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BOUNDARY-VALUE PROBLEMS FOR NONLINEAR THIRD-ORDER q-DIFFERENCE EQUATIONS

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Abstract. This article shows existence results for a boundary-value problem of nonlinear third-order q-difference equations. Our results are based on Leray-Schauder degree theory and some standard fixed point theorems.

1. Introduction

The subject of q-difference equations, initiated in the beginning of the 19th century [1, 6, 19, 22], has evolved into a multidisciplinary subject; see for example [8, 9, 10, 11, 12, 13, 14, 15, 18, 20, 21] and references therein. For some recent work on q-difference equations, we refer the reader to [2, 3, 5, 7, 16, 17, 23]. However, the theory of boundary-value problems for nonlinear q-difference equations is still in the initial stages and many aspects of this theory need to be explored. To the best of our knowledge, the theory of boundary-value problems for third-order nonlinear q-difference equations is yet to be developed.

In this paper, we discuss the existence of solutions for the nonlinear boundary-value problem (BVP) of third-order q-difference equation

$$\begin{gathered} D_q^3 u(t) = f(t, u(t)), \quad 0 \le t \le 1, \\ u(0) = 0, \quad D_q u(0) = 0, \quad u(1) = 0, \end{gathered} \tag{1.1}$$

where f is a given continuous function.

2. Preliminaries

Let us recall some basic concepts of q-calculus [15, 21].

For $0 < q < 1$, we define the q-derivative of a real valued function f as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad D_q f(0) = \lim_{t \to 0} D_q f(t).$$

Higher order q-derivatives are given by

$$D_q^0 f(t) = f(t), \quad D_q^n f(t) = D_q D_q^{n-1} f(t), \quad n \in \mathbb{N}.$$

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The q-integral of a function f defined in the interval $[a,b]$ is given by

$$\int_a^x f(t)d_qt := \sum_{n=0}^{\infty} x(1-q)q^n f(xq^n) - af(q^n a), \quad x \in [a,b],$$

and for $a = 0$, we denote

$$I_q f(x) = \int_0^x f(t)d_qt = \sum_{n=0}^{\infty} x(1-q)q^n f(xq^n),$$

provided the series converges. If $a \in [0,b]$ and f is defined on the interval $[0,b]$, then

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt.$$

Similarly, we have

$$I_q^0 f(t) = f(t), \quad I_q^n f(t) = I_q I_q^{n-1} f(t), \quad n \in \mathbb{N}.$$

Observe that

$$D_q I_q f(x) = f(x), \tag{2.1}$$

and if f is continuous at $x = 0$, then $I_q D_q f(x) = f(x) - f(0)$. In q-calculus, the product rule and integration by parts formula are

$$D_q(gh)(t) = D_q g(t)h(t) + g(qt)D_q h(t), \tag{2.2}$$

$$\int_0^x f(t)D_q g(t)dqt = \Big[f(t)g(t)\Big]_0^x - \int_0^x D_q f(t)g(qt)d_qt. \tag{2.3}$$

In the limit $q \to 1$ the above results correspond to their counterparts in standard calculus.

Motivated by the solution of a classical third-order ordinary differential equation (see Remark 2.2), we can write the solution of the third-order q-difference equation $D_q^3 u(t) = v(t)$ in the form

$$u = \int_0^t \Big(\alpha_1(q)t^2 + \alpha_2(q)ts + \alpha_3(q)s^2\Big)v(s)d_qs + a_0 + a_1 t + a_2 t^2, \tag{2.4}$$

where a_0, a_1, a_2 are arbitrary constants and $\alpha_1(q), \alpha_2(q), \alpha_3(q)$ can be fixed appropriately.

Choosing $\alpha_1(q) = 1/(1+q)$, $\alpha_2(q) = -q$, $\alpha_3(q) = q^3/(1+q)$ and using (2.1) and (2.2), we find that

$$D_q u(t) = \int_0^t tv(s)d_qs - \int_0^t qsv(s)d_qs, \quad D_q^2 u(t) = \int_0^t v(s)d_qs, \quad D_q^3 u(t) = v(t).$$

Thus, the solution (2.4) of $D_q^3 u(t) = v(t)$ takes the form

$$u = \int_0^t \Big(\frac{t^2 + q^3 s^2}{1+q} - qts\Big)v(s)d_qs + a_0 + a_1 t + a_2 t^2. \tag{2.5}$$

Lemma 2.1. *The BVP* (1.1) *is equivalent to the integral equation*

$$u = \Gamma u, \tag{2.6}$$

where

$$\Gamma u = \int_0^1 G(t,s;q)f(s,u(s))d_qs,$$

and $G(t,s;q)$ is the Green's function given by

$$G(t,s;q) = \frac{1}{(1+q)}\begin{cases} qs(1-t)[q^2s(1+t)-(1+q)t], & 0 \le s < t \le 1,\\ t^2(1-qs)(q^2s-1), & 0 \le t \le s \le 1. \end{cases} \tag{2.7}$$

Proof. In view of (2.5), the solution of $D_q^3 u = f(t,u)$ can be written as

$$u = \int_0^t \Big(\frac{t^2+q^3s^2}{1+q} - qts\Big) f(s,u(s))d_qs + a_0 + a_1t + a_2t^2, \tag{2.8}$$

where a_1, a_2, a_2 are arbitrary constants. Using the boundary conditions of (1.1) in (2.8), we find that $a_0 = 0, a_1 = 0$ and

$$a_2 = -\int_0^1 \Big(\frac{1+q^3s^2}{1+q} - qs\Big) f(s,u(s))d_qs.$$

Substituting the values of a_0, a_1 and a_2 in (2.8), we obtain

$$\begin{aligned} u &= \int_0^t \Big(\frac{t^2+q^3s^2}{1+q} - qts\Big) f(s,u(s))d_qs - t^2\int_0^1 \Big(\frac{1+q^3s^2}{1+q} - qs\Big) f(s,u(s))d_qs\\ &= \int_0^1 G(t,s;q)f(s,u(s))d_qs, \end{aligned}$$

where $G(t,s;q)$ is given by (2.7). □

We define

$$G_1 = \max_{t\in[0,1]} \Big|\int_0^1 G(t,s;q)d_qs\Big| = \frac{(1+q)q^2}{(1+q+q^2)^4}. \tag{2.9}$$

Remark 2.2. For $q \to 1$, equation (2.8) takes the form

$$u = \frac{1}{2}\int_0^t (t-s)^2 f(s,u(s))ds + a_0 + a_1t + a_2t^2,$$

which is the solution of a classical third-order ordinary differential equation $u'''(t) = f(t,u(t))$ and the associated form of Green's function for the classical case is

$$G(t,s) = \frac{1}{2}\begin{cases} s(1-t)[s(1+t)-2t], & \text{if } 0 \le s < t \le 1,\\ -t^2(1-s)^2, & \text{if } 0 \le t \le s \le 1. \end{cases}$$

3. Some existence results

Theorem 3.1. *Assume that there exist constants $M_1 \ge 0$ and $M_2 > 0$ such that $M_1G_1 < 1$ and $|f(t,u)| \le M_1|u| + M_2$ for all $t \in [0,1], u \in C([0,1])$, where G_1 is given by (2.9). Then the BVP (1.1) has at least one solution.*

Proof. In view of Lemma 2.1, we just need to prove the existence of at least one solution $u \in C([0,1])$ such that $u = \Gamma u$. Thus, it is sufficient to show that $\Gamma : \overline{B}_R \to C([0,1])$ satisfies

$$u \ne \lambda\Gamma u, \quad \forall u \in \partial B_R \quad \forall \lambda \in [0,1], \tag{3.1}$$

where $B_R \subset C([0,1])$ is a suitable ball with radius $R > 0$. Let us define

$$H(\lambda,u) = \lambda\Gamma u, \quad u \in C([0,1]),\ \lambda \in [0,1].$$

Then, by Arzela-Ascoli theorem, $h_\lambda(u) = u - H(\lambda, u) = u - \lambda\Gamma u$ is completely continuous. If (3.1) is true, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned}\deg(h_\lambda, B_R, 0) &= \deg(I - \lambda\Gamma, B_R, 0) = \deg(h_1, B_R, 0)\\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \quad 0 \in B_r,\end{aligned}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, $h_1(t) = u - \lambda\Gamma u = 0$ for at least one $u \in B_R$. Let us set

$$B_R = \{u \in C([0,1]) : \max_{t\in[0,1]} |u(t)| < R\},$$

where R will be fixed later. In order to prove (3.1), we assume that $u = \lambda\Gamma u$ for some $\lambda \in [0,1]$ and for all $t \in [0,1]$ so that

$$\begin{aligned}|u(t)| = |\lambda\Gamma u(t)| &\leq \Big| \int_0^1 |G(t,s;q) f(s,u(s)) d_q s\Big|\\ &\leq \Big| \int_0^1 G(t,s;q)(M_1|u(s)| + M_2) d_q s\Big|\\ &\leq (M_1\|u\| + M_2) \max_{t\in[0,1]} \Big| \int_0^1 G(t,s;q) d_q s\Big|\\ &\leq (M_1\|u\| + M_2) G_1,\end{aligned}$$

which implies

$$\|u\| \leq \frac{M_2 G_1}{1 - M_1 G_1}.$$

Letting $R = \frac{M_2G_1}{1-M_1G_1} + 1$, (3.1) holds. This completes the proof. □

Example 3.2. Consider the following problem

$$\begin{gathered} D^3_{1/2} u(t) = \frac{M_1}{(2\pi)} \sin(2\pi u) + \frac{|u|}{1+|u|}, \quad 0 \leq t \leq 1,\\ u(0) = 0, \quad D_{1/2}u(0) = 0, \quad u(1) = 0.\end{gathered} \tag{3.2}$$

Here $q = 1/2$ and M_1 will be fixed later. Observe that

$$|f(t,u)| = \Big| \frac{M_1}{(2\pi)} \sin(2\pi u) + \frac{|u|}{1+|u|}\Big| \leq M_1|u| + 1,$$

and

$$G_1 = \frac{q^2(1+q)}{(1+q+q^2)^4}\Big|_{q=1/2} = \frac{96}{2401}.$$

Clearly $M_2 = 1$ and and we can choose $M_1 < \frac{1}{G_1} = \frac{2401}{96}$; that is, $M_1 \leq 25$. Thus, Theorem 3.1 applies to the problem (3.2).

To prove the next existence result, we need the following known fixed point theorem [4].

Theorem 3.3. *Let Ω be an open bounded subset of a Banach space E with $0 \in \Omega$ and $B : \overline{\Omega} \to E$ be a compact operator. Then B has a fixed point in $\overline{\Omega}$ provided $\|Bu - u\|^2 \geq \|Bu\|^2 - \|u\|^2$, $u \in \partial\Omega$.*

Theorem 3.4. *If there exists a constant M_3 such that*

$$|f(t,u)| \le \frac{M_3}{G_1}, \quad \forall t \in [0,1],\ u \in [-M_3, M_3],$$

where G_1 is given by (2.9). Then (1.1) has at least one solution.

Proof. Let us define $B_{M_3} = \{u \in C([0,1]) : \max_{t\in[0,1]} |u(t)| < M_3\}$. In view of Theorem 3.3, we just need to show that

$$\|\Gamma u\| \le \|u\|, \quad \forall u \in \partial B_{M_3}. \tag{3.3}$$

For all $t \in [0,1]$, $u \in \partial B_{M_3}$, we have

$$|\Gamma u(t)| = \Big| \int_0^1 G(t,s;q) f(s,u(s)) d_q s \Big| \le \frac{M_3}{G_1} \Big| \int_0^1 G(t,s;q) d_q s \Big| \le M_3.$$

Thus (3.3) holds, which completes the proof. □

In view of the assumption $|f(t,u)| \le M_1|u| + M_2$ of Theorem 3.1, we find that $M_3 = M_2 G_1 (1 - M_1 G_1)^{-1}$.

Theorem 3.5. *Suppose that f is of class C^1 in the second variable and there exists a constant $0 \le M_4 < \frac{1}{G_1}$ (G_1 is given by (2.9)) such that $|f_u(t,u)| \le M_4$ for all $t \in [0,1], u \in C([0,1])$, then (1.1) has at least one solution.*

Proof. For all $t \in [0,1]$, we find that

$$\begin{aligned} |\Gamma u(t)| &= \Big| \int_0^1 G(t,s;q) f(s,u(s)) d_q s \Big| \le \Big| \int_0^1 G(t,s;q) (f_u(s,u(s))u(s) + \nu) d_q s \Big| \\ &\le \Big| \int_0^1 G(t,s;q) d_q s \Big| (M_4 \|u\| + \nu) \le M_4 G_1 \|u\| + \nu_1, \end{aligned}$$

where $\nu_1 = G_1 \nu$ (ν is a positive constant). For $R > 0$, we define

$$B_R = \{u \in C([0,1]) : \max_{t\in[0,1]} |u(t)| < R\},$$

so that

$$\|\Gamma u\| \le M_4 G_1 R + \nu_1 = R\Big(M_4 G_1 + \frac{\nu_1}{R}\Big) \le R,$$

for sufficiently large R. Therefore, by Schauder fixed point theorem, Γ has a fixed point. This completes the proof. □

Example 3.6. Consider the problem

$$\begin{gathered} D^3_{\frac14} u(t) = \frac{1}{12}\big(\frac{1-u^2}{1+u^2}\big)\sin(2\pi t), \quad 0 \le t \le 1, \\ u(0) = 0, \quad D_{\frac14} u(0) = 0, \quad u(1) = 0. \end{gathered} \tag{3.4}$$

Clearly $f(t,u) = \frac{1}{12}\big(\frac{1-u^2}{1+u^2}\big)\sin(2\pi t)$ and

$$G_1 = \frac{q^2(1+q)}{(1+q+q^2)^4}\Big|_{q=1/4} = \frac{5120}{194481}.$$

Furthermore,

$$|f_u(t,u)| \le \frac{1}{3}\Big(\frac{|u|}{(1+u^2)^2}\Big) < \frac{1}{G_1} = \frac{194481}{5120}.$$

Thus, by Theorem 3.5, there exists one solution for problem (3.4).

Our final result deals with the uniqueness of solutions to (1.1).

Theorem 3.7. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a jointly continuous function satisfying the condition*

$$|f(t,u) - f(t,v)| \le L|u-v|, \quad \forall t \in [0,1],\ u,v \in \mathbb{R},$$

where L is a Lipschitz constant. Then (1.1) has a unique solution provided that $L < 1/G_1$, where G_1 is given by (2.9).

Proof. For $t \in [0,1]$, we define $\Gamma : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ by

$$\Gamma u = \int_0^1 G(t,s;q) f(s,u(s)) d_q s,$$

where $G(t,s;q)$ is the Green's function given by (2.7).

Let us set $M = \max_{t \in [0,1]} |f(t,0)|$ and choose

$$r \ge \frac{MG_1}{1 - LG_1}. \tag{3.5}$$

Now we show that $\Gamma B_r \subset B_r$, where $B_r = \{u \in C([0,1],\mathbb{R}) : \|u\| \le r\}$. For $u \in B_r$, we have

$$\begin{aligned}
\|\Gamma u\| &= \max_{t \in [0,1]} \Big| \int_0^1 G(t,s;q) f(s,u(s)) d_q s \Big| \\
&= \max_{t \in [0,1]} \Big| \int_0^1 G(t,s;q) [(f(s,u(s)) - f(s,0)) + f(s,0)] d_q s \Big| \\
&\le \max_{t \in [0,1]} \Big| \int_0^1 G(t,s;q) d_q s \Big| (L\|u\| + M|) \\
&\le G_1(Lr + M) \le r.
\end{aligned}$$

where we have used (3.5). Now, for $u, v \in \mathbb{R}$ and for each $t \in [0,1]$, we obtain

$$\begin{aligned}
\|(\Gamma u) - (\Gamma v)\| &= \max_{t \in [0,1]} |(\Gamma u)(t) - (\Gamma v)(t)| \\
&\le \max_{t \in [0,1]} \Big| \int_0^1 G(t,s;q) [f(s,u(s)) - f(s,v(s))] d_q s \Big| \\
&\le L \max_{t \in [0,1]} \Big| \int_0^1 G(t,s;q) d_q s \Big| \|u - v\| \\
&\le LG_1 \|u - v\|.
\end{aligned}$$

As $L < 1/G_1$, therefore Γ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. This completes the proof. $\square$

Example 3.8. Consider

$$\begin{gathered}
D^3_{\frac{3}{4}} u(t) = L\big(\cos t + \tan^{-1} u\big), \quad 0 \le t \le 1, \\
u(0) = 0, \quad D_{\frac{3}{4}} u(0) = 0, \quad u(1) = 0.
\end{gathered} \tag{3.6}$$

With $f(t,u) = L(\cos t + \tan^{-1} u)$, we find that

$$|f(t,u) - f(t,v)| \le L|\tan^{-1} u - \tan^{-1} v| \le L|u - v|$$

and

$$G_1 = \frac{q^2(1+q)}{(1+q+q^2)^4}\Big|_{q=3/4} = \frac{64512}{1874161}.$$

Fixing $L < \frac{1}{G_1} = \frac{1874161}{64512}$, it follows by Theorem 3.7 that (3.6) has a unique solution.

Remark 3.9. In the limit as $q \to 1$, our results reduce to the ones for the classical third-order boundary-value problem

$$u'''(t) = f(t, u(t)) \quad t \in [0, 1]$$
$$u(0) = 0, \quad u'(0) = 0, \quad u(1) = 0.$$

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References

[1] C. R. Adams; On the linear ordinary q-difference equation, *Annals Math.* **30** (1928), 195-205.
[2] M. H. Annaby, Z. S. Mansour; q-Taylor and interpolation series for Jackson q-difference operators, *J. Math. Anal. Appl.*, **344** (2008), 472-483.
[3] G. Bangerezako; Variational q-calculus, *J. Math. Anal. Appl.* **289** (2004), 650-665.
[4] M. Altman, A fixed point theorem in Banach space. Bull. Polish acad. Sci. 5 (1957), 19-22.
[5] B. Ahmad, S. K. Ntouyas; Boundary value problems for q-difference inclusions, *Abstr. Appl. Anal.* Vol. 2011, Article ID 292860, 15 pages.
[6] R. D. Carmichael; The general theory of linear q-difference equations, *American J. Math.* **34** (1912), 147-168.
[7] A. Dobrogowska, A. Odzijewicz; Second order q-difference equations solvable by factorization method, *J. Comput. Appl. Math.* **193** (2006), 319-346.
[8] T. Ernst; A new notation for q-calculus and a new q-Taylor formula, U.U.D.M. Report 1999:25, ISSN 1101-3591, *Department of Mathematics, Uppsala University*, 1999.
[9] R. Finkelstein, E. Marcus; Transformation theory of the q-oscillator, *J. Math. Phys.* **36** (1995), 2652-2672.
[10] R. Finkelstein; The q-Coulomb problem, *J. Math. Phys.* **37** (1996), 2628-2636.
[11] R. Floreanini, L. Vinet; Automorphisms of the q-oscillator algebra and basic orthogonal polynomials, *Phys. Lett. A* **180** (1993), 393-401.
[12] R. Floreanini, L. Vinet; Symmetries of the q-difference heat equation, *Lett. Math. Phys.* 32 (1994), 37-44.
[13] R. Floreanini, L. Vinet; Quantum symmetries of q-difference equations, *J. Math. Phys.* **36** (1995), 3134-3156.
[14] P. G. O. Freund, A. V. Zabrodin; The spectral problem for the q-Knizhnik-Zamolodchikov equation and continuous q-Jacobi polynomials, *Comm. Math. Phys.* **173** (1995), 17-42.
[15] G. Gasper, M. Rahman; *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.
[16] G. Gasper, M. Rahman; Some Some systems of multivariable orthogonal q-Racah polynomials. *Ramanujan J.* **13** (2007), 389-405.
[17] M. E. H. Ismail, P. Simeonov; q-difference operators for orthogonal polynomials, *J. Computat. Appl. Math.* **233** (2009), 749-761.
[18] G. N. Han, J. Zeng; On a q-sequence that generalizes the median Genocchi numbers, *Ann. Sci. Math. Quebec* **23** (1999), 63-72.
[19] F. H. Jackson; On q-difference equations, *American J. Math.* **32**(1910), 305- 314.
[20] M. Jaulent, I. Miodek; Nonlinear evolution equations associated with 'energy-dependent Schrodinger potentials', *Lett. Math. Phys.* **1** (1976), 243-250.
[21] V. Kac, P. Cheung; *Quantum Calculus*, Springer, New York, 2002.
[22] T. E. Mason; On properties of the solutions of linear q-difference equations with entire function coefficients, *American J. Math.* **37** (1915), 439-444.
[23] M. El-Shahed, H. A. Hassan; Positive solutions of q-difference equation, *Proc. Amer. Math. Soc.* 138 (2010), 1733-1738.

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