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EXISTENCE OF THREE SOLUTIONS FOR A KIRCHHOFF-TYPE BOUNDARY-VALUE PROBLEM

SHAPOUR HEIDARKHANI, GHASEM ALIZADEH AFROUZI, DONAL O'REGAN

Abstract. In this note, we establish the existence of two intervals of positive real parameters λ for which the boundary-value problem of Kirchhoff-type

$$-K\Big(\int_a^b |u'(x)|^2 dx\Big)u'' = \lambda f(x,u),$$
$$u(a) = u(b) = 0$$

admits three weak solutions whose norms are uniformly bounded with respect to λ belonging to one of the two intervals. Our main tool is a three critical point theorem by Bonanno.

1. Introduction

In the literature many results focus on the existence of multiple solutions to boundary-value problems. For example, certain chemical reactions in tubular reactors can be mathematically described by a nonlinear two-point boundary-value problem and one is interested if multiple steady-states exist. For a recent treatment of chemical reactor theory and multiple solutions see [1, section 7] and the references therein.

Bonanno in [3] established the existence of two intervals of positive real parameters λ for which the functional $\Phi - \lambda\Psi$ has three critical points whose norms are uniformly bounded in respect to λ belonging to one of the two intervals and he obtained multiplicity results for a two point boundary-value problem. In the present paper as an application, we shall illustrate these results for a Kirchhoff-type problem.

Problems of Kirchhoff-type have been widely investigated. We refer the reader to the papers [2, 5, 7, 9, 10, 11, 15] and the references therein. Ricceri [13] established the existence of at least three weak solutions to a class of Kirchhoff-type doubly eigenvalue boundary value problem using [12, Theorem 2].

Consider the Kirchhoff-type problem

$$\begin{gathered}-K\Big(\int_a^b |u'(x)|^2 dx\Big)u'' = \lambda f(x,u),\\ u(a) = u(b) = 0\end{gathered} \tag{1.1}$$

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where $K : [0, +\infty[\to \mathbb{R}$ is a continuous function, $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $\lambda > 0$.

In the present paper, our approach is based on a three critical points theorem proved in [3], which is recalled in the next section for the reader's convenience (Theorem 2.1). Our main result is Theorem 2.2 which, under suitable assumptions, ensures the existence of two intervals Λ_1 and Λ_2 such that, for each $\lambda \in \Lambda_1 \cup \Lambda_2$, the problem (1.1) admits at least three classical solutions whose norms are uniformly bounded in respect to $\lambda \in \Lambda_2$.

Let X the the Sobolev space $H_0^1([a, b])$ with the norm

$$\|u\| = \Big(\int_a^b (|u'(x)|^2) dx \Big)^{1/2}.$$

We say that u is a weak solution to (1.1) if $u \in X$ and

$$K\Big(\int_a^b |u'(x)|^2 dx \Big) \int_a^b u'(x)v'(x) dx - \lambda \int_a^b f(x, u(x))v(x) dx = 0$$

for every $v \in X$.

For other basic notations and definitions, we refer the reader to [4, 6, 8, 14].

2. Results

For the reader's convenience, dirst we here recall [3, Theorem 2.1].

Theorem 2.1. *Let X be a separable and reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a nonnegative continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $J : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exists $x_0 \in X$ such that $\Phi(x_0) = J(x_0) = 0$ and that*

$$\lim_{\|x\| \to +\infty} (\Phi(x) - \lambda J(x)) = +\infty \quad \text{for all } \lambda \in [0, +\infty[.$$

Further, assume that there are $r > 0$, $x_1 \in X$ such that $r < \Phi(x_1)$ and

$$\sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} J(x) < \frac{r}{r + \Phi(x_1)} J(x_1);$$

here $\overline{\Phi^{-1}(] - \infty, r[)}^w$ denotes the closure of $\Phi^{-1}(] - \infty, r[)$ in the weak topology (in particular note $J(x_1) \geq 0$ since $x_0 \in \overline{\Phi^{-1}(] - \infty, r[)}^w$ (note $J(x_0) = 0$) so $\sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} J(x) \geq 0$). Then, for each

$$\lambda \in \Lambda_1 = \Big] \frac{\Phi(x_1)}{J(x_1) - \sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} J(x)}, \frac{r}{\sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} J(x)} \Big[,$$

the equation

$$\Phi'(u) + \lambda J'(u) = 0 \tag{2.1}$$

has at least three solutions in X and, moreover, for each $\eta > 1$, there exist an open interval

$$\Lambda_2 \subseteq \Big[0, \frac{\eta r}{r \frac{J(x_1)}{\Phi(x_1)} - \sup_{x \in \overline{\Phi^{-1}(-\infty, r[)}^w} J(x)} \Big]$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2$, the equation (2.2) has at least three solutions in X whose norms are less than σ.

Let $K : [0, +\infty[\to \mathbb{R}$ be a continuous function such that there exists a positive number m with $K(t) \geq m$ for all $t \geq 0$, and let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $\sup_{|\xi| \leq s} |f(., \xi)| \in L^1(a, b)$ for all $s > 0$. Corresponding to K and f we introduce the functions $\tilde{K} : [0, +\infty[\to \mathbb{R}$ and $F : [a, b] \times \mathbb{R} \to \mathbb{R}$, respectively as follows

$$\tilde{K}(t) = \int_0^t K(s)ds \quad \text{for all } t \geq 0 \tag{2.2}$$

and

$$F(x, t) = \int_0^t f(x, s)ds \quad \text{for all } (x, t) \in [a, b] \times \mathbb{R}. \tag{2.3}$$

Now, we state our main result.

Theorem 2.2. *Assume that there exist positive constants r and θ, and a function $w \in X$ such that:*

(i) $\tilde{K}(\|w\|^2) > 2r$,

(ii)

$$\int_a^b \sup_{t\in[-\sqrt{\frac{r(b-a)}{2m}},\sqrt{\frac{r(b-a)}{2m}}]} F(x, t)dx < r\frac{\int_a^b F(x, w(x))dx}{r + \frac{1}{2}\tilde{K}(\|w\|^2)},$$

(iii) $\frac{(b-a)^2}{2m} \limsup_{|t|\to+\infty} \frac{F(x,t)}{t^2} < \frac{1}{\theta}$ *uniformly with respect to $x \in [a, b]$.*

Further, assume that there exists a continuous function $h : [0, +\infty[\to \mathbb{R}$ such that $h(tK(t^2)) = t$ for all $t \geq 0$. Then, for each λ in the interval

$$\Lambda_1 =]\frac{\frac{1}{2}\tilde{K}(\|w\|^2)}{\int_a^b F(x, w(x))dx - \int_a^b \sup_{t\in[-\sqrt{\frac{r(b-a)}{2m}},\sqrt{\frac{r(b-a)}{2m}}]} F(x, t)dx}, \frac{r}{\int_a^b \sup_{t\in[-\sqrt{\frac{r(b-a)}{2m}},\sqrt{\frac{r(b-a)}{2m}}]} F(x, t)dx}[,$$

problem (1.1) *admits at least three weak solutions in X and, moreover, for each $\eta > 1$, there exist an open interval*

$$\Lambda_2 \subseteq [0, \frac{\eta r}{2r\frac{\int_a^b F(x,w(x))dx}{\tilde{K}(\|w\|^2)} - \int_a^b \sup_{t\in[-\sqrt{\frac{r(b-a)}{2m}},\sqrt{\frac{r(b-a)}{2m}}]} F(x, t)dx}]$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2$, problem (1.1) *admits at least three weak solutions in X whose norms are less than σ.*

Let us first give a particular consequence of Theorem 2.2 for a fixed test function w.

Corollary 2.3. *Assume that there exist positive constants c, d, α, β and θ with $\beta - \alpha < b - a$ such that Assumption (ii) in Theorem 2.2 holds, and*

(i) $\tilde{K}(d^2(\frac{\alpha+\beta}{\alpha\beta})) > \frac{4mc^2}{b-a}$,

(ii) $F(x, t) \geq 0$ *for each* $(x, t) \in ([a, a + \alpha] \cup [b - \beta, b]) \times [0, d]$,

(iii) $\int_a^b \sup_{t\in[-c,c]} F(x, t)dx < \frac{2mc^2}{b-a} \frac{\int_{a+\alpha}^{b-\beta} F(x,d)dx}{\frac{2mc^2}{b-a} + \frac{1}{2}\tilde{K}(d^2(\frac{\alpha+\beta}{\alpha\beta}))}$.

Further, assume that there exists a continuous function $h : [0, +\infty[\to \mathbb{R}$ *such that* $h(tK(t^2)) = t$ *for all* $t \geq 0$. *Then, for each*

$$\lambda \in \Lambda_1' =]\frac{\frac{1}{2}\tilde{K}(d^2(\frac{\alpha+\beta}{\alpha\beta}))}{\int_{a+\alpha}^{b-\beta} F(x,d)dx - \int_a^b \sup_{t\in[-c,c]} F(x,t)dx}, \frac{\frac{2mc^2}{b-a}}{\int_a^b \sup_{t\in[-c,c]} F(x,t)dx}[,$$

problem (1.1) *admits at least three weak solutions in* X *and, moreover, for each* $\eta > 1$, *there exist an open interval*

$$\begin{aligned}\Lambda_2 \subseteq [0, &\big(\frac{2\eta mc^2}{b-a}\big) \\ &\div \Big(\frac{4mc^2}{b-a} \frac{\int_a^{a+\alpha} F(x, \frac{d}{\alpha}(x-a))dx + \int_{a+\alpha}^{b-\beta} F(x,d)dx + \int_{b-\beta}^{b} F(x, \frac{d}{\beta}(b-x))dx}{\tilde{K}(d^2(\frac{\alpha+\beta}{\alpha\beta}))} \\ &- \int_a^b \sup_{t\in[-c,c]} F(x,t)dx\Big)]\end{aligned}$$

and a positive real number σ *such that, for each* $\lambda \in \Lambda_2$, *problem* (1.1) *admits at least three weak solutions in* X *whose norms are less than* σ.

Proof. We claim that the all the assumptions of Theorem 2.2 are fulfilled with

$$w(x) = \begin{cases} \frac{d}{\alpha}(x-a) & \text{if } a \leq x < a+\alpha, \\ d & \text{if } a+\alpha \leq x \leq b-\beta, \\ \frac{d}{\beta}(b-x) & \text{if } b-\beta < x \leq b \end{cases} \tag{2.4}$$

and $r = 2mc^2/(b-a)$ where constants c, d, α and β are given in the statement of the theorem.

It is clear from (2.4) that $w \in X$ and, in particular, one has

$$\|w\|^2 = d^2(\frac{\alpha+\beta}{\alpha\beta}). \tag{2.5}$$

Moreover with this choice of w and taking into account (2.5), from (i) we get (i) of Theorem 2.2. Since $0 \leq w(x) \leq d$ for each $x \in [a,b]$, condition (ii) ensures that

$$\int_a^{a+\alpha} F(x, w(x))dx + \int_{b-\beta}^{b} F(x, w(x))dx \geq 0,$$

so from (iii) we have

$$\begin{aligned}\int_a^b \sup_{t\in[-c,c]} F(x,t)dx &< \frac{2mc^2}{b-a} \frac{\int_{a+\alpha}^{b-\beta} F(x,d)dx}{\frac{2mc^2}{b-a} + \frac{1}{2}\tilde{K}(d^2(\frac{\alpha+\beta}{\alpha\beta}))} \\ &\leq \frac{2mc^2}{b-a} \frac{\int_a^b F(x,w(x))dx}{\frac{2mc^2}{b-a} + \frac{1}{2}\tilde{K}(\|w\|^2)} \\ &= r\frac{\int_a^b F(x,w(x))dx}{r + \frac{1}{2}\tilde{K}(\|w\|^2)},\end{aligned}$$

so (ii) of Theorem 2.2 holds (note $c^2 = \frac{r(b-a)}{2m}$). Next notice that

$$\frac{\frac{1}{2}\tilde{K}(\|w\|^2)}{\int_a^b F(x,w(x))dx - \int_a^b \sup_{t\in[-\sqrt{\frac{r(b-a)}{2m}}, \sqrt{\frac{r(b-a)}{2m}}]} F(x,t)dx}$$

$$\le \frac{\frac{1}{2}\tilde{K}(d^2(\frac{\alpha+\beta}{\alpha\beta}))}{\int_{a+\alpha}^{b-\beta} F(x,d)dx - \int_a^b \sup_{t\in[-c,c]} F(x,t)dx}$$

and

$$\frac{r}{\int_a^b \sup_{t\in[-\sqrt{\frac{r(b-a)}{2m}},\sqrt{\frac{r(b-a)}{2m}}]} F(x,t)dx} = \frac{\frac{2mc^2}{b-a}}{\int_a^b \sup_{t\in[-c,c]} F(x,t)dx}.$$

In addition note that

$$\begin{aligned}
&\frac{\frac{1}{2}\tilde{K}(d^2(\frac{\alpha+\beta}{\alpha\beta}))}{\int_{a+\alpha}^{b-\beta} F(x,d)dx - \int_a^b \sup_{t\in[-c,c]} F(x,t)dx}\\
&< \frac{\frac{1}{2}\tilde{K}(d^2(\frac{\alpha+\beta}{\alpha\beta}))}{\big(\frac{\frac{2mc^2}{b-a}+\frac{1}{2}\tilde{K}(d^2(\frac{\alpha+\beta}{\alpha\beta}))}{\frac{2mc^2}{b-a}} - 1\big)\int_a^b \sup_{t\in[-c,c]} F(x,t)dx}\\
&= \frac{\frac{2mc^2}{b-a}}{\int_a^b \sup_{t\in[-c,c]} F(x,t)dx}.
\end{aligned}$$

Finally note that

$$\begin{aligned}
&\frac{hr}{2r\frac{\int_a^b F(x,w(x))dx}{\tilde{K}(\|w\|^2)} - \int_a^b \sup_{t\in[-\sqrt{\frac{r(b-a)}{2m}},\sqrt{\frac{r(b-a)}{2m}}]} F(x,t)dx}\\
&= \big(\frac{2hmc^2}{b-a}\big)\\
&\div \Big(\frac{4mc^2}{b-a}\frac{\int_a^{a+\alpha} F(x,\frac{d}{\alpha}(x-a))dx + \int_{a+\alpha}^{b-\beta} F(x,d)dx + \int_{b-\beta}^b F(x,\frac{d}{\beta}(b-x))dx}{\tilde{K}(d^2(\frac{\alpha+\beta}{\alpha\beta}))}\\
&- \int_a^b \sup_{t\in[-c,c]} F(x,t)dx\Big),
\end{aligned}$$

and taking into account that $\Lambda_1' \subseteq \Lambda_1$ we have the desired conclusion directly from Theorem 2.2. □

It is of interest to list some special cases of Corollary 2.3.

Corollary 2.4. *Assume that there exist positive constants c, d, p_1, p_2, α, β and θ with $\beta - \alpha < b - a$ such that Assumption (ii) of Corollary 2.3 holds, and*

(i) $p_1 d^2(\frac{\alpha+\beta}{\alpha\beta}) + \frac{p_2}{2} d^4(\frac{\alpha+\beta}{\alpha\beta})^2 > \frac{4p_1c^2}{b-a}$,

(ii)
$$\int_a^b \sup_{t\in[-c,c]} F(x,t)dx < \frac{2p_1c^2}{b-a}\frac{\int_{a+\alpha}^{b-\beta} F(x,d)dx}{\frac{2p_1c^2}{b-a} + \frac{p_1}{2}d^2(\frac{\alpha+\beta}{\alpha\beta}) + \frac{p_2}{4}d^4(\frac{\alpha+\beta}{\alpha\beta})^2},$$

(iii) $\frac{(b-a)^2}{2p_1}\limsup_{|t|\to+\infty}\frac{F(x,t)}{t^2} < \frac{1}{\theta}$ *uniformly with respect to $x \in [a,b]$.*

Then, for each

$$\lambda \in \Lambda_1'' = \Big]\frac{\frac{p_1}{2}d^2(\frac{\alpha+\beta}{\alpha\beta}) + \frac{p_2}{4}d^4(\frac{\alpha+\beta}{\alpha\beta})^2}{\int_{a+\alpha}^{b-\beta} F(x,d)dx - \int_a^b \sup_{t\in[-c,c]} F(x,t)dx}, \frac{\frac{2p_1c^2}{b-a}}{\int_a^b \sup_{t\in[-c,c]} F(x,t)dx}\Big[,$$

the problem

$$-(p_1+p_2\int_a^b |u'(x)|^2dx)u''=\lambda f(x,u),\\ u(a)=u(b)=0 \tag{2.6}$$

admits at least three weak solutions in X *and, moreover, for each* $\eta>1$, *there exist an open interval*

$$\begin{aligned}\Lambda_2 \subseteq [0, (\frac{2\eta p_1 c^2}{b-a})\\ \div\Big(\frac{4p_1c^2}{b-a}\frac{\int_a^{a+\alpha}F(x,\frac{d}{\alpha}(x-a))dx+\int_{a+\alpha}^{b-\beta}F(x,d)dx+\int_{b-\beta}^{b}F(x,\frac{d}{\beta}(b-x))dx}{p_1d^2(\frac{\alpha+\beta}{\alpha\beta})+\frac{p_2}{2}d^4(\frac{\alpha+\beta}{\alpha\beta})^2}\\ -\int_a^b\sup_{t\in[-c,c]}F(x,t)dx\Big)]\end{aligned}$$

and a positive real number σ *such that, for each* $\lambda\in\Lambda_2$, *problem* (2.6) *admits at least three weak solutions in* X *whose norms are less than* σ.

Proof. For fixed $p_1,p_2>0$, set $K(t)=p_1+p_2t$ for all $t\geq 0$. Bearing in mind that $m=p_1$, from (i)–(iii), we see that (i)–(iii) of Corollary 2.4 hold respectively. Also we note that there exists a continuous function $h:[0,+\infty[\to\mathbb{R}$ such that $h(tK(t^2))=t$ for all $t\geq 0$ because the function K is nondecreasing in $[0,+\infty[$ with $K(0)>0$ and $t\to tK(t^2)$ $(t\geq 0)$ is increasing and onto $[0,+\infty[$. Hence, Corollary 2.3 yields the conclusion. □

Corollary 2.5. *Assume that there exist positive constants* c, d, α, β *and* θ *with* $\beta-\alpha<b-a$ *such that Assumption (ii) in Corollary 2.3 holds, and*

(i) $d^2(\frac{\alpha+\beta}{\alpha\beta})>\frac{4c^2}{b-a}$,

(ii)
$$\int_a^b\sup_{t\in[-c,c]}F(x,t)dx<\frac{2c^2}{b-a}\frac{\int_{a+\alpha}^{b-\beta}F(x,d)dx}{\frac{2c^2}{b-a}+\frac{d^2}{2}(\frac{\alpha+\beta}{\alpha\beta})},$$

(iii) $\frac{(b-a)^2}{2}\limsup_{|t|\to+\infty}\frac{F(x,t)}{t^2}<\frac{1}{\theta}$ *uniformly with respect to* $x\in[a,b]$.

Then, for each

$$\lambda\in\Lambda_1'''=]\frac{\frac{d^2}{2}(\frac{\alpha+\beta}{\alpha\beta})}{\int_{a+\alpha}^{b-\beta}F(x,d)dx-\int_a^b\sup_{t\in[-c,c]}F(x,t)dx},\frac{\frac{2c^2}{b-a}}{\int_a^b\sup_{t\in[-c,c]}F(x,t)dx}[,$$

the problem

$$-u''=\lambda f(x,u),\\ u(a)=u(b)=0 \tag{2.7}$$

admits at least three weak solutions in X *and, moreover, for each* $\eta>1$, *there exist an open interval*

$$\begin{aligned}\Lambda_2 \subseteq [0, (\frac{2\eta c^2}{b-a})\\ \div\Big(\frac{4c^2}{b-a}\frac{\int_a^{a+\alpha}F(x,\frac{d}{\alpha}(x-a))dx+\int_{a+\alpha}^{b-\beta}F(x,d)dx+\int_{b-\beta}^{b}F(x,\frac{d}{\beta}(b-x))dx}{d^2(\frac{\alpha+\beta}{\alpha\beta})}\end{aligned}$$

$$- \int_a^b \sup_{t\in[-c,c]} F(x,t)dx\Big)]$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2$, the problem (2.7) admits at least three weak solutions in X whose norms are less than σ.

We conclude this section by giving an example to illustrate our results applying by Corollary 2.4.

Example 2.6. Consider the problem

$$\begin{gathered} -(\frac{1}{128} + \frac{1}{64}\int_0^1 |u'(x)|^2 dx)u'' = \lambda(e^{-u}u^{11}(12-u)), \\ u(0) = u(1) = 0 \end{gathered} \tag{2.8}$$

where $\lambda > 0$. Set $p_1 = \frac{1}{128}$, $p_2 = \frac{1}{64}$ and $f(x,t) = e^{-t}t^{11}(12-t)$ for all $(x,t) \in [0,1]\times\mathbb{R}$. A direct calculation yields $F(x,t) = e^{-t}t^{12}$ for all $(x,t) \in [0,1]\times\mathbb{R}$. Assumptions (i) and (ii) of Corollary 2.4 are satisfied by choosing, for example $d = 2$, $c = 1$, $[a,b] = [0,1]$ and $\alpha = \beta = 1/4$. Also, since $\limsup_{|t|\to+\infty} \frac{F(x,t)}{t^2} = 0$, Assumption (iii) of Corollary 2.4 is fulfilled. Now we can apply Corollary 2.4. Then, for each

$$\lambda \in \Lambda_1'' =]\frac{33}{2^{14}e^{-2} - 8e}, \frac{1}{64e}[$$

problem (2.8) admits at least three weak solutions in $H_0^1([0,1])$ and, moreover, for each $\eta > 1$, there exist an open interval

$$\Lambda_2 \subseteq [0, \frac{\eta}{\frac{8}{33}\Big(8^{12}\int_0^{\frac14} e^{-8t}t^{12}dt + 2^{11}e^{-2} + 8^{12}\int_{\frac34}^1 e^{-8(1-t)}(1-t)^{12}dt\Big) - 64e}]$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2$, problem (2.8) admits at least three weak solutions in $H_0^1([0,1])$ whose norms are less than σ.

3. Proof of Theorem 2.2

We begin by setting

$$\Phi(u) = \frac{1}{2}\tilde{K}(\|u\|^2), \tag{3.1}$$

$$J(u) = \int_a^b F(x,u(x))dx \tag{3.2}$$

for each $u \in X$, where $\tilde{K}$ and F are given in (2.2) and (2.3), respectively. It is well known that J is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $J'(u) \in X^*$, given by

$$J'(u)v = \int_a^b f(x,u(x))v(x)dx$$

for every $v \in X$, and that $J' : X \to X^*$ is a continuous and compact operator. Moreover, Φ is a continuously Gâteaux differentiable and sequentially weakly lower semi continuous functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)v = K(\int_a^b |u'(x)|^2dx)\int_a^b u'(x)v'(x)dx$$

for every $v \in X$. We claim that Φ' admits a continuous inverse on X (we identity X with X^*). To prove this fact, arguing as in [13] we need to find a continuous operator $T : X \to X$ such that $T(\Phi'(u)) = u$ for all $u \in X$. Let $T : X \to X$ be the operator defined by

$$T(v) = \begin{cases} \frac{h(\|v\|)}{\|v\|}v & \text{if } v \neq 0 \\ 0 & \text{if } v = 0, \end{cases}$$

where h is defined in the statement of Theorem 2.2. Since, h is continuous and $h(0) = 0$, we have that the operator T is continuous in X. For every $u \in X$, taking into account that $\inf_{t\geq 0} K(t) \geq m > 0$, we have since $h(t\,K(t^2)) = t$ for all $t \geq 0$ that

$$\begin{aligned} T(\Phi'(u)) &= T(K(\|u\|^2)u) \\ &= \frac{h(K(\|u\|^2)\|u\|)}{K(\|u\|^2)\|u\|}K(\|u\|^2)u \\ &= \frac{\|u\|}{K(\|u\|^2)\|u\|}K(\|u\|^2)u = u, \end{aligned}$$

so our claim is true. Moreover, since $m \leq K(s)$ for all $s \in [0, +\infty[$, from (3.1) we have

$$\Phi(u) \geq \frac{m}{2}\|u\|^2 \quad \text{for all } u \in X. \tag{3.3}$$

Furthermore from (iii), there exist two constants $\gamma, \tau \in \mathbb{R}$ with $0 < \gamma < 1/\theta$ such that

$$\frac{(b-a)^2}{2m}F(x,t) \leq \gamma t^2 + \tau \quad \text{for all } x \in (a,b) \text{ and all } t \in \mathbb{R}.$$

Fix $u \in X$. Then

$$F(x,u(x)) \leq \frac{2m}{(b-a)^2}(\gamma|u(x)|^2 + \tau) \quad \text{for all } x \in (a,b). \tag{3.4}$$

Fix $\lambda \in]0, +\infty[$. Then there exists $\theta > 0$ with $\lambda \in]0, \theta]$. Now since

$$\max_{x\in[a,b]} |u(x)| \leq \frac{(b-a)^{1/2}}{2}\|u\|, \tag{3.5}$$

from (3.3), (3.4) and (3.5), we have

$$\begin{aligned} \Phi(u) - \lambda J(u) &= \frac{1}{2}\tilde{K}(\|u\|^2) - \lambda\int_a^b F(x,u(x))dx \\ &\geq \frac{m}{2}\|u\|^2 - \frac{2\theta m}{(b-a)^2}\Big(\gamma\int_a^b |u(x)|^2 + \tau(b-a)\Big) \\ &\geq \frac{m}{2}\|u\|^2 - \frac{2\theta m}{(b-a)^2}\Big(\gamma\frac{(b-a)^2}{4}\|u\|^2 + \tau(b-a)\Big) \\ &= \frac{m}{2}(1-\gamma\theta)\|u\|^2 - \frac{2\theta\tau m}{b-a}, \end{aligned}$$

and so

$$\lim_{\|u\|\to+\infty}(\Phi(u) - \lambda J(u)) = +\infty.$$

Also from (3.1) and (i) we have $\Phi(w) > r$. Using (3.3) and (3.5), we obtain

$$\begin{aligned} \Phi^{-1}(]-\infty, r[) &= \big\{u \in X; \Phi(u) < r\big\} \\ &\subseteq \big\{u \in X; \|u\| < \sqrt{2r/m}\big\} \end{aligned}$$

$$\subseteq \big\{u \in X; |u(x)| \leq \sqrt{r(b-a)/(2m)},\ \text{for all}\ \ x \in [a,b]\big\},$$

so, we have

$$\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^w} J(u) \leq \int_a^b \sup_{t\in[-\sqrt{\frac{r(b-a)}{2m}},\sqrt{\frac{r(b-a)}{2m}}]} F(x,t)dx.$$

Therefore, from (ii), we have

$$\begin{aligned}\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^w} J(u) &\leq \int_a^b \sup_{t\in[-\sqrt{\frac{r(b-a)}{2m}},\sqrt{\frac{r(b-a)}{2m}}]} F(x,t)dx \\ &< \frac{r}{r+\frac{1}{2}\tilde{K}(\|w\|^2)}\int_a^b F(x,w(x))dx \\ &= \frac{r}{r+\Phi(w)}J(w).\end{aligned}$$

Now, we can apply Theorem 2.1. Note for each $x \in [a,b]$,

$$\frac{r}{\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^w} J(u)} \geq \frac{r}{\int_a^b \sup_{t\in[-\sqrt{\frac{r(b-a)}{2m}},\sqrt{\frac{r(b-a)}{2m}}]} F(x,t)dx}$$

and

$$\begin{aligned}&\frac{\Phi(w)}{J(w)-\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^w} J(u)} \\ &\leq \frac{\frac{1}{2}\tilde{K}(\|w\|^2)}{\int_a^b F(x,w(x))dx - \int_a^b \sup_{t\in[-\sqrt{\frac{r(b-a)}{2m}},\sqrt{\frac{r(b-a)}{2m}}]} F(x,t)dx}.\end{aligned}$$

Note also that (ii) immediately implies

$$\begin{aligned}&\frac{\frac{1}{2}\tilde{K}(\|w\|^2)}{\int_a^b F(x,w(x))dx - \int_a^b \sup_{t\in[-\sqrt{\frac{r(b-a)}{2m}},\sqrt{\frac{r(b-a)}{2m}}]} F(x,t)dx} \\ &< \frac{\frac{1}{2}\tilde{K}(\|w\|^2)}{\big(\frac{r+\frac{1}{2}\tilde{K}(\|w\|^2)}{r}\ -\ 1\big)\int_a^b \sup_{t\in[-\sqrt{\frac{r(b-a)}{2m}},\sqrt{\frac{r(b-a)}{2m}}]} F(x,t)dx} \\ &= \frac{r}{\int_a^b \sup_{t\in[-\sqrt{\frac{r(b-a)}{2m}},\sqrt{\frac{r(b-a)}{2m}}]} F(x,t)dx}.\end{aligned}$$

Also

$$\begin{aligned}&\frac{\eta r}{r\frac{J(w)}{\Phi(w)} - \sup_{u\in\overline{\Phi^{-1}(-\infty,r[)}^w} J(u)} \\ &\leq \frac{\eta r}{2r\frac{\int_a^b F(x,w(x))dx}{\tilde{K}(\|w\|^2)} - \int_a^b \sup_{t\in[-\sqrt{\frac{r(b-a)}{2m}},\sqrt{\frac{r(b-a)}{2m}}]} F(x,t)dx} = \rho.\end{aligned}$$

Note from (ii) that

$$2r\frac{\int_a^b F(x,w(x))dx}{\tilde{K}(\|w\|^2)} - \int_a^b \sup_{t\in[-\sqrt{\frac{r(b-a)}{2m}},\sqrt{\frac{r(b-a)}{2m}}]} F(x,t)dx$$

$$
\begin{aligned}
&> \Big(\frac{2r}{\tilde{K}(\|w\|^2)} - \frac{r}{r+\frac{1}{2}\tilde{K}(\|w\|^2)}\Big)\int_a^b F(x,w(x))dx \\
&\geq \Big(\frac{2r}{\tilde{K}(\|w\|^2)} - \frac{2r}{\tilde{K}(\|w\|^2)}\Big)\int_a^b F(x,w(x))dx = 0
\end{aligned}
$$

since $\int_a^b F(x,w(x))dx \geq 0$ (note $F(x,0)=0$ so

$$
\int_a^b \sup_{t\in[-\sqrt{\frac{r(b-a)}{2m}},\sqrt{\frac{r(b-a)}{2m}}]} F(x,t)dx \geq 0
$$

and now apply (ii). Now with $x_0 = 0$, $x_1 = w$ from Theorem 2.1 (note $J(0) = 0$ from (2.3)) it follows that, for each $\lambda \in \Lambda_1$, the problem (1.1) admits at least three weak solutions and there exist an open interval $\Lambda_2 \subseteq [0,\rho]$ and a real positive number σ such that, for each $\lambda \in \Lambda_2$, the problem (1.1) admits at least three weak solutions that whose norms in X are less than σ.

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Shapour Heidarkhani

Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran

E-mail address: s.heidarkhani@razi.ac.ir

Ghasem Alizadeh Afrouzi
Department of Mathematics, Faculty of Basic Sciences, University of Mazandaran, 47416-1467 Babolsar, Iran
E-mail address: afrouzi@umz.ac.ir

Donal O'Regan
Department of Mathematics, National University of Ireland, Galway, Ireland
E-mail address: donal.oregan@nuigalway.ie