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UNIFORM DECAY OF SOLUTIONS TO CAUCHY VISCOELASTIC PROBLEMS WITH DENSITY

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Abstract. In this article we consider the decay of solutions to a linear Cauchy viscoelastic problem with density. This study includes the exponential and polynomial rates as particular cases. To compensate for the lack of Poincare's inequality in the whole space, we consider the solutions in spaces weighted by the density.

1. Introduction

In this article we are concerned with the initial-value problem

$$\begin{gathered} \rho(x)u_{tt} - \Delta u(x) + \int_0^t g(t-s)\Delta u(x,s)ds = 0, \quad x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \end{gathered} \tag{1.1}$$

where u_0, u_1 are initial data chosen in suitable spaces and g is the relaxation function subjected to some conditions to be specified later. The density $\rho(x)$ satisfying the following conditions

(H1) $\rho : \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$, $\rho(x) > 0$, $\rho(x) \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma \in (0,1)$ and $\rho \in L^{n/2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

In the whole space case, Poincare's inequality and some Lebesgue and Sobolev embedding inequalities are not valid. To overcome this difficulty in this case, we exploit the density to introduce weighted spaces for solutions of our problem.

The work with weighted spaces was studied by many authors. Papadopoulos and Stavarakakis [11] established existence of a global solutions and blow up results for the non local quasilinear hyperbolic problem of Kirchhoff type

$$\begin{gathered} u_{tt} - \phi(x)\|\nabla u(t)\|^2\Delta u + \delta u_t = |u|^a u, \\ x \in \mathbb{R}^n,\ t \geq 0, \end{gathered}$$

in the case where $n \geq 3$, $\delta \geq 0$ and $\rho(x) = (\phi(x))^{-1}$ is a positive function lying in $L^{n/2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Brown and Stavarakakis [1] proved the existence of positive

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solutions for the semilinear elliptic equation

$$
\begin{gathered}
-\Delta u(x) = \lambda g(x) f(u(x)), \quad 0 < u < 1,\ x \in \mathbb{R} \\
\lim_{|x| \to +\infty} u(x) = 0
\end{gathered}
$$

and for $g \in L^{n/2}(\mathbb{R}^n)$ for $n = 1, 2, 3$. Karachalios and Stavarakakis [6] proved a local existence of solutions and global attractor in the energy space $D^{1,2}(\mathbb{R}^n) \times L^2_g(\mathbb{R}^n)$ for a semilinear hyperbolic problem

$$
u_{tt} - \phi(x)\Delta u + \delta u_t + \lambda f(u) = \eta(x), \quad x \in \mathbb{R}^n,\ t > 0,
$$

in the case where $\delta > 0$, $n \geq 3$ and $\rho(x) = (\phi(x))^{-1}$ lies in $L^{n/2}(\mathbb{R}^n)$.

It is also worth mentioning the work of Zhou [12] and Cavalcanti citec2. In this work, the following nonlinear wave equation with damping and source terms of the form

$$
u_{tt} - \phi(x)\Delta u + a|u_t|^{m-1}u_t = f(x, u), \quad x \in \mathbb{R}^n,\ t > 0,
$$

was considered. Where the author proved, in the linear damping case, that the solution blows up in finite time even for vanishing initial energy. Criteria to guarantee blow up of solutions with positive initial energy were established for both linear and nonlinear cases. Global existence and large time behavior also proved.

In [9], a class of abstract viscoelastic systems of the form

$$
\begin{gathered}
u_{tt}(t) + \mathcal{A}u(t) + \beta u(t) - (g * \mathcal{A}^{\alpha} u)(t) = 0 \\
u(0) = u_0, \quad u_t(0) = u_1,
\end{gathered} \tag{1.2}
$$

for $0 \leq \alpha \leq 1$, $\beta \geq 0$, were investigated. The main focus was on the case when $0 < \alpha < 1$ and the main result was that solutions for (1.2) decay polynomially even if the kernel g decay exponentially. This result is sharp (see [9, Theorem 12]. This result has been improved by Rivera *et al.* [10], where the authors studied a more general abstract problem than (1.2) and established a necessary and sufficient condition to obtain an exponential decay. In the case of lack of exponential decay, a polynomial decay has been proved.

Kafini and Messaoudi [4] looked into the following Cauchy viscoelastic problem

$$
\begin{gathered}
u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds = 0, \quad x \in \mathbb{R}^n,\ t > 0 \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n
\end{gathered}
$$

and showed that, for compactly supported initial data u_0, u_1 and for an exponentially decaying relaxation function g, the decay of the first energy of solution is polynomial. The finite-speed propagation is used to compensate for the lack of Poincaré's inequality in $\mathbb{R}^n$. For nonexistence, the same authors [5] established a blow-up result to the following Cauchy viscoelastic problem

$$
\begin{gathered}
u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + u_t = |u|^{p-1}u, \quad x \in \mathbb{R}^n,\ t > 0 \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n,
\end{gathered}
$$

under the conditions

$$\int_0^t g(s)ds < \frac{2p-2}{2p-1}, \quad g'(t) \le 0,$$
$$E(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\nabla u_0\|_2^2 - \frac{1}{p+1}\|u_0\|_{p+1}^{p+1} \le 0.$$

This result extends the one of [13], established for the wave equation in bounded domain.

In this article, we will extend the result in [12] to our viscoelastic problem. We aim to study the effect of the density to the decay rates. In the case $\rho(x) = 1$ (as in [4]), the best decay obtained is polynomial. Here, we establish a general decay result for solutions where the exponential and polynomial are only special cases. This result does not contradict the past results in [4, 9, 10]. The choice of spaces of solutions and the density one make it possible to get an exponential decay rate. Where we have $\rho(x)$ is a continuous and $L^{n/2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ function that make most of its contribution concentrate at the early time in the contrast of the late time ($t \to \infty$). Obviously, $\rho(x)$ cannot be a constant here. In our proof, we use the multiplier method together with the Lyapunov functional method as in [7] with some necessary modification due to the nature of the problem. The paper organized as follows. In Section 2, we define our function space and the assumptions on the kernel g. In section 3, we state and prove our main result.

2. Preliminaries

To achieve our result, we assume the following assumptions on the relaxation function g:

(G1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function such that

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)ds = l > 0.$$

(G2) There exists a nonincreasing differentiable function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$g'(t) \le -\xi(t)g(t), \quad t \ge 0, \quad \int_0^\infty \xi(t)dt = +\infty.$$

There are many functions satisfying (G1) and (G2), for example

$$g_1(t) = \frac{\alpha}{(1+t)^\nu}, \quad \nu > 1$$
$$g_2(t) = \alpha e^{-\beta(t+1)^p}, \quad 0 < p \le 1$$
$$g_3(t) = \frac{\alpha}{(1+t)[\ln(1+t)]^\nu}, \quad \nu > 1$$

where α and β are chosen properly.

Remark 2.1. Condition (G1) is necessary to guarantee the hyperbolicity of the system.

We define the function space of our problem and its norm, as in [1, 11], as follows:
(A) The function space for (1.1) is $X = D^{1,2}(\mathbb{R}^n) \times L^2_\rho(\mathbb{R}^n)$, with

$$D^{1,2}(\mathbb{R}^n) = \{f \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) : \nabla f \in L^2(\mathbb{R}^n)\}.$$

(B) The space $L^2_\rho(\mathbb{R}^n)$ is defined to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(f,h)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h \, dx.$$

One can easily check that $L^2_\rho(\mathbb{R}^n)$ is a separable Hilbert space and

$$\|f\|^2_{L^2_\rho(\mathbb{R}^n)} = (f,f)_{L^2_\rho(\mathbb{R}^n)}.$$

(C) For $1 < p < \infty$, if f is a measurable function on $\mathbb{R}^n$, we define

$$\|f\|_{L^p_\rho} = \Big(\int_{\mathbb{R}^n} \rho |f|^p dx \Big)^{1/p}$$

and let $L^p_\rho(\mathbb{R}^n)$ consist of all f for which $\|f\|_{L^p_\rho(\mathbb{R}^n)} < \infty$.

For the weighted space $L^p_\rho(\mathbb{R}^n)$, we have the following lemma

Lemma 2.2 ([6])**.** *Let ρ satisfies* (H1)*, then for any $u \in D^{1,2}(\mathbb{R}^n)$,*

$$\|u\|_{L^q_\rho} \le \|\rho\|_{L^s} \|\nabla u\|_{L^2}, \quad \text{with} \quad s = \frac{2n}{2n - qn + 2q}, \quad 2 \le q \le \frac{2n}{n-2}.$$

Corollary 2.3. *If $q = 2$, then Lemma 2.2. yields*

$$\|u\|_{L^2_\rho} \le \|\rho\|_{L^{n/2}} \|\nabla u\|_{L^2},$$

where we can assume $\|\rho\|_{L^{n/2}} = C_0 > 0$ to get

$$\|u\|_{L^2_\rho(\mathbb{R}^n)} \le C_0 \|\nabla u\|_{L^2}. \tag{2.1}$$

Theorem 2.4 ([12])**.** *Suppose that* (H1) *holds and g satisfies* (G1)*. Assume that $1 \le p \le \frac{n+2}{n-2}$ if $n \ge 2$ or $1 \le p$ if $n = 2$. Then for any initial data*

$$u_0 \in \mathrm{D}^{1,2}(\mathbb{R}^n) \quad \text{and} \quad u_1 \in L^2_\rho(\mathbb{R}^n),$$

problem (1.1) *has a unique solution*

$$u \in C([0,T); D^{1,2}(\mathbb{R}^n)) \quad \text{and} \quad u_t \in C([0,T); L^2_\rho(\mathbb{R}^n)),$$

for T small enough.

We now introduce the 'modified' energy functional

$$E(t) = \frac{1}{2}\|u_t\|^2_{L^2_\rho} + \frac{1}{2}\Big(1 - \int_0^t g(s)ds\Big)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u), \tag{2.2}$$

where

$$(g \circ v)(t) = \int_0^t g(t-s)\|v(t) - v(s)\|_2^2 ds, \quad \forall v \in L^2(\mathbb{R}^n).$$

3. Decay of solutions

In this section, we state and prove our main result. For this purpose, we set

$$F(t) := E(t) + \varepsilon_1 \Psi(t) + \varepsilon_2 \chi(t), \tag{3.1}$$

where ε_1 and ε_2 are positive constants and

$$\Psi(t) := \int_{\mathbb{R}^n} \rho u u_t \, dx, \tag{3.2}$$

$$\chi(t) := -\int_{\mathbb{R}^n} \rho u_t \int_0^t g(t-s)\big(u(t) - u(s)\big)\, ds\, dx. \tag{3.3}$$

Lemma 3.1. *Along the solution of* (1.1)*, the 'modified' energy satisfies*
$$E'(t) = \frac{1}{2}(g' \circ \nabla u) - \frac{1}{2}g(t)\|\nabla u\|_2^2 \le \frac{1}{2}(g' \circ \nabla u) \le 0. \tag{3.4}$$

Proof. By multiplying (1.1) by u_t and integrating over $\mathbb{R}^n$, using integration by parts, hypotheses (G1) and (G2) and some manipulations as in [2, 3, 8], we reach the result. □

Lemma 3.2. *For any ε_1 and ε_2 small enough,*
$$\alpha_1 F(t) \le E(t) \le \alpha_2 F(t), \tag{3.5}$$
holds for two positive constants α_1 and α_2.

Proof. By applying Young's inequality to (3.1) and using (3.2) and (3.3), we obtain
$$\begin{aligned}
F(t) &\le E(t) + \varepsilon_1\big(\delta\|u_t\|_{L^2_\rho}^2 + \frac{1}{4\delta}\|u\|_{L^2_\rho}^2\big)\\
&\quad + \varepsilon_2\Big(\delta\|u_t\|_{L^2_\rho}^2 + \frac{1}{4\delta}\|\int_0^t g(t-s)\big(u(t)-u(s)\big)ds\|_{L^2_\rho}^2\big)\\
&\le E(t) + \varepsilon_1\Big(\delta\|u_t\|_{L^2_\rho}^2 + \frac{C_0}{4\delta}\|\nabla u\|_2^2\Big) + \varepsilon_2\Big(\delta\|u_t\|_{L^2_\rho}^2 + \frac{C_0}{4\delta}(1-l)(g\circ\nabla u)\Big)\\
&\le E(t) + \delta(\varepsilon_1+\varepsilon_2)\|u_t\|_{L^2_\rho}^2 + \frac{\varepsilon_1 C_0}{4\delta}\|\nabla u\|_2^2 + \frac{\varepsilon_2 C_0}{4\delta}(1-l)(g\circ\nabla u)\\
&\le E(t) + \beta E(t) \le \alpha_2 F(t).
\end{aligned}$$
Also, for ε_1 and ε_2 small enough, we have
$$\begin{aligned}
F(t) &\ge E(t) - \varepsilon_1\Psi(t) - \varepsilon_2\chi(t)\\
&\ge E(t) - \delta(\varepsilon_1+\varepsilon_2)\|u_t\|_{L^2_\rho}^2 - \frac{\varepsilon_1 C_0}{4\delta}\|\nabla u\|_2^2 - \frac{\varepsilon_2 C_0}{4\delta}(1-l)(g\circ\nabla u)\\
&\ge E(t) - \beta E(t) \ge \alpha_1 F(t).
\end{aligned}$$
Consequently, (3.5) follows. □

Lemma 3.3. *Assume* (H1), (G1), (G2)*. Along the solution of* (1.1)*, the functional*
$$\Psi(t) := \int_{\mathbb{R}^n} \rho u u_t\,dx,$$
satisfies
$$\Psi'(t) \le \|u_t\|_{L^2_\rho}^2 - (l-\delta)\|\nabla u\|_2^2 + \frac{1}{4\delta}(1-l)(g\circ\nabla u)(t). \tag{3.6}$$

Proof. From the definition of $\Psi(t)$ in (3.2) we have
$$\Psi'(t) = \int_{\mathbb{R}^n} \rho u_t^2 dx + \int_{\mathbb{R}^n} \rho u u_{tt} dx. \tag{3.7}$$

To estimate the last term in (3.7), we multiply (1.1) by u and integrate by parts over $\mathbb{R}^n$. So, we obtain
$$\int_{\mathbb{R}^n} \rho u u_{tt} dx = \int_{\mathbb{R}^n} \nabla u(t)\cdot\int_0^t g(t-s)\nabla u(s)\,ds\,dx - \int_{\mathbb{R}^n} |\nabla u(t)|^2 dx. \tag{3.8}$$

The first term in (3.8) can be estimated as follows

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \nabla u(t)\cdot \int_0^t g(t-s)\nabla u(s)\,ds\,dx \\
&= \int_{\mathbb{R}^n} \nabla u(t)\cdot \int_0^t g(t-s)(\nabla u(s)-\nabla u(t)+\nabla u(t))\,ds\,dx \\
&= \Big(\int_0^t g(s)ds\Big)\int_{\mathbb{R}^n} |\nabla u(t)|^2dx + \int_{\mathbb{R}^n} \nabla u(t)\cdot \int_0^t g(t-s)(\nabla u(s)-\nabla u(t))\,ds\,dx \\
&\le \Big(\int_0^t g(s)ds\Big)\int_{\mathbb{R}^n} |\nabla u(t)|^2dx + \delta\int_{\mathbb{R}^n} |\nabla u(t)|^2dx + \frac{1}{4\delta}\Big(\int_0^t g(s)ds\Big)(g\circ\nabla u)(t).
\end{aligned} \tag{3.9}
$$

Recalling that

$$
\int_0^t g(s)ds \le \int_0^\infty g(s)ds = 1-l,
$$

we obtain the result. □

Lemma 3.4. *Assume* (G1), (G2). *Along the solution of* (1.1), *for any* $\delta>0$, *the functional*

$$
\chi(t) := -\int_{\mathbb{R}^n} \rho u_t \int_0^t g(t-s)(u(t)-u(s))\,ds\,dx
$$

satisfies

$$
\begin{aligned}
\chi'(t) &\le \delta(1+2(1-l)^2)\|\nabla u\|_2^2 + (1-l)\big[(2\delta+\frac{1}{2\delta})+\frac{1}{4\delta}\big](g\circ\nabla u)(t) \\
&\quad \frac{g(0)}{4\delta}C_0(-(g'\circ\nabla u)(t)) - \Big(\int_0^t g(s)ds-\delta\Big)\int_{\mathbb{R}^n}\rho u_t^2dx.
\end{aligned} \tag{3.10}
$$

Proof. From the definition of $\chi(t)$ in (3.3), we have

$$
\begin{aligned}
\chi'(t) = &-\int_{\mathbb{R}^n} \rho u_{tt}\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\
&-\int_{\mathbb{R}^n} \rho u_t\int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx - \Big(\int_0^t g(s)ds\Big)\int_{\mathbb{R}^n}\rho u_t^2dx.
\end{aligned} \tag{3.11}
$$

To simplify the first term in (3.11), we multiply (1.1) by $\int_0^t g(t-s)(u(t)-u(s))ds$ and integrate by parts over $\mathbb{R}^n$. So we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \rho u_{tt}\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\
&= \int_{\mathbb{R}^n} \Delta u(x)\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\
&- \int_{\mathbb{R}^n} (\int_0^t g(t-s)(u(t)-u(s))ds\int_0^t g(t-s)\Delta u(s)ds)dx.
\end{aligned} \tag{3.12}
$$

The first term in the right side of (3.12) is estimated as follows

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Delta u(x) \int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\
&\le -\int_{\mathbb{R}^n} \nabla u(x)\cdot \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&\le \int_{\mathbb{R}^n} \nabla u(x)\cdot \int_0^t g(t-s)(\nabla u(s)-\nabla u(t))\,ds\,dx \\
&\le \delta \int_{\mathbb{R}^n} |\nabla u(t)|^2 dx + \frac{1}{4\delta}(\int_0^t g(s)ds)(g\circ\nabla u)(t) \\
&\le \delta \int_{\mathbb{R}^n} |\nabla u(t)|^2 dx + \frac{1}{4\delta}(1-l)(g\circ\nabla u)(t),
\end{aligned} \tag{3.13}$$

while the second term becomes, as in (3.9),

$$\begin{aligned}
&-\int_{\mathbb{R}^n} (\int_0^t g(t-s)(u(t)-u(s))ds \int_0^t g(t-s)\Delta u(x,s)ds)dx \\
&= \int_{\mathbb{R}^n} (\int_0^t g(t-s)\nabla u(s)ds)\cdot(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds)dx \\
&\le \delta \int_{\mathbb{R}^n} |\int_0^t g(t-s)\nabla u(s)ds|^2 dx + \frac{1}{4\delta}\int_{\mathbb{R}^n} |\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds|^2 dx \\
&\le (2\delta + \frac{1}{4\delta}) \int_{\mathbb{R}^n} (\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds)^2 dx \\
&\quad 2\delta(1-l)^2 \int_{\mathbb{R}^n} |\nabla u(t)|^2 dx \\
&\le (2\delta + \frac{1}{4\delta})(1-l)(g\circ\nabla u)(t) + 2\delta(1-l)^2 \int_{\mathbb{R}^n} |\nabla u(t)|^2 dx.
\end{aligned} \tag{3.14}$$

Back to (3.11), the second term can be estimated as follows

$$\begin{aligned}
&-\int_{\mathbb{R}^n} \rho u_t \int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx \\
&\le \delta \int_{\mathbb{R}^n} \rho u_t^2 dx + \frac{1}{4\delta}\| \int_0^t -g'(t-s)(u(t)-u(s))ds\|_{L^2_\rho}^2 \\
&\le \delta \int_{\mathbb{R}^n} \rho u_t^2 dx + \frac{g(0)}{4\delta} C_0(-(g'\circ\nabla u)(t)).
\end{aligned} \tag{3.15}$$

By combining (3.11)-(3.15), the assertion of the lemma is established. □

Theorem 3.5. *Let $u_0 \in D^{1,2}(\mathbb{R}^n)$ and $u_1 \in L^2_\rho(\mathbb{R}^n)$ be given. Assume that* (H1) *holds and g satisfies* (G1) *and* (G2). *Then, for each $t_0 > 0$, there exist strictly positive constants C_1 and C_2 such that the energy of solution given by* (1.1) *satisfies, for all $t \ge t_0$,*

$$E(t) \le C_1 E(t_0) e^{-C_2 \int_{t_0}^t \xi(s)ds}, \quad \forall t > t_0. \tag{3.16}$$

Proof. Since g is positive and $g(0) > 0$, then for any $t_0 > 0$ we have

$$\int_0^t g(s)ds \ge \int_0^{t_0} g(s)ds = g_0 > 0, \quad \forall t \ge t_0.$$

Differentiation of (2.1) using (3.4), (3.6) and (3.10), yields

$$\begin{aligned} F'(t) &= E'(t) + \varepsilon_1 \Psi'(t) + \varepsilon_2 \chi'(t) \\ &\leq -(\varepsilon_2(g_0-\delta)-\varepsilon_1)\int_{\mathbb{R}^n} \rho u_t^2 dx + [\frac{1}{2} - \varepsilon_2 \frac{g(0)}{4\delta} C_0]((g' \circ \nabla u)(t)) \\ &\quad - [\varepsilon_1(l-\delta) - \varepsilon_2 \delta(1+2(1-l)^2)]\|\nabla u\|_2^2 \\ &\quad \{\frac{\varepsilon_1}{4\delta}(1-l) + \varepsilon_2(1-l)[\big(2\delta + \frac{1}{2\delta}\big) + \frac{1}{4\delta}]\}(g\circ \nabla u)(t) \end{aligned} \tag{3.17}$$

At this point we choose δ so small that

$$\begin{gathered} \max\{g_0-\delta, l-\delta\} > \frac{1}{2} g_0, \\ \delta(1+2(1-l)^2) < \frac{1}{4} g_0. \end{gathered}$$

Whence δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\frac{1}{4} g_0 \varepsilon_2 < \varepsilon_1 < \frac{1}{2} g_0 \varepsilon_2, \tag{3.18}$$

will make

$$\begin{gathered} k_1 = \varepsilon_2(g_0-\delta) - \varepsilon_1 > 0, \\ k_2 = \varepsilon_1(l-\delta) - \varepsilon_2\delta(1+2(1-l)^2) > 0. \end{gathered}$$

Then we pick ε_1 and ε_2 so small that (3.5) and (3.18) remain valid and

$$\frac{1}{2} - \varepsilon_2 \frac{g(0)}{4\delta} C_0 > 0.$$

Therefore, for some positive constants β, β_1 and β_2, we have

$$\begin{aligned} F'(t) &\leq -\beta(\int_{\mathbb{R}^n} \rho {u_t}^2 dx + \|\nabla u\|_2^2) + \beta_1 (g\circ\nabla u)(t), \\ &\leq -\beta E(t) + \beta_2 (g \circ \nabla u)(t) \quad \forall t \geq t_0. \end{aligned} \tag{3.19}$$

Multiplying (3.19) by $\xi(t)$ gives

$$\xi(t) F'(t) \leq -\beta \xi(t) E(t) + \beta_2 \xi(t) (g\circ\nabla u)(t), \quad \forall t \geq t_0.$$

The last term can be estimated, using (H2), as follows

$$\beta_2 \xi(t)(g\circ\nabla u)(t) \leq -\beta_2 (g'\circ\nabla u)(t) \leq -2\beta_2 E'(t).$$

Thus, (3.19) becomes

$$\begin{gathered} \xi(t)F'(t) \leq -\beta\xi(t)E(t) - 2\beta_2 E'(t) \\ \xi(t)F'(t) + 2\beta_2 E'(t) \leq -\beta \xi(t) E(t). \end{gathered} \tag{3.20}$$

It is clear that

$$F_1(t) = \xi(t)F(t) + 2\beta_2 E(t) \sim E(t).$$

Therefore, using (3.20) and the fact that $\xi'(t) \leq 0$, we arrive at

$$F_1'(t) = (\xi(t)F(t) + 2\beta_2 E(t))' \leq -\beta\xi(t)E(t). \tag{3.21}$$

Integration over (t_0, t) leads to, for some constant $C_2 > 0$ such that

$$F_1(t) \leq F_1(t_0) e^{-C_2 \int_{t_0}^t \xi(s) ds}, \quad \forall t > t_0.$$

Recalling (3.5), estimate (3.11) yields the desired result (3.16). $\square$

Remark 3.6. Our result is established without using the condition $\int_0^\infty \xi(s)ds = +\infty$, which is crucial for obtaining uniform stability.

Exponential decay is obtained for $\xi(t) \equiv a$, and polynomial decay for $\xi(t) = a(1+t)^{-1}$, where a is a positive constant.

Estimate (3.16) is also true for $t \in [0, t_0]$ by virtue of continuity and boundedness of $E(t)$ and $\xi(t)$.

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