u'|^{p(t)-2}u'\big)' + t^{-\theta(t)}g(t,u) = 0, \quad t > 0$$

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was treated by Zhang [16], and the half-linear elliptic inequality with $p(x)$-Laplacian ($p(x) = \alpha(x) + 1$, $\alpha(x) > 0$)

$$vQ[v] \le 0$$

was investigated by Yoshida [14] via Riccati method, where

$$\begin{aligned} Q[v] &:= \nabla \cdot \big(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\big) - A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v \\ &\quad + |\nabla v|^{\alpha(x)-1}B(x)\cdot\nabla v + C(x)|v|^{\alpha(x)-1}v. \end{aligned}$$

We note that $vQ[v] \le 0$ is *half-linear* in the sense that any constant multiple of a solution $v$ of $vQ[v] \le 0$ is also a solution of $vQ[v] \le 0$. In fact, it can be shown that

$$(kv)Q[kv] = |k|^{\alpha(x)+1}vQ[v] \quad (k \in \mathbb{R})$$

(cf. Yoshida [14, Proposition 2.1]). For Sturmian comparison theorems for half-linear elliptic inequalities with $p(x)$-Laplacians we refer to Yoshida [15]. We mention, in particular, the paper [2] by Allegretto in which Picone identity arguments are used.

The objective of this paper is to establish Picone-type inequalities for the half-linear elliptic operator $q$ defined by

$$\begin{aligned} q[u] &:= \nabla \cdot \big(a(x)|\nabla u|^{\alpha(x)-1}\nabla u\big) - a(x)(\log|u|)|\nabla u|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla u \\ &\quad + |\nabla u|^{\alpha(x)-1}b(x)\cdot\nabla u + c(x)|u|^{\alpha(x)-1}u, \end{aligned}$$

and the quasilinear elliptic operator $\tilde{Q}$ defined by

$$\begin{aligned} \tilde{Q}[v] &:= \nabla \cdot \big(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\big) - A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v \\ &\quad + |\nabla v|^{\alpha(x)-1}B(x)\cdot\nabla v \\ &\quad + C(x)|v|^{\alpha(x)-1}v + D(x)|v|^{\beta(x)-1}v + E(x)|v|^{\gamma(x)-1}v, \end{aligned} \tag{1.1}$$

and derive Sturmian comparison theorems for $q$ and $\tilde{Q}$ by using the Picone-type inequality obtained. We remark that $\log|v|$ in (1.1) has singularities at zeros of $v$, but $v\log|v|$ becomes continuous at the zeros of $v$ if we define $v\log|v| = 0$ at the zeros, in light of the fact that $\lim_{\varepsilon\to+0}\varepsilon\log\varepsilon = 0$. Therefore, we conclude that $v\tilde{Q}[v]$ has no singularities and is continuous in $\Omega$. We give the same remarks about $q$. We note that $v\tilde{Q}[v] \le 0$ is not half-linear.

In Section 2 Picone-type inequalities are established for quasilinear elliptic operators with $p(x)$-Laplacians. In Section 3 we obtain Sturmian comparison theorems by utilizing the Picone-type inequality in Section 2. In Section 4 we present extensions to more general quasilinear elliptic operators with $p(x)$-Laplacians, and specializations to quasilinear ordinary differential operators with $p(t)$-Laplacians.

## 2. Picone-type inequality

In this section we establish Picone-type inequalities when $Q[v]$ has the super-half-linear term $D(x)|v|^{\beta(x)-1}v$ and the sub-half-linear term $E(x)|v|^{\gamma(x)-1}v$, where $\beta(x) > \alpha(x) > \gamma(x) > 0$.

Let $G$ be a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary $\partial G$. It is assumed that $a(x), A(x) \in C(\overline{G}; (0,\infty))$, $b(x), B(x) \in C(\overline{G}; \mathbb{R}^n)$, $c(x), C(x) \in C(\overline{G}; \mathbb{R})$, $D(x), E(x) \in C(\overline{G}; [0,\infty))$, $\alpha(x) \in C^1(\overline{G}; (0,\infty))$ and $\beta(x), \gamma(x)$ belong to $C(\overline{G}; (0,\infty))$.

The domain $\mathcal{D}_q(G)$ of $q$ is defined as the set of all functions $u$ of class $C^1(\overline{G};\mathbb{R})$ such that $a(x)|\nabla u|^{\alpha(x)-1}\nabla u \in C^1(G;\mathbb{R}^n)\cap C(\overline{G};\mathbb{R}^n)$. The domain $\mathcal{D}_{\tilde{Q}}(G)$ of $\tilde{Q}$ is defined similarly.

**Theorem 2.1** (Picone-type inequality for $\tilde{Q}$)**.** *If $v \in \mathcal{D}_{\tilde{Q}}(G)$ and $v$ has no zero in $G$, then we obtain the following Picone-type inequality for any $u \in C^1(G;\mathbb{R})$ which has no zeros in $G$:*

$$\begin{aligned}
&-\nabla\cdot\Big(u\varphi(u)\frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{\varphi(v)}\Big)\\
&\geq -A(x)\Big|\nabla u+\frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x)-\frac{u}{(\alpha(x)+1)A(x)}B(x)\Big|^{\alpha(x)+1}\\
&\quad+\big(C(x)+\tilde{C}(x)\big)|u|^{\alpha(x)+1}\\
&\quad+A(x)\Big[\Big|\nabla u+\frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x)-\frac{u}{(\alpha(x)+1)A(x)}B(x)\Big|^{\alpha(x)+1}\\
&\quad+\alpha(x)|\frac{u}{v}\nabla v|^{\alpha(x)+1}-(\alpha(x)+1)|\frac{u}{v}\nabla v|^{\alpha(x)-1}\Big(\nabla u+\frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x)\\
&\quad-\frac{u}{(\alpha(x)+1)A(x)}B(x)\Big)\cdot\big(\frac{u}{v}\nabla v\big)\Big]-\frac{|u|^{\alpha(x)+1}}{|v|^{\alpha(x)+1}}\big(v\tilde{Q}[v]\big)\quad \text{in } G,
\end{aligned}\tag{2.1}$$

*where $\varphi(u)=|u|^{\alpha(x)-1}u$ and*

$$\tilde{C}(x)=\Big(\frac{\beta(x)-\gamma(x)}{\alpha(x)-\gamma(x)}\Big)\Big(\frac{\beta(x)-\alpha(x)}{\alpha(x)-\gamma(x)}\Big)^{\frac{\alpha(x)-\beta(x)}{\beta(x)-\gamma(x)}}D(x)^{\frac{\alpha(x)-\gamma(x)}{\beta(x)-\gamma(x)}}E(x)^{\frac{\beta(x)-\alpha(x)}{\beta(x)-\gamma(x)}}.$$

*Proof.* The following relations are obtained by Yoshida [15, proof of Theorem 2.1]:

$$\begin{aligned}
-\nabla\cdot\Big(u\varphi(u)\frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{\varphi(v)}\Big)&=-\nabla(u\varphi(u))\cdot\frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{\varphi(v)}\\
&\quad-u\varphi(u)A(x)|\nabla v|^{\alpha(x)-1}\nabla\big(\frac{1}{\varphi(v)}\big)\cdot\nabla v\\
&\quad-\frac{u\varphi(u)}{\varphi(v)}\nabla\cdot\big(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\big),
\end{aligned}\tag{2.2}$$

$$\begin{aligned}
&\nabla(u\varphi(u))\cdot\frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{\varphi(v)}\\
&=(\alpha(x)+1)A(x)|\frac{u}{v}\nabla v|^{\alpha(x)-1}(\nabla u)\cdot\Big(\frac{u}{v}\nabla v\Big)\\
&\quad+A(x)u(\log|u|)\frac{\varphi(u)}{\varphi(v)}|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v
\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}
&u\varphi(u)A(x)|\nabla v|^{\alpha(x)-1}\nabla\big(\frac{1}{\varphi(v)}\big)\cdot\nabla v\\
&=-A(x)\alpha(x)|\frac{u}{v}\nabla v|^{\alpha(x)+1}-\frac{u\varphi(u)}{\varphi(v)}A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v.
\end{aligned}\tag{2.4}$$

From (1.1) it follows that

$$\begin{aligned}
&\frac{u\varphi(u)}{\varphi(v)}\nabla\cdot\big(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\big)\\
&=\frac{u\varphi(u)}{\varphi(v)}\Big(\tilde{Q}[v]+A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v\\
&-|\nabla v|^{\alpha(x)-1}B(x)\cdot\nabla v-C(x)|v|^{\alpha(x)-1}v-D(x)|v|^{\beta(x)-1}v-E(x)|v|^{\gamma(x)-1}v\Big)\\
&=\frac{u\varphi(u)}{\varphi(v)}\tilde{Q}[v]+\frac{u\varphi(u)}{\varphi(v)}A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v\\
&\quad-\frac{u\varphi(u)}{\varphi(v)}|\nabla v|^{\alpha(x)-1}B(x)\cdot\nabla v\\
&\quad-C(x)|u|^{\alpha(x)+1}-|u|^{\alpha(x)+1}\Big(D(x)|v|^{\beta(x)-\alpha(x)}+\frac{E(x)}{|v|^{\alpha(x)-\gamma(x)}}\Big).
\end{aligned}\tag{2.5}$$

We remark that for any fixed $x\in G$, Young's inequality

$$ab\le\frac{a^{p(x)}}{p(x)}+\frac{b^{q(x)}}{q(x)}\quad\Big(p(x)>1,\ \frac{1}{p(x)}+\frac{1}{q(x)}=1\Big)$$

holds for any $a\ge0$, $b\ge0$. Hence, the following inequality holds:

$$D(x)|v|^{\beta(x)-\alpha(x)}+\frac{E(x)}{|v|^{\alpha(x)-\gamma(x)}}\ge\tilde{C}(x)\tag{2.6}$$

(cf. Jaroš, Kusano and Yoshida [9, p.717]). Combining (2.5) with (2.6) yields the inequality

$$\begin{aligned}
&\frac{u\varphi(u)}{\varphi(v)}\nabla\cdot\big(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\big)\\
&\le\frac{u\varphi(u)}{\varphi(v)}\tilde{Q}[v]+\frac{u\varphi(u)}{\varphi(v)}A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v\\
&\quad-\frac{u\varphi(u)}{\varphi(v)}|\nabla v|^{\alpha(x)-1}B(x)\cdot\nabla v-\big(C(x)+\tilde{C}(x)\big)|u|^{\alpha(x)+1}.
\end{aligned}\tag{2.7}$$

Combining (2.2)–(2.4) and (2.7), we derive

$$\begin{aligned}
&-\nabla\cdot\Big(u\varphi(u)\frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{\varphi(v)}\Big)\\
&\ge\big(C(x)+\tilde{C}(x)\big)|u|^{\alpha(x)+1}+A(x)\Big[\alpha(x)|\frac{u}{v}\nabla v|^{\alpha(x)+1}\\
&-(\alpha(x)+1)|\frac{u}{v}\nabla v|^{\alpha(x)-1}\Big(\nabla u+\frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x)\\
&\quad-\frac{u}{(\alpha(x)+1)A(x)}B(x)\Big)\cdot\big(\frac{u}{v}\nabla v\big)\Big]-\frac{u\varphi(u)}{v\varphi(v)}\big(v\tilde{Q}[v]\big),
\end{aligned}$$

which is equivalent to (2.1). $\square$

**Theorem 2.2** (Picone-type inequality for $q$ and $\tilde{Q}$)**.** *Assume that $\alpha(x)$ belongs to $C^2(G;(0,\infty))$, $b(x)/a(x)\in C^1(G;\mathbb{R}^n)$, and that $u\in C^1(G;\mathbb{R})$, $u$ has no zero in $G$, and the following hypothesis holds:*

(H1) *there is a function* $f \in C(\overline{G};\mathbb{R})$ *such that* $f \in C^1(G;\mathbb{R})$ *and*

$$\nabla f = \frac{\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{b(x)}{(\alpha(x)+1)a(x)} \quad \text{in } G.$$

*If* $e^f u \in \mathcal{D}_q(G)$, $v \in \mathcal{D}_Q(G)$ *and* $v$ *has no zero in* $G$, *then we obtain the Picone-type inequality:*

$$\begin{aligned}
&\nabla\cdot\Big(e^{-(\alpha(x)+1)f}(e^f u)a(x)|\nabla(e^f u)|^{\alpha(x)-1}\nabla(e^f u) - \frac{u\varphi(u)}{\varphi(v)}A(x)|\nabla v|^{\alpha(x)-1}\nabla v\Big)\\
&\geq a(x)\Big|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)a(x)}b(x)\Big|^{\alpha(x)+1}\\
&\quad - A(x)\Big|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)}B(x)\Big|^{\alpha(x)+1}\\
&\quad + \big(C(x)+\tilde{C}(x)-c(x)\big)|u|^{\alpha(x)+1}\\
&\quad + A(x)\Big[\Big|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)}B(x)\Big|^{\alpha(x)+1}\\
&\quad + \alpha(x)\Big|\frac{u}{v}\nabla v\Big|^{\alpha(x)+1}\\
&\quad - (\alpha(x)+1)\Big|\frac{u}{v}\nabla v\Big|^{\alpha(x)-1}\Big(\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x)\\
&\quad - \frac{u}{(\alpha(x)+1)A(x)}B(x)\Big)\cdot\big(\frac{u}{v}\nabla v\big)\Big]\\
&\quad + e^{-(\alpha(x)+1)f}(e^f u)q[e^f u] - \frac{|u|^{\alpha(x)+1}}{|v|^{\alpha(x)+1}}\big(v\tilde{Q}[v]\big) \quad \text{in } G.
\end{aligned} \tag{2.8}$$

*Proof.* The following identity holds:

$$\begin{aligned}
&\nabla\cdot\Big(e^{-(\alpha(x)+1)f}(e^f u)a(x)|\nabla(e^f u)|^{\alpha(x)-1}\nabla(e^f u)\Big)\\
&= a(x)\big|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)a(x)}b(x)\big|^{\alpha(x)+1}\\
&\quad - c(x)|u|^{\alpha(x)+1} + e^{-(\alpha(x)+1)f}(e^f u)q[e^f u]
\end{aligned} \tag{2.9}$$

(see Yoshida [15, proof of Theorem 2.2]). The Picone-type inequality (2.8) follows by combining (2.1) with (2.9). ☐

## 3. Sturmian comparison theorems

On the basis of the Picone-type inequality (2.8), we can establish Sturmian comparison theorems for $q$ and $\tilde{Q}$.

**Lemma 3.1.** *The inequality*

$$|\xi|^{\alpha(x)+1} + \alpha(x)\,|\eta|^{\alpha(x)+1} - (\alpha(x)+1)|\eta|^{\alpha(x)-1}\xi\cdot\eta \geq 0$$

*is valid for* $x \in G$, $\xi,\eta \in \mathbb{R}^n$, *where the equality holds if and only if* $\xi = \eta$.

**Theorem 3.2** (Sturmian comparison theorem)**.** *Let* $\alpha(x) \in C^2(G;(0,\infty))$ *and* $b(x)/a(x), B(x)/A(x) \in C^1(G;\mathbb{R}^n)$. *Assume that there exists* $u \in C^1(\overline{G};\mathbb{R})$ *such that* $u = 0$ *on* $\partial G$, $u$ *has no zero in* $G$, *the hypothesis* (H1) *of Theorem 2.2 holds and that*

(H2) *there is a function* $F \in C(\overline{G};\mathbb{R})$ *such that* $F \in C^1(G;\mathbb{R})$ *and*

$$\nabla F = \frac{\log |u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{B(x)}{(\alpha(x)+1)A(x)} \quad \text{in } G.$$

*If the following conditions are satisfied*

(i) $e^f u \in \mathcal{D}_q(G)$ *and* $(e^f u)q[e^f u] \ge 0$ *in* $G$*;*

(ii)

$$\begin{aligned} V_G[u] := \int_G \Big[ & a(x)\Big|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)a(x)}b(x)\Big|^{\alpha(x)+1} \\ & - A(x)\Big|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)}B(x)\Big|^{\alpha(x)+1} \\ & + \big(C(x) + \tilde{C}(x) - c(x)\big)|u|^{\alpha(x)+1}\Big]\,dx \ge 0, \end{aligned}$$

*then every solution* $v \in \mathcal{D}_{\tilde{Q}}(G)$ *of* $v\tilde{Q}[v] \le 0$ *must vanish at some point of* $\overline{G}$*.*

*Proof.* Suppose to the contrary that there exists a solution $v \in \mathcal{D}_{\tilde{Q}}(G)$ of $v\tilde{Q}[v] \le 0$ such that $v$ has no zero on $\overline{G}$. Integrating the Picone-type inequality (2.8) over $G$ and using the divergence theorem, we obtain

$$0 \ge V_G[u] + \int_G W(u,v)\,dx \ge 0,$$

from which we observe

$$\int_G W(u,v)\,dx = 0,$$

where

$$\begin{aligned} W(u,v) := A(x)\Big[&\Big|\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)}B(x)\Big|^{\alpha(x)+1} \\ & + \alpha(x)\Big|\frac{u}{v}\nabla v\Big|^{\alpha(x)+1} - (\alpha(x)+1)\Big|\frac{u}{v}\nabla v\Big|^{\alpha(x)-1}\Big(\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) \\ & - \frac{u}{(\alpha(x)+1)A(x)}B(x)\Big)\cdot\big(\frac{u}{v}\nabla v\big)\Big]. \end{aligned}$$

From Lemma 3.1 we see that

$$\nabla u + \frac{u\log|u|}{\alpha(x)+1}\nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)}B(x) \equiv \frac{u}{v}\nabla v \quad \text{in } G;$$

that is,

$$\nabla u + u\nabla F \equiv \frac{u}{v}\nabla v \quad \text{in } G,$$

which is equivalent to

$$e^{-F}v\nabla\big(e^F\frac{u}{v}\big) \equiv 0 \quad \text{in } G.$$

Therefore, there exists a constant $k_0$ such that $e^F u/v = k_0$ in $G$ and hence on $\overline{G}$ by continuity. Since $u = 0$ on $\partial G$, we see that $k_0 = 0$, which contradicts the hypothesis that $u$ is nontrivial. The proof is complete. ☐

**Corollary 3.3.** *Let* $\alpha(x) \in C^2(G;(0,\infty))$*,* $b(x)/a(x), B(x)/A(x) \in C^1(G;\mathbb{R}^n)$*. Assume that*

(i) $\dfrac{b(x)}{a(x)} = \dfrac{B(x)}{A(x)}$ *in* $G$*;*

(ii) *$a(x) \ge A(x)$, $C(x) + \tilde{C}(x) \ge c(x)$ in $G$.*

*If there exists a function $u \in C^1(\overline{G};\mathbb{R})$ with the properties that $u = 0$ on $\partial G$, $u$ has no zero in $G$, the hypothesis* (H1) *of Theorem 2.2 holds and* (i) *of Theorem* 3.2 *holds, then every solution $v \in \mathcal{D}_{\tilde{Q}}(G)$ of $v\tilde{Q}[v] \le 0$ must vanish at some point of $\overline{G}$.*

*Proof.* Conditions (i), (ii) imply that $V_G[u] \ge 0$ for any $u \in C^1(\overline{G};\mathbb{R})$ and (H2) is the same as (H1). The conclusion follows from Theorem 3.2. □

**Remark 3.4.** When specialized to the case where $D(x) = E(x) \equiv 0$, our results reduce to those of Yoshida [15, Theorem 3.1 and Corollary 3.1].

## 4. Generalizations and specializations

First we derive extensions to more general quasilinear elliptic operators with $p(x)$-Laplacians. Let the quasilinear elliptic operator $\hat{Q}$ be defined by

$$\begin{aligned}
\hat{Q}[v] := &\nabla \cdot \big(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\big) - A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v \\
&+ |\nabla v|^{\alpha(x)-1}B(x)\cdot\nabla v + C(x)|v|^{\alpha(x)-1}v \\
&+ \sum_{i=1}^{\ell} D_i(x)|v|^{\beta_i(x)-1}v + \sum_{j=1}^{m} E_j(x)|v|^{\gamma_j(x)-1}v,
\end{aligned}$$

where $\beta_i(x) > \alpha(x) > \gamma_j(x) > 0$, and $D_i(x), E_j(x) \in C(\overline{G};[0,\infty))$ $(i = 1,2,\dots,\ell;$ $j = 1,2,\dots,m)$. The domain $\mathcal{D}_{\hat{Q}}(G)$ of $\hat{Q}$ is defined as the same as $\mathcal{D}_{\tilde{Q}}(G)$. Let $N = \min\{\ell, m\}$ and we define

$$\hat{C}(x) = \sum_{i=1}^{N} H(\beta_i(x),\alpha(x),\gamma_i(x);D_i(x),E_i(x)),$$

where

$$\begin{aligned}
&H(\beta(x),\alpha(x),\gamma(x);D(x),E(x)) \\
&= \Big(\frac{\beta(x)-\gamma(x)}{\alpha(x)-\gamma(x)}\Big)\Big(\frac{\beta(x)-\alpha(x)}{\alpha(x)-\gamma(x)}\Big)^{\frac{\alpha(x)-\beta(x)}{\beta(x)-\gamma(x)}} D(x)^{\frac{\alpha(x)-\gamma(x)}{\beta(x)-\gamma(x)}} E(x)^{\frac{\beta(x)-\alpha(x)}{\beta(x)-\gamma(x)}}.
\end{aligned}$$

Applying Young's inequality, we obtain

$$\begin{aligned}
&\sum_{i=1}^{\ell} D_i(x)|v|^{\beta_i(x)-\alpha(x)} + \sum_{j=1}^{m} E_j(x)|v|^{\gamma_j(x)-\alpha(x)} \\
&\ge \sum_{i=1}^{N}\Big(D_i(x)|v|^{\beta_i(x)-\alpha(x)} + \frac{E_i(x)}{|v|^{\alpha(x)-\gamma_i(x)}}\Big) \\
&\ge \sum_{i=1}^{N} H(\beta_i(x),\alpha(x),\gamma_i(x);D_i(x),E_i(x)) = \hat{C}(x).
\end{aligned}$$

In the proof of Theorem 2.1, we use the above inequality instead of (2.6), and observe that the Picone-type inequality (2.1) holds for $\tilde{Q}[v]$ and $\tilde{C}(x)$ replaced by $\hat{Q}[v]$ and $\hat{C}(x)$, respectively. Therefore we conclude that Theorems 2.1–3.2, Corollary 3.3 remain true if we replace $\tilde{Q}[v]$ and $\tilde{C}(x)$ by $\hat{Q}[v]$ and $\hat{C}(x)$, respectively.

For example, we state the analogue of Corollary 3.3.

**Corollary 4.1.** *Let $\alpha(x) \in C^2(G;(0,\infty))$, $b(x)/a(x), B(x)/A(x) \in C^1(G;\mathbb{R}^n)$. Assume that*

(i) $\dfrac{b(x)}{a(x)} = \dfrac{B(x)}{A(x)}$ *in $G$;*

(ii) $a(x) \geq A(x)$, $C(x) + \hat{C}(x) \geq c(x)$ *in $G$.*

*If there exists a function $u \in C^1(\overline{G};\mathbb{R})$ with the properties that $u = 0$ on $\partial G$, $u$ has no zero in $G$, the hypothesis* (H1) *of Theorem 2.2 holds and* (i) *of Theorem 3.2 holds, then every solution $v \in \mathcal{D}_{\hat{Q}}(G)$ of $v\hat{Q}[v] \leq 0$ must vanish at some point of $\overline{G}$.*

Next we consider the special case where $n = 1$, $b(x) = B(x) \equiv 0$. We let $x_1 = t$, $G = (t_1, t_2)$, and define $q_0$ and $Q_0$ by

$$q_0[y] := \big(a(t)|y'|^{\alpha(t)-1}y'\big)' - a(t)(\log|y|)|y'|^{\alpha(t)-1}\alpha'(t)y' + c(t)|y|^{\alpha(t)-1}y, \tag{4.1}$$

$$\begin{aligned} Q_0[z] := {}& \big(A(t)|z'|^{\alpha(t)-1}z'\big)' - A(t)(\log|z|)|z'|^{\alpha(t)-1}\alpha'(t)z' \\ &+ C(t)|z|^{\alpha(t)-1}z + D(t)|z|^{\beta(t)-1}z + E(t)|z|^{\gamma(t)-1}z, \end{aligned} \tag{4.2}$$

where the coefficients appearing in (4.1) and (4.2) are supposed to satisfy the same conditions as in Section 2. The domains $\mathcal{D}_{q_0}(I)$, $\mathcal{D}_{Q_0}(I)$ are defined as in Section 2, where $I = (t_1, t_2)$.

**Theorem 4.2.** *Let $\alpha(t) \in C^2(I;(0,\infty)) \cap C^1(\overline{I};(0,\infty))$. Assume that there exists a function $y \in C^1(\overline{I};\mathbb{R})$ such that $y(t_1) = y(t_2) = 0$, $y$ has no zero in $I$, and the following hypothesis is satisfied:*

(H1') *there is a function $f \in C(\overline{I};\mathbb{R})$ such that $f \in C^1(I;\mathbb{R})$ and*

$$f'(t) = \frac{\log|y|}{\alpha(t)+1}\alpha'(t) \quad \text{in } I.$$

*If $e^f y \in \mathcal{D}_{q_0}(I)$, $(e^f y)q_0[e^f y] \geq 0$ in $I$, and*

$$\begin{aligned} V_I[u] = \int_I \Big[ &\big(a(t) - A(t)\big)\Big|y' + \frac{y\log|y|}{\alpha(t)+1}\alpha'(t)\Big|^{\alpha(t)+1} \\ &+ \big(C(t) + \tilde{C}(t) - c(t)\big)|y|^{\alpha(t)+1}\Big] dt \geq 0, \end{aligned}$$

*then every solution $z \in \mathcal{D}_{Q_0}(I)$ of $zQ_0[z] \leq 0$ must vanish at some point of $\overline{I}$.*

The proof of the above theorem follows from Theorem 3.2.

**Corollary 4.3.** *Let $\alpha(t) \in C^2(I;(0,\infty)) \cap C^1(\overline{I};(0,\infty))$. Assume that there is a function $y \in C^1(\overline{I};\mathbb{R})$ such that $y(t_1) = y(t_2) = 0$, $y$ has no zero in $I$, and the hypothesis* (H1') *of Theorem 4.2 holds. If $e^f y \in \mathcal{D}_{q_0}(I)$, $(e^f y)q_0[e^f y] \geq 0$ in $I$, and*

$$a(t) \geq A(t), \quad C(t) + \tilde{C}(t) \geq c(t) \quad \text{in } I,$$

*then every solution $z \in \mathcal{D}_{Q_0}(I)$ of $zQ_0[z] \leq 0$ must vanish at some point of $\overline{I}$.*

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Norio Yoshida
Department of Mathematics, University of Toyama, Toyama, 930-8555 Japan
*E-mail address*: nori@sci.u-toyama.ac.jp