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MULTIPLE PERIODIC SOLUTIONS FOR IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS WITH FEEDBACK CONTROL

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Abstract. Using the well known Leggett-Williams fixed point theorem, we study the existence of periodic solutions for a class of impulsive functional equations with feedback control. The main results are illustrated with two examples.

1. Introduction

Consider the impulsive functional differential equation with feedback control

$$\begin{gathered} x'(t) = -r(t)x(t) + F(t, x(t-\tau_1(t)), x(t-\tau_2(t)), \dots, x(t-\tau_n(t)), u(t-\zeta(t))); \\ u'(t) = -\eta(t)u(t) + h(t)x(t-\sigma(t)); \\ I_k(x(t_k)) = \Delta x|_{t=t_k}, \quad k = 1, 2, \dots, m, \end{gathered} \tag{1.1}$$

where $\Delta x\big|_{t=t_k} = x(t_k+0) - x(t_k-0)$, $0 \le t_1 < t_2 < \dots < t_m < \omega$, and $F \in C(\mathbb{R}^{n+2}, [0,\infty))$, $r, \eta, h \in C(\mathbb{R}, (0,\infty))$, $\sigma, \zeta, \tau_i \in C(\mathbb{R}, \mathbb{R})$, $i = 1, 2, \dots, n$, $I_k \in C(\mathbb{R}, [0, +\infty))$, $k = 1, 2, \dots, m$. All of the above functions are ω-periodic in t, $\omega > 0$ is a constant.

The existence of periodic solutions of functional differential equations with feed back has been studied extensively by many authors and the methods used are coincidence degree theory, Schauder fixed point theorems, Krasnoselskii's fixed point theorem, and upper-lower solutions method (see [1, 4, 5, 6, 9, 8, 11, 14], and the references therein).

Using the Krasnoselskii's fixed point theorem, Li and Wang [4] investigated the existence of positive periodic solutions of the delay differential system with feedback control

$$\begin{gathered} \frac{dx}{dt} = -b(t)x(t) + F(t, x(t-\tau_1(t)), x(t-\tau_2(t)), \dots, x(t-\tau_n(t)), u(t-\delta(t))); \\ \frac{du}{dt} = -\eta(t)u(t) + a(t)x(t-\sigma(t)). \end{gathered} \tag{1.2}$$

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They obtained the existence of at least one positive ω-periodic solution.

Liu and Li [9] employed Avery-Henderson fixed point theorem to study the existence of positive periodic solutions to the following nonlinear nonautonomous functional differential system with feed back

$$\begin{aligned} \frac{dx}{dt} &= -r(t)x(t) + F(t, x_t, u(t-\delta(t))); \\ \frac{du}{dt} &= -h(t)u(t) + g(t)x(t-\sigma(t)). \end{aligned} \tag{1.3}$$

They obtained the existence of at least two positive solutions under some complicated assumptions such as

(A) $F(t,\phi_t,(\Phi\phi)(t-\delta(t))) > \frac{c}{\lambda_\eta}$ for $\beta c e^{-\int_0^\omega r(s)ds} \le \phi_t(\theta) \le \frac{c}{\beta} e^{-\int_0^\eta r(s)ds}$, $n\beta\omega g_* c e^{-\int_0^\omega r(s)ds} \le (\Phi\phi)(t-\delta(t)) \le \frac{m\omega g^* c}{\beta} e^{-\int_0^\eta r(s)ds}$, $t\in[\eta,\omega]$, $\theta\in\mathbb{R}$;

(B) $F(t,\phi_t,(\Phi\phi)(t-\delta(t))) > \frac{b}{\xi_\eta}$ for $\beta b e^{-\int_0^\eta r(s)ds} \le \phi_t(\theta) \le \frac{b}{\beta} e^{-\int_0^\eta r(s)ds}$, $n\beta\omega g_* c e^{-\int_0^\eta r(s)ds} \le (\Phi\phi)(t-\delta(t)) \le \frac{m\omega g^* b}{\beta} e^{-\int_0^\eta r(s)ds}$, $t\in[0,\omega]$, $\theta\in\mathbb{R}$;

(C) $F(t,\phi_t,(\Phi\phi)(t-\delta(t))) > \frac{a}{\lambda_l}$ for $\beta a e^{-\int_0^\omega r(s)ds} \le \phi_t(\theta) \le \frac{a}{\beta} e^{-\int_0^l r(s)ds}$, $n\beta\omega g_* a e^{-\int_0^\omega r(s)ds} \le (\Phi\phi)(t-\delta(t)) \le \frac{m\omega g^* a}{\beta} e^{-\int_0^l r(s)ds}$, $t\in[l,\omega]$, $\theta\in\mathbb{R}$.

We notice that the above conditions are not applicable since they are too complicated to confirm, and there was no example in [8] to demonstrate their conclusions.

On the other hand, many physical systems undergo abrupt changes at certain moments due to instantaneous perturbations which lead to impulse effects, and a lot of such equations arise in many mathematical models of real processes and phenomena, for example, physics, population dynamics, biotechnology, and economics (see [2, 3, 7, 10, 13], and the references therein). So, in recent years, impulsive differential equations have received a lot of attention. As far as we know, there is no paper to study the existence of triple solutions for impulsive functional equations with feedback control. The goal of present paper is to attempt to fill this gap. And we shall show the impulsive effect plays a crucial role in some cases (see Remark 3.1).

Comparing with (1.2) and (1.3), we note that (1.1) has the impulsive effects. The following are the main features of present paper. First, the result we obtain is the existence of three nonnegative ω-periodic solutions. Second, the method used here is Leggett-Williams fixed point theorem. Furthermore, the assumptions here are easily checked. Finally, two examples illustrate the applications of the main result.

The organization of this paper is as follows. In the next section, some lemmas are presented. In section 3, we state and prove our main result about the existence of triple periodic solutions of (1.1). At last, two examples are given to show the applications of our main result in section 4.

At the end of this section, we state the Leggett-Williams fixed point theorem which will be used in section 3.

Let E be a real Banach space with norm $\|\cdot\|$ and $P \subset E$ be a cone of E, $P_r = \{x \in P : \|x\| < r\}(r > 0)$. Consider a nonnegative continuous and concave functional $\alpha(x)$ defined on P, i.e. $\alpha : P \to \mathbb{R}^+ = [0,\infty)$ is continuous and satisfies $\alpha(tx + (1-t)y) \ge t\alpha(x) + (1-t)\alpha(y)$, for $x, y \in P$ and $t \in [0,\omega]$.

Let

$$P(\alpha; a, b) = \{x \in P : a \le \alpha(x), \|x\| \le b\},$$

where $0 < a < b$. It is not difficult to see that $P(\alpha; a, b)$ is a bounded closed convex subset of P.

Lemma 1.1 (Leggett-Williams Fixed Point Theorem). *Let operator $A : \overline{P_c} \to P_c$ be completely continuous and let α be a nonnegative concave functional on P such that $\alpha(x) \le \|x\|$ for every $x \in \overline{P_c}$. Suppose that there exist $0 < d < a < b \le c$ such that*

- (A1) $\{x : x \in P(\alpha; a, b), \alpha(x) > a\} \neq \emptyset$ *and* $\alpha(Ax) > a$ *for each* $x \in P(\alpha, a, b)$;
- (A2) $\|Ax\| < d$ *for* $x \in \overline{P_d}$;
- (A3) $\alpha(Ax) > a$ *for* $x \in P(\alpha; a, c)$ *with* $\|Ax\| > b$.

Then A has at least three fixed points x_1, x_2, x_3 satisfying $x_1 \in P_d$, $x_2 \in U$, $x_3 \in \overline{P_c} \backslash (P_d \cup U)$, where $U = \{x : x \in P(\alpha; a, c), \alpha(x) > a\}$.

2. Preliminaries

Let $PC(\mathbb{R}) = \{x : x$ is a map from $[0, \omega]$ into $\mathbb{R}$ such that $x(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and the right limit $x(t_k + 0)$ exists for $k = 1, 2, \dots, m\}$.

Evidently, $PC_\omega(\mathbb{R}) = \{x \in PC(\mathbb{R}) : x(t) = x(t + \omega), \forall t \in \mathbb{R}\}$ is a Banach space with the norm $\|x\| = \sup_{t\in[0,\omega]} |x(t)|, \forall x \in PC_\omega(\mathbb{R})$.

Let $P := \{x \in PC_\omega(\mathbb{R}) : x(t) \ge \lambda\|x\|, t \in [0, \omega]\}$, where $\lambda = \exp\{-\int_0^\omega r(\upsilon)d\upsilon\}$. Obviously, P is a cone of Banach space $PC_\omega(\mathbb{R})$.

First, we transform (1.1) into another form. Suppose (x, u) is a solution of (1.1) and x is ω-periodic. By integrating the second equation of (1.1) from t to $t + \omega$, we obtain that

$$u(t) = \int_t^{t+\omega} g(t, s)h(s)x(s - \sigma(s))ds := (\Phi x)(t), \tag{2.1}$$

where

$$g(t, s) = \frac{\exp\{\int_t^s \eta(\xi)d\xi\}}{\exp\{\int_0^\omega \eta(\xi)d\xi\} - 1}.$$

From this, we know

$$\begin{aligned}
u(t + \omega) &= \int_{t+\omega}^{t+2\omega} g(t + \omega, s)h(s)x(s - \sigma(s))ds \\
&= \int_t^{t+\omega} g(t + \omega, \upsilon + \omega)h(\upsilon + \omega)x(\upsilon + \omega - \sigma(\upsilon + \omega))d\nu \\
&= \int_t^{t+\omega} g(t + \omega, \upsilon + \omega)h(\upsilon)x(\upsilon - \sigma(\upsilon))d\nu \\
&= \int_t^{t+\omega} g(t, \upsilon)h(\upsilon)x(\upsilon - \sigma(\upsilon))d\nu \\
&= u(t).
\end{aligned}$$

Therefore, the existence of ω-periodic solution of (1.1) is equivalent to that of the equation

$$\begin{aligned}
\frac{dx}{dt} &= -r(t)x(t) + F(t, x(t - \tau_1(t)), x(t - \tau_2(t)), \dots, x(t - \tau_n(t)), (\Phi x)(t - \zeta(t))) \\
&= -r(t)x(t) + F(t, Ux(t)),
\end{aligned}$$

where

$$Ux(t) = (x(t-\tau_1(t)), x(t-\tau_2(t)), \dots, x(t-\tau_n(t)), (\Phi x)(t-\zeta(t))). \tag{2.2}$$

To solve the above equation, we transform it into

$$(\frac{dx}{dt} + r(t)x(t))e^{\int_0^t r(\nu)d\nu} = (F(t, Ux(t))e^{\int_0^t r(\nu)d\nu};$$

that is,

$$(x(t)e^{\int_0^t r(\nu)d\nu})' = (F(t, Ux(t))e^{\int_0^t r(\nu)d\nu}.$$

So, integrating the above equality from t to $t+\omega$, and noticing that $x(t) = x(t+\omega)$, we have

$$x(t) = \int_t^{t+\omega} G(t,s)F(s, Ux(s))ds + \sum_{j=1}^m G(t, t_{k_j})I_j(x(t_j)),$$

where

$$G(t,s) = \frac{\exp\{\int_t^s r(\nu)d\nu\}}{\exp\{\int_0^\omega r(\nu)d\nu\} - 1}$$

for $(t,s) \in \mathbb{R}^2$ and t_{k_j} satisfies $t_{k_j} = t_j + k\omega$, $t_{k_j} \in [t, t+\omega]$, $I_{k_j}(x(t_{k_j})) = I_j(x(t_j))$, $j = 1, 2, \dots, m$. It is clear that $G(t,s) > 0$ and $G(t,s) = G(t+\omega, s+\omega)$ for all $(t,s) \in \mathbb{R}^2$.

Now we define an operator Ψ on P as

$$\begin{aligned}
&(\Psi x)(t) \\
&= \int_t^{t+\omega} G(t,s)F(s, x(s-\tau_1(s)), \dots, x(s-\tau_n(s)), (\Phi x)(s-\zeta(s)))ds \\
&\quad + \sum_{j=1}^m G(t, t_{k_j})I_j(x(t_j)), \quad \forall x \in P,\ t \in \mathbb{R}.
\end{aligned} \tag{2.3}$$

It is obvious that x is an ω-periodic solution for (1.1) if and only if x is a fixed point of the operator Ψ.

Lemma 2.1. *$\Psi : P \to P$ is well defined.*

Proof. First, it is easy to see $\Psi : PC(\mathbb{R}) \to PC(\mathbb{R})$. Next, since

$$\begin{aligned}
&(\Psi x)(t+\omega) \\
&= \int_{t+\omega}^{t+2\omega} G(t+\omega, s)F(s, Ux(s))ds + \sum_{j=1}^m G(t+\omega, t_{k_j}+\omega)I_j(x(t_j)) \\
&= \int_t^{t+\omega} G(t+\omega, \upsilon+\omega)F(\upsilon+\omega, Ux(\upsilon+\omega))d\upsilon + \sum_{j=1}^m G(t, t_{k_j})I_j(x(t_j)) \\
&= \int_t^{t+\omega} G(t, \upsilon)F(\upsilon, Ux(\upsilon))d\upsilon + \sum_{j=1}^m G(t, t_{k_j})I_j(x(t_j)) = (\Psi x)(t),
\end{aligned}$$

we have $\Psi \in PC_\omega(\mathbb{R})$. Observe that

$$\begin{aligned}
p := \frac{1}{\exp\{\int_0^\omega r(\nu)d\nu\} - 1} &\le G(t,s) = \frac{\exp\{\int_t^s r(\nu)d\nu\}}{\exp\{\int_0^\omega r(\nu)d\nu\} - 1} \\
&\le \frac{\exp\{\int_0^\omega r(\nu)d\nu\}}{\exp\{\int_0^\omega r(\nu)d\nu\} - 1} := q
\end{aligned} \tag{2.4}$$

for all $s \in [t, t+\omega]$. Hence, we obtain that, for $x \in P$,

$$\|\Psi x\| \le q \int_0^\omega F(s, Ux(s))ds + q \sum_{j=1}^m I_j(x(t_j)) \tag{2.5}$$

and

$$(\Psi x)(t) \ge p \int_0^\omega F(s, Ux(s))ds + p \sum_{j=1}^m I_j(x(t_j)) \ge \frac{p}{q}\|\Psi x\| = \lambda \|\Psi x\|.$$

Thus, $\Psi x \in P$. □

Lemma 2.2. $\Psi : P \to P$ *is completely continuous.*

Proof. First we show that Ψ is continuous. According to our assumptions and (2.1), we know that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $x, y \in PC_\omega(\mathbb{R})$ with $\|x - y\| < \delta$,

$$\sup_{0 \le s \le \omega} |F(s, Ux(s)) - F(s, Uy(s))| < \frac{\varepsilon}{2q\omega},$$
$$\max_{0 \le t \le \omega} |I_k(x(t_k)) - I_k(y(t_k))| < \frac{\varepsilon}{2qm}, \quad k = 1, 2, \dots, m,$$

where q is defined in (2.4). So for each $t \in \mathbb{R}$, we have

$$\begin{aligned} &|(\Psi x)(t) - (\Psi y)(t)| \\ &\le q \int_t^{t+\omega} |F(s, Ux(s)) - F(s, Uy(s))|ds + q \sum_{k=1}^m |I_k(x(t_k)) - I_k(y(t_k))| \\ &\le q \frac{\varepsilon}{2q\omega}\omega + q \sum_{k=1}^m \frac{\varepsilon}{2qm} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This yields $\|\Psi x - \Psi y\| < \varepsilon$ when $\|x - y\| < \delta$. Hence Ψ is continuous.

Next, we show that Ψ is compact. Let $S \subset PC_\omega(\mathbb{R})$ be a bounded subset, that is, there exists $M > 0$ such that $\|x\| \le M$ for $\forall x \in S$. Therefore,

$$\begin{aligned} |(\Phi x)(t - \zeta(t))| &= |\int_{t-\zeta(t)}^{t-\zeta(t)+\omega} g(t - \zeta(t), s)h(s)x(s - \sigma(s))ds| \\ &\le M \frac{\exp(\int_0^\omega \eta(\xi)d\xi)}{\exp(\int_0^\omega \eta(\xi)d\xi) - 1} \int_t^{t+\omega} h(s)ds < \infty \end{aligned}$$

and

$$|x(t - \tau_i(t))| \le \|x\| \le M, \ i = 1, 2, \dots, m, \quad \forall t \in \mathbb{R}.$$

From the continuity of F and I_k, there exist $M_1 > 0$ and $M_2 > 0$ such that

$$|F(s, Ux(s))| \le M_1, \quad s \in [0, \omega], \quad \forall x \in S,$$
$$|I_k(x(t_k))| \le M_2, \quad \forall x \in S.$$

Then,

$$\|\Psi x\| \le q \int_0^\omega |F(s, Ux(s))|ds + q \sum_{j=1}^m I_j(x(t_j)) \le q\omega M_1 + qmM_2,$$

which implies $\Psi(S)$ is uniformly bounded. Finally, notice that

$$\begin{aligned}\frac{d}{dt}(\Psi x)(t) &= -r(t)(\Psi x)(t) + G(t,t+\omega)F(t+\omega, Ux(t+\omega)) - G(t,t)F(t,Ux(t))\\ &= -r(t)(\Psi x)(t) + [G(t,t+\omega) - G(t,t)]F(t,Ux(t))\\ &= -r(t)(\Psi x)(t) + F(t,Ux(t)).\end{aligned}$$

This guarantees that, for each $x \in S$, we have

$$\begin{aligned}|\frac{d}{dt}(\Psi x)(t)| &\le \|r(t)\|(q\int_0^\omega |F(s,Ux(s))|ds + q\sum_{j=1}^m I_j(x(t_j))) + |F(t,Ux(t))|\\ &\le \|r(t)\|(q\omega M_1 + qmM_2) + M_1 < \infty.\end{aligned}$$

Consequently, $\Psi(S)$ is equi-continuous. By Ascoli-Arzela theorem, the function Ψ is completely continuous from P to P. $\square$

3. Main Results

For convenience, first let us list the following assumptions.

(H1) $\lim_{|v|\to+\infty}\max_{0\le t\le\omega}\frac{F(t,v)}{|v|} = 0$, $\lim_{x\to+\infty}\frac{I_i(x)}{x} = 0$, $i = 1,2,\dots,m$, where $v = (v_1, v_2, \dots, v_{n+1}) \in \mathbb{R}^{n+1}$, $|v| = \max_{1\le j\le n+1}|v_j|$;

(H2) There exist two positive numbers a and b with $ab^{-1} < \lambda$ such that one of the following two conditions holds:
  - (i) for some j ($j = 1,2,\dots,n$), $F(t,v) > \frac{1}{p\omega}v_j$ as $a \le \min_{1\le i\le n}\{v_i\} \le \max_{1\le i\le n}\{v_i\} \le b$ and $ap_1\overline{h} \le v_{n+1} \le bq_1\overline{h}$, where

$$q_1 = \frac{\exp(\int_0^\omega \eta(\xi)d\xi)}{\exp(\int_0^\omega \eta(\xi)d\xi) - 1}, \quad p_1 = \frac{1}{\exp(\int_0^\omega \eta(\xi)d\xi) - 1}, \quad \overline{h} = \int_0^\omega h(s)ds;$$

  - (ii) there exists $1 \le j \le m$ such that $I_j(x) > lx$ for $x \in [a,b]$, where $l = 1/p$;

(H3) $\lim_{|v|\to 0}\max_{0\le t\le\omega} F(t,v)/|v| = 0$, $\lim_{x\to 0} I_i(x)/x = 0$, $i = 1,2,\dots,m$.

Now we state our main result on the existence of three nonnegative ω-periodic solutions for (1.1).

Theorem 3.1. *Suppose* (H1)–(H3) *hold. Then* (1.1) *has at least three nonnegative periodic solutions.*

Proof. Define a functional $\alpha(x)$ on cone P by

$$\alpha(x) = \min\{x(t) : t \in [0,\omega]\}, \ \forall x \in P. \tag{3.1}$$

Evidently, $\alpha : P \to \mathbb{R}^+ = [0,\infty)$ is nonnegative continuous and concave. Moreover, $\alpha(x) \le \|x\|$ for each $x \in P$. Notice that

$$P(\alpha; a, b) = \{x \in P : a \le \alpha(x), \|x\| \le b\} = \{x \in P : a \le x(t) \le b, \ \forall t \in [0,\omega]\}. \tag{3.2}$$

For using Lemma 1.1, we first prove that there exists a positive number c with $c \ge b$ such that $\Psi : \overline{P}_c \to P_c$. By (H1), we know there exist ε with $0 < \varepsilon < (q\omega + qm + qq_1\omega\int_0^\omega h(s)ds)^{-1}$ and $L > 0$ such that

$$\frac{F(t,v)}{|v|} < \varepsilon \quad \text{and} \quad \frac{I_k(x)}{x} < \varepsilon \quad \text{for } |v| > L, \ x > L, \ \forall t \in [0,\omega].$$

Notice that F is continuous on $[0,\omega]\times\mathbb{R}^+$ and $I_k(x)$ is also continuous on $\mathbb{R}^+$. Thus, there exists $M>0$ such that

$$F(t,v)<\varepsilon|v|+M,$$
$$I_k(x)<\varepsilon x+M,\quad k=1,2,\dots,m,\ \forall t\in[0,\omega].$$

Choose

$$c=\max\{b,\frac{qM\omega+qmM}{1-q\varepsilon(\omega+m+q_1\omega\int_0^\omega h(s)ds)}\}.$$

So for each $x\in\overline{P}_c$, $t\in[0,\omega]$, we have

$$|\Phi x(t-\zeta(t))|\le c\frac{\exp(\int_0^\omega\eta(\xi)d\xi)}{\exp(\int_0^\omega\eta(\xi)d\xi)-1}\int_{t-\zeta(t)}^{t-\zeta(t)+\omega}h(s)ds=cq_1\int_0^\omega h(s)ds,$$

where

$$q_1=\frac{\exp(\int_0^\omega\eta(\xi)d\xi)}{\exp(\int_0^\omega\eta(\xi)d\xi)-1}.$$

Then,

$$|Ux(t)|=\max\{\max_{1\le i\le n}|x(t-\tau_i(t)|,|\Phi x(t-\zeta(t))|\}\le c+cq_1\int_0^\omega h(s)ds. \tag{3.3}$$

This together with (2.3) guarantees that

$$\begin{aligned}|(\Psi x)(t)|&\le q\int_t^{t+\omega}|F(s,Ux(s))|ds+q\sum_{k=1}^m I_k(x(t_k))\\&\le q\omega(\varepsilon c+\varepsilon cq_1\int_0^\omega h(s)ds+M)+qm(\varepsilon c+M)<c,\quad\forall t\in[0,\omega].\end{aligned}$$

Combining this inequality with Lemma 2.2, we obtain that $\Psi:\overline{P}_c\to P_c$ is completely continuous.

Next we show that the condition (A1) of Lemma 1.1 holds. From (3.1), (3.2), (H2), we know that $b\in\{x:x\in P(\alpha;a,b),\alpha(x)>a\}$. Suppose first (i) of (H2) holds. Then, for each $x\in P(\alpha;a,b)=\{x\in P:a\le x(t)\le b,\ \forall t\in[0,\omega]\}$, we have

$$\begin{aligned}(\Psi x)(t)&\ge p\int_t^{t+\omega}F(s,Ux(s))ds+p\sum_{k=1}^m I_k(x(t_k))\\&\ge p\int_t^{t+\omega}F(s,Ux(s))ds>p\frac{1}{p\omega}\int_t^{t+\omega}x(t-\tau_j(t))ds\\&\ge p\frac{1}{p\omega}a\omega=a,\quad\forall t\in[0,\omega].\end{aligned}$$

It is easy to see that (A1) of Lemma 1.1 holds.

Next, assume (ii) of (H2) is satisfied. It follows that

$$\begin{aligned}(\Psi x)(t)&\ge p\int_t^{t+\omega}F(s,Ux(s))ds+p\sum_{k=1}^m I_k(x(t_k))\\&\ge pI_j(x(t_j))>plx(t_j)\\&\ge pla=a,\quad\forall t\in[0,\omega],\end{aligned}$$

which also means that (A1) of Lemma 1.1 holds.

In addition, for each $x \in P(\alpha; a, c)$ with $\|\Psi x\| \geq b$, by (2.3) and (H2), we know that

$$(\Psi x)(t) \geq \lambda \|\Psi x\| \geq \lambda b > a, \quad \forall t \in [0, \omega]. \tag{3.4}$$

This guarantees that (A3) of Lemma 1.1 is satisfied.

Finally, from (H3), there exist ε satisfying $0 < \varepsilon < (q\omega + qm + qq_1\omega \int_0^\omega h(s)ds)^{-1}$ and $\delta < a$ such that

$$\frac{F(t,v)}{|v|} < \varepsilon \quad \text{and} \quad \frac{I_k(x)}{x} < \varepsilon \quad \text{for } |v| < \delta,\ 0 < x < \delta,\ k = 1, 2, \dots, m,\ \forall t \in [0, \omega].$$

Choose $d = \min\{a, \delta, \delta/(1 + q_1 \int_0^\omega h(s)ds)\}$. This together with (2.2) guarantees that, for each $x \in \overline{P}_d$,

$$|Ux(t)| \leq d + dq_1 \int_0^\omega h(s)ds < \delta.$$

Thus,

$$\begin{aligned}
|(\Psi x)(t)| &\leq q \int_t^{t+\omega} |F(s, Ux(s))|ds + q \sum_{j=1}^m I_k(x(t_k)) \\
&\leq q\omega\varepsilon\Big(d + dq_1 \int_0^\omega h(s)ds\Big) + qm\varepsilon d \\
&= \varepsilon d\Big(q\omega + qq_1\omega \int_0^\omega h(s)ds + qm\Big) < d,
\end{aligned}$$

which implies $\|\Psi x\| \leq d$, that is, (A2) of Lemma 1.1 holds.

In conclusion, by Lemma 1.1, the operator Ψ has at least three fixed points x_1, x_2 and x_3, that is, (1.1) has at least three nonnegative periodic solutions x_1, x_2 and x_3 satisfying $x_1 \in P_d$, $x_2 \in U = \{x : x \in P(\alpha; a, c), \alpha(x) > a\}$, and $x_3 \in \overline{P_c} \backslash (P_d \cup U)$. □

We remark that Condition (H2) indicates that the impulse plays an important role.

4. Examples

In this section, two examples illustrate the application of our main result obtained in section 3.

Example 4.1. Consider the impulsive functional differential equations with feedback control

$$\begin{gathered}
x'(t) = 9\ln(1 + x^2(t - \sin(2\pi t))) + \sin(2\pi t)\sqrt{x(t - \sin(2\pi t))} \\
\times \ln(1 + u(t - \sin(2\pi t)) - (\ln(\frac{5}{4}e) + \cos(2\pi t))x(t); \\
u'(t) = -(\frac{3}{2} + \cos(2\pi t))u(t) + (\frac{3}{2} + \cos(2\pi t))x(t - \sin(2\pi t)); \\
\Delta x|_{t=1/2} = I_1(x(\frac{1}{2})),
\end{gathered} \tag{4.1}$$

where $I_1(x) = x^2 e^{-x}$, $x \in [0, +\infty)$. Then, (4.1) has three nonnegative periodic solutions.

Proof. Equation (4.1) can be regarded as of the form (1.1), where

$$F(t, v_1, v_2, v_3) = 9\ln(1 + v_1^2) + \sin(2\pi t)\sqrt{v_2}\ln(1 + v_3), \tag{4.2}$$

$\tau_1(t) = \tau_2(t) = \sigma(t) = \zeta(t) = \sin(2\pi t)$, $\eta(t) = h(t) = \frac{3}{2} + \cos(2\pi t)$, $r(t) = \ln(\frac{5}{4}e) + \cos(2\pi t)$, $\omega = 1$, $t_1 = 1/2$, $p = 1/(\frac{5}{4}e - 1)$, $\frac{1}{p\omega} = \frac{5}{4}e - 1$.

Now we prove that (H1)–(H3) hold. From (4.2), we have

$$|F(t, v_1, v_2, v_3| \le 9\ln(1 + |v|^2) + \sqrt{|v|}\ln(1 + 2|v|), \quad \forall t \in [0, 1],\ v \in \mathbb{R}^3, \tag{4.3}$$

where $|v| = \max_{1\le j\le 3}|v_j|$. obviously, (H1) and (H3) are satisfied.

Next, we show that (i) of (H2) holds. Choose $a = \sqrt{e^2 - 1}$ and $b = e^3$. Then $a^{-1}b = \frac{e^3}{\sqrt{e^2-1}} > \frac{5}{4}e$. Notice that the function $\frac{\ln(1+x^2)}{x}$ defined on $[\sqrt{e^2 - 1}, e^3]$ takes its minimum at $x = e^3$. Therefore, when $a \le \min\{v_1, v_2\} \le \max\{v_1, v_2\} \le b$, we have

$$F(t, v_1, v_2, v_3) \ge 9\ln(1 + v_1^2) \ge 9\frac{\ln(1 + e^6)}{e^3}v_1 > (\frac{5}{4}e - 1)v_1 = \frac{1}{p\omega}v_1.$$

This means (i) of (H2) is satisfied. Consequently, by Theorem 3.1, (4.1) has three nonnegative periodic solutions. □

Example 4.2. Consider the system

$$\begin{gathered}
x'(t) = 8\sin(2\pi t)\sqrt{x(t - 2\sin(2\pi t)) + 2u(t - 2\sin(2\pi t))} \\
\times \ln(1 + x(t - 2\sin(2\pi t))) - (\ln(\frac{3}{2}e) + \cos(2\pi t))x(t); \\
u'(t) = -(2 + \sin(2\pi t))u(t) + (2 + \sin(2\pi t))x(t - 2\sin(2\pi t)); \\
\Delta x|_{t=1/2} = I_1(x(\frac{1}{2})),
\end{gathered} \tag{4.4}$$

where $I_1(x) = 150x^2e^{-x}$, $x \in [0, +\infty)$. Then, (4.2) has three nonnegative periodic solutions.

Proof. Equation (4.4) can be regarded as of the form (1.1), where

$$F(t, v_1, v_2, v_3) = 8\sin(2\pi t)\sqrt{v_1 + 2v_3}\ln(1 + v_2), \tag{4.5}$$

$\tau_1(t) = \tau_2(t) = \sigma(t) = \zeta(t) = 2\sin(2\pi t)$, $\eta(t) = h(t) = 2 + \sin(2\pi t)$, $r(t) = \ln(\frac{3}{2}e) + \cos(2\pi t)$, $\omega = 1$, $t_1 = 1/2$, $p = 1/(\frac{3}{2}e - 1)$, $l = \frac{3}{2}e - 1$.

As in the proof of Example 4.1, it is easy to see that (H1) and (H3) are satisfied. Now it remains to show that (ii) of (H2) holds. Choose $a = 1$ and $b = 2e$. Then $a^{-1}b = 2e > \frac{3}{2}e$. Notice the function xe^{-x} defined on $[1, 2e]$ takes its minimum at $x = 2e$. So, it is not difficult to see

$$I_1(x) = 150x^2e^{-x} \ge 150(2e)e^{-(2e)}x = \frac{150(2e)}{e^{2e}}x > (\frac{3}{2}e - 1)x = lx, \quad \forall x \in [1, 2e],$$

which implies (ii) of (H2) is satisfied. Consequently, by Theorem 3.1, (4.4) has three nonnegative periodic solutions. □

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