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COMPACT INVERSES OF MULTIPOINT NORMAL DIFFERENTIAL OPERATORS FOR FIRST ORDER

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Abstract. In this work, we describe all normal extensions of a multipoint minimal operators generated by linear multipoint differential-operator expressions for first order in the Hilbert space of vector functions, in terms of boundary values at the endpoints of infinitely many separated subintervals. Also we investigate compactness properties of the inverses of such extensions.

1. Introduction

It is known that the traditional infinite direct sum of Hilbert spaces H_n, $n \geq 1$ and infinite direct sum of operators A_n in H_n, $n \geq 1$ are defined as

$$H = \oplus_{n=1}^{\infty} H_n = \{u = (u_n) : u_n \in H_n,\ n \geq 1,\ \sum_{n=1}^{\infty} \|u_n\|_{H_n}^2 < +\infty\},$$
$$A = \oplus_{n=1}^{\infty} A_n,$$
$$D(A) = \{u = (u_n) \in H : u_n \in D(A_n),\ n \geq 1,\ Au = (A_n u_n) \in H\}.$$

Note that H is a Hilbert space with norm induced by the inner product

$$(u, v)_H = \sum_{n=1}^{\infty} (u_n, v_n)_{H_n}, \quad u, v \in H\,;$$

see [2, 9, 10]. The general theory of linear closed operators in Hilbert spaces and its applications to physical problems have been investigated by many researches (see for example [2, 11]). Furthermore, many physical problems require studying the theory of linear operators in direct sums in Hilbert spaces. This is the case in [3, 5, 7, 12, 13, 14] and their references, which is the motivation for this work. We note that a detail analysis of normal subspaces and operators in Hilbert spaces have been studied in [1] and the references there in.

Besides the introduction, this study contains three sections. In section 2, the multipoint minimal and maximal operators for the first order differential-operator expression are determined. In section 3, all normal extensions of multipoint formally normal operators are described in terms of boundary values in the endpoints of the

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infinitely many separated subintervals. Finally in section 4, compactness properties of the inverses of such extensions have been established.

2. Minimal and maximal operators

Throughout this work (a_n) and (b_n) will be sequences of real numbers such that

$$-\infty < a_n < b_n < a_{n+1} < \dots < +\infty,$$

H_n is a Hilbert space, $\Delta_n = (a_n, b_n)$, $L_n^2 = L^2(H_n, \Delta_n)$, $L^2 = \oplus_{n=1}^{\infty} L^2(H_n, \Delta_n)$, $n \geq 1$, $\sup_{n\geq 1}(b_n - a_n) < +\infty$, $W_2^1 = \oplus_{n=1}^{\infty} W_2^1(H_n, \Delta_n)$, $\overset{\circ}{W}{}_2^1 = \oplus_{n=1}^{\infty} \overset{\circ}{W}{}_2^1(H_n, \Delta_n)$, $H = \oplus_{n=1}^{\infty} H_n$, $\mathrm{cl}(T)$ is the closure of the operator T. $l(\cdot)$ is a linear multipoint differential-operator expression for first order in L^2 in the following form:

$$l(u) = (l_n(u_n)), \tag{2.1}$$

and for each $n \geq 1$,

$$l_n(u_n) = u_n' + A_n u_n, \tag{2.2}$$

where $A_n : D(A_n) \subset H_n \to H_n$ is a linear positive defined selfadjoint operator in H_n. It is clear that formally adjoint expression to (2.2) in the Hilbert space L_n^2 is in the form

$$l_n^+(v_n) = -v_n' + A_n v_n, n \geq 1. \tag{2.3}$$

We define an operator L_{n0}' on the dense manifold of vector functions D_{n0}' in L_n^2 as

$$\begin{aligned} D_{n0}' := \Big\{ u_n \in L_n^2 : u_n = \sum_{k=1}^{m} \phi_k f_k, \phi_k \in C_0^{\infty}(\Delta_n), \\ f_k \in D(A_n), k = 1, 2, \dots, m; m \in \mathbb{N} \Big\} \end{aligned}$$

with $L_{n0}' u_n := l_n(u_n), n \geq 1$. Since the operator $A_n > 0$, $n \geq 1$, then the relation

$$\mathrm{Re}(L_{n0}' u_n, u_n)_{L_n^2} = 2(A_n u_n, u_n)_{L_n^2} \geq 0, \quad u_n \in D_{n0}'$$

implies that L_{n0}' is an accretive in L_n^2, $n \geq 1$. Hence the operator L_{n0}' has a closure in L_n^2, $n \geq 1$. The closure $\mathrm{cl}(L_{n0}')$ of the operator L_{n0}' is called the minimal operator generated by differential-operator expression (2.2) and is denoted by L_{n0} in L_n^2, $n \geq 1$. The operator L_0 defined by

$$D(L_0) := \{u = (u_n) : u_n \in D(L_{n0}),\ n \geq 1,\ \sum_{n=1}^{\infty} \|L_{n0} u_n\|_{L_n^2}^2 < +\infty\}$$

with $L_0 u := (L_{n0} u_n)$, $u \in D(L_0)$, $L_0 : D(L_0) \subset L^2 \to L^2$ is called a minimal operator (multipoint) generated by differential-operator expression (2.1) in Hilbert space L^2 and denoted by $L_0 = \oplus_{n=1}^{\infty} L_{n0}$. In a similar way the minimal operator for two points denoted by L_{n0}^+ in $L_n^2, n \geq 1$ for the formally adjoint linear differential-operator expression (2.3) can be constructed. In this case the operator L_0^+ defined by

$$D(L_0^+) := \{v := (v_n) : v_n \in D(L_{n0}^+),\ n \geq 1,\ \sum_{n=1}^{\infty} \|L_{n0}^+ v_n\|_{L_n^2}^2 < +\infty\}$$

with $L_0^+ v := (L_{n0}^+ v_n)$, $v \in D(L_0^+)$, $L_0^+ : D(L_0^+) \subset L^2 \to L^2$ is called a minimal operator (multipoint) generated by $l^+(v) = (l_n^+(v_n))$ in the Hilbert space L^2 and denoted by $L_0^+ = \oplus_{n=1}^{\infty} L_{n0}^+$.

Now we state the following relevant result.

Theorem 2.1. *The minimal operators L_0 and L_0^+ are densely defined closed operators in L^2.*

Proof. Let $w = (w_n)$ be any element in $L^2 = \oplus_{n=1}^{\infty} L_n^2$ and ε be an arbitrary positive number. In this case $w_n \in L_n^2$, $n \geq 1$ and $\sum_{n=1}^{\infty} \|w_n\|_{L_n^2}^2 < +\infty$. Since the linear manifold $D(L_{n0})$, $n \geq 1$ is densely defined in L_n^2, $n \geq 1$, then there exist $w_n(\varepsilon) \in D(L_{n0})$ such that

$$\|w_n - w_n(\varepsilon)\|_{L_n^2} < \frac{\varepsilon}{n}, \quad n \geq 1.$$

Then for the element $w(\varepsilon) = (w_n(\varepsilon))$ we have

$$\begin{aligned}
\|w(\varepsilon)\|_{L^2}^2 &= \sum_{n=1}^{\infty} \|w_n(\varepsilon)\|_{L_n^2}^2 \\
&\leq \sum_{n=1}^{\infty} (\|w_n - w_n(\varepsilon)\|_{L_n^2} + \|w_n(\varepsilon)\|_{L_n^2})^2 \\
&\leq 2\sum_{n=1}^{\infty} \|w_n - w_n(\varepsilon)\|_{L_n^2}^2 + 2\sum_{n=1}^{\infty} \|w_n(\varepsilon)\|_{L_n^2}^2 \\
&\leq 2\varepsilon^2 \sum_{n=1}^{\infty} \frac{1}{n^2} + 2\sum_{n=1}^{\infty} \|w_n(\varepsilon)\|_{L_n^2}^2 < +\infty;
\end{aligned}$$

that is, $w(\varepsilon) \in L^2$. On the other hand since

$$\|w - w(\varepsilon)\|_{L^2}^2 = \sum_{n=1}^{\infty} \|w_n - w_n(\varepsilon)\|_{L_n^2}^2 \leq \varepsilon^2 \sum_{n=1}^{\infty} \frac{1}{n^2} < 2\varepsilon^2,$$

the linear manifold $D(L_0)$ is dense in L^2.

Now we show that the minimal operator L_0 is closed in L^2. Let $w^{(m)} \subset D(L_0)$ be any sequence such that for $w, z \in L^2$, $w^{(m)} \to w$ as $m \to \infty$ and $L_0 w^{(m)} \to z$ as $m \to \infty$ in L^2. In this situation, in the space L_n^2, $w_n^{(m)} \to w_n$ as $m \to \infty$ and $L_{n0} w_n^{(m)} \to z_n$ as $m \to \infty$, $n \geq 1$, where $w = (w_n)$ and $z = (z_n)$. Since the operator L_{n0} is closed in L_n^2, then $w_n \in D(L_{n0})$ and $z_n = L_{n0} w_n, n \geq 1$.

The above relations and $(w_n), (z_n) \in L^2$ imply that $w = (w_n) \in D(L_0)$ and $z = L_0 w$. In similar way it could be shown that the minimal operator L_0^+ is densely defined and closed operator in $\in L^2$.

The operators in L^2 defined by $L := (L_0^+)^* = \oplus_{n=1}^{\infty} L_n$ and $L^+ := (L_0)^* = \oplus_{n=1}^{\infty} L_n^+$ are called maximal operators (multipoint) for the differential-operator expression $l(\cdot)$ and $l^+(\cdot)$ respectively.It is clear that $Lu = (l_n(u_n)), u \in D(L)$,

$$D(L) := \{u = (u_n) \in L^2 : u_n \in D(L_n),\ n \geq 1,\ \sum_{n=1}^{\infty} \|L_n u_n\|_{L_n^2}^2 < \infty\},$$

$$L^+ v = (l_n^+(v_n)), v \in D(L^+),$$

$$D(L^+) := \{v = (v_n) \in L^2 : v_n \in D(L_n^+),\ n \geq 1,\ \sum_{n=1}^{\infty} \|L_n^+ v_n\|_{L_n^2}^2 < \infty\}$$

and $L_0 \subset L$, $L_0^+ \subset L^+$. □

From [8] and the definition of direct sum of operators, the validity of following theorem is clear.

Theorem 2.2. *The domain of the operator L consists of $u = (u_n) \in L^2$ such that: (1) for each $n \geq 1$ vector function $u_n \in L^2_n$, u_n is absolutely continuous in the interval Δ_n; (2) $l_n(u_n) \in L^2_n$, $n \geq 1$; (3) $l(u) = (l_n(u_n)) \in L^2\}$; i.e.,*

$$D(L) = \{u = (u_n) \in L^2 : u_n \in D(L_n),\ n \geq 1,\ l(u) = (l_n(u_n)) \in L^2\}.$$

The domain of L_0 is $D(L_0) = \{u = (u_n) \in D(L) : u_n(a_n) = u_n(b_n) = 0,\ n \geq 1\}$.

Remark 2.3. If $A_n \in B(H)$, $n \geq 1$ and $\sup_{n\geq 1} \|A_n\| \leq c < +\infty$, then for any $u = (u_n) \in L^2$ we have $(Au) = (A_n u_n) \in L^2$.

Theorem 2.4. *If a minimal operator L_0 is formally normal in L^2, then $D(L_0) \subset \overset{\circ}{W}{}^1_2$ and $AD(L_0) \subset L^2$.*

Proof. In this case for every $u = (u_n) \in D(L_0) \subset D(L^+)$ we have $u' + Au \in L^2$ and $-u' + Au \in L^2$. From this it is obtained that $u' \in L^2, Au \in L^2$. This means that $D(L_0) \subset \overset{\circ}{W}{}^1_2$ and $AD(L_0) \subset L^2$. $\square$

Theorem 2.5. *If $A^{1/2}W^1_2 \subset W^1_2$, then minimal operator L_0 is formally normal in L^2.*

Proof. In this case from the relations

$$L_0^+ u = L_0 u - 2Au, u \in D(L_0), \quad L_0 u = L_0^+ u + 2Au, u \in D(L_0^+)$$

imply that $D(L_0) = D(L_0^+)$. Since $D(L_0^+) \subset D(L_0^*) = D(L^+)$, it is obtained that $D(L_0) \subset D(L^+)$.

On the other hand for any $u \in D(L_0)$,

$$\begin{aligned}
\|L_0 u\|^2_{L^2} &= (u' + Au, u' + Au)_{L^2} \\
&= \|u'\|^2_{L^2} + [(u', Au)_{L^2} + (Au, u')_{L^2}] + \|Au\|^2_{L^2} \\
&= \|u'\|^2_{L^2} + \|Au\|^2_{L^2}
\end{aligned}$$

and

$$\begin{aligned}
\|L^+ u\|^2_{L^2} &= (-u' + Au, -u' + Au)_{L^2} \\
&= \|u'\|^2_{L^2} - [(u', Au)_{L^2} + (Au, u')_{L^2}] + \|Au\|^2_{L^2} \\
&= \|u'\|^2_{L^2} + \|Au\|^2_{L^2}.
\end{aligned}$$

Thus, it is established that operator L_0 is formally normal in L^2. $\square$

Remark 2.6. If $A_n \in B(H), n \geq 1$ and $\sup_{n\geq 1} \|A_n\| \leq c < +\infty$, then $D(L_0) = D(L_0^+)$ and $D(L) = D(L^+)$.

If $AW^1_2 \subset L^2$, then $D(L_0) = D(L_0^+)$ and $D(L) = D(L^+)$.

3. Description of normal extensions of the minimal operator

In this section the main purpose is to describe all normal extensions of the minimal operator L_0 in L^2 in terms in the boundary values of the endpoints of the subintervals. Firstly, we will show that there exists normal extension of the minimal operator L_0. Consider the following extension of the minimal operator L_0,

$$\widetilde{L}u := u' + Au, \quad AW^1_2 \subset W^1_2,$$

$$D(\widetilde{L}) = \{u = (u_n) \in W^1_2 : u_n(a_n) = u_n(b_n), n \geq 1\}.$$

Under the condition on the coefficient A, we have

$$\begin{aligned}(\widetilde{L}u, v)_{L^2} &= (u', v)_{L^2} + (Au, v)_{L^2}\\ &= (u, v)'_{L^2} + (u, -v' + Av)_{L^2}\\ &= \sum_{n=1}^{\infty}[(u_n(b_n), v_n(b_n))_{H_n} - (u_n(a_n), v_n(a_n))_{H_n}] + (u, -v' + Av)_{L^2}\end{aligned}$$

From this it is obtained that

$$\widetilde{L}^*v := -v' + Av,$$
$$D(\widetilde{L}^*) = \{v = (v_n) \in W_2^1 : v_n(a_n) = v_n(b_n), n \geq 1\}.$$

In this case it is clear that $D(\widetilde{L}) = D(\widetilde{L}^*)$. On the other hand, since for each $u \in D(\widetilde{L})$,

$$\|\widetilde{L}u\|_{L^2}^2 = \|u'\|_{L^2}^2 + [(u', Au)_{L^2} + (Au, u')_{L^2}] + \|Au\|_{L^2}^2,$$
$$\|\widetilde{L}^*u\|_{L^2}^2 = \|u'\|_{L^2}^2 - [(u', Au)_{L^2} + (Au, u')_{L^2}] + \|Au\|_{L^2}^2$$

and

$$\begin{aligned}(u', Au)_{L^2} + (Au, u')_{L^2} &= (u, Au)'_{L^2}\\ &= \sum_{n=1}^{\infty}[(u_n(b_n), A_nu_n(b_n))_{H_n} - (u_n(a_n), A_nu_n(a_n))_{H_n}] = 0.\end{aligned}$$

Then $\|\widetilde{L}u\|_{L^2} = \|\widetilde{L}^*u\|_{L^2}$ for every $u \in D(\widetilde{L})$. Consequently, $\widetilde{L}$ is a normal extension of the minimal operator L_0.

The following result establishes the relationship between normal extensions of L_0 and normal extensions of L_{n0}, $n \geq 1$.

Theorem 3.1. *The extension $\widetilde{L} = \oplus_{n=1}^{\infty}\widetilde{L_n}$ of the minimal operator L_0 in L^2 is a normal if and only if for any $n \geq 1$, $\widetilde{L_n}$ is so in L_n^2.*

Proof. Let $\widetilde{L} = \oplus_{n=1}^{\infty}\widetilde{L_n}$ be a normal extension of the minimal operator $L_0 = \oplus_{n=1}^{\infty}L_{n0}$. In this case it is clear that for each $n \geq 1$ an operator $\widetilde{L_n} : D(\widetilde{L_n}) \subset L_n^2 \to L_n^2$ defined by

$$\widetilde{L_n}u_n = L_nu_n, \quad D(\widetilde{L_n}) = P_nD(\widetilde{L}),$$

where P_n is an orthogonal projection operator from L^2 to L_n^2, is a normal extension of the minimal operator L_{n0}. On the contrary, assume that $\widetilde{L_n} : D(\widetilde{L_n}) \subset L_n^2 \to L_n^2$ is any normal extension of the minimal operator for each $n \geq 1$, and $\widetilde{L} := \oplus_{n=1}^{\infty}\widetilde{L_n}$. In this case for any $u = (u_n) \in D(\widetilde{L})$ we have $\|\widetilde{L}u\|_{L^2} = \sum_{n=1}^{\infty}\|\widetilde{L_n}u_n\|_{L_n^2}^2 < +\infty$. Since for any $n \geq 1$, $\widetilde{L_n}$ is a normal extension, it follows that $u_n \in D(\widetilde{L_n})$, $n \geq 1$ and $\sum_{n=1}^{\infty}\|\widetilde{L_n}^*u_n\|_{L_n^2}^2 = \sum_{n=1}^{\infty}\|\widetilde{L_n}u_n\|^2 < +\infty$. From this for every $u = (u_n) \in D(\widetilde{L})$ it is obtained that $(u_n) \in D(\widetilde{L}^*)$; i.e., $D(\widetilde{L}) \subset D(\widetilde{L}^*)$.

In the similar way it can be shown that $D(\widetilde{L}^*) \subset D(\widetilde{L})$. Hence $D(\widetilde{L}) = D(\widetilde{L}^*)$. On the other hand for any $n \geq 1$ since $\widetilde{L_n}$ is a normal extension, then for each $u = (u_n) \in D(\widetilde{L})$,

$$\|\widetilde{L}u\|_{L^2}^2 = \sum_{n=1}^{\infty}\|\widetilde{L_n}u_n\|_{L_n^2}^2 = \sum_{n=1}^{\infty}\|\widetilde{L_n}^*u_n\|_{L_n^2}^2 = \|\widetilde{L}^*u\|_{L^2}^2 < +\infty.$$

This complete the proof. $\square$

Now using Theorem 3.1 and [8] we can formulate the following main result of this section, where it is given a description of all normal extension of the minimal operator L_0 in L^2 in terms of boundary values of vector functions at the endpoints of subintervals.

Theorem 3.2. *Let $A^{1/2}W_2^1 \subset W_2^1$. If $\widetilde{L} = \oplus_{n=1}^{\infty}\widetilde{L_n}$ is a normal extension of the minimal operator L_0 in L^2, then it is generated by differential-operator expression* (2.1) *with boundary conditions*

$$u_n(b_n) = W_n u_n(a_n), u_n \in D(L_n), \tag{3.1}$$

where W_n is a unitary operator in H_n and $W_n A_n^{-1} = A_n^{-1} W_n, n \geq 1$. The unitary operator $W = \oplus_{n=1}^{\infty} W_n$ in $H = \oplus_{n=1}^{\infty} H_n$ is determined uniquely by the extension $\widetilde{L}$; that is, $\widetilde{L} = L_W$.

On the contrary, the restriction of the maximal operator L to the linear manifold $u \in D(L)$ satisfying the condition (3.1) with any unitary operator $W = \oplus_{n=1}^{\infty} W_n$ in H with property $WA^{-1} = A^{-1}W$ is a normal extension of the minimal operator L_0 in L^2.

4. Some compactness properties of the normal extensions

In this work $\mathscr{H}_n, \mathfrak{H}_\mathfrak{n}, H_n, n \geq 1$ will denoted Hilbert spaces. First, we prove the following result.

Theorem 4.1. *For the point spectrum of $\mathscr{A} = \oplus_{n=1}^{\infty}\mathscr{A}_n$ in the direct sum $\mathscr{H} = \oplus_{n=1}^{\infty}\mathscr{H}_n$ of Hilbert spaces $\mathscr{H}_n, n \geq 1$, it is true that*

$$\sigma_p(\mathscr{A}) = \cup_{n=1}^{\infty}\sigma_p(\mathscr{A}_n)$$

Proof. Let $\lambda \in \sigma_p(\mathscr{A})$. In this case there exist non zero element $u = (u_n) \in D(\mathscr{A})$ such that $Au = \lambda u, u \neq 0$; i.e., $A_n u_n = \lambda u_n, n \geq 1$. Since $u \neq 0$, then there exist some $m \in \mathbb{N}$ such that $u_m \neq 0$ and $A_m u_m = \lambda u_m$. This means that $\lambda \in \sigma_p(\mathscr{A}_m)$. From this it is obtained that $\sigma_p(\mathscr{A}) \subset \cup_{n=1}^{\infty}\sigma_p(\mathscr{A}_n)$. On the contrary, if for any $m \in \mathbb{N}$, $\lambda \in \sigma_p(A_m)$, then there exist $u_m \in D(A_m)$ such that $u_m \neq 0$ and $A_m u_m = \lambda u_m$. In this case, if we choose the element $u_* := \{0, 0, \dots, u_m, 0, \dots\}$, then $u_* \in D(\mathscr{A}), u_* \neq 0$ and $Au_* = \lambda u_*$. Hence $\lambda \in \sigma_p(\mathscr{A})$. This implies that $\cup_{n=1}^{\infty}\sigma_p(\mathscr{A}_n) \subset \sigma_p(\mathscr{A})$. Therefore, the claim of the theorem is valid. $\square$

Theorem 4.2 ([10]). *Let $\mathscr{A}_n \in B(\mathscr{H}_n)$, $n \geq 1$, $\mathscr{A} = \oplus_{n=1}^{\infty}\mathscr{A}_n$ and $\mathscr{H} = \oplus_{n=1}^{\infty}\mathscr{H}_n$. In order to $\mathscr{A} \in B(\mathscr{H})$ the necessary and sufficient condition is $\sup_{n\geq 1}\|\mathscr{A}_n\| < +\infty$. In this case $\|\mathscr{A}\| = \sup_{n\geq 1}\|\mathscr{A}_n\|$.*

Let $C_\infty(\cdot)$ and $C_p(\cdot), 1 \leq p < \infty$ denote the class of compact operators and the Schatten-von Neumann subclasses of compact operators in corresponding spaces respectively.

Definition 4.3 ([4]). *Let T be a linear closed and densely defined operator in any Hilbert space $\mathscr{H}$. If $\rho(T) \neq \emptyset$ and for $\lambda \in \rho(T)$ the resolvent operator $R_\lambda(T) \in C_\infty(\mathscr{H})$, then operator $T : D(T) \subset \mathscr{H} \to \mathscr{H}$ is called an operator with discrete spectrum.*

First we note the following result.

Theorem 4.4. *If the operator $\mathscr{A} = \oplus_{n=1}^{\infty}\mathscr{A}_n$ as an operator with discrete spectrum in $\mathscr{H} = \oplus_{n=1}^{\infty}\mathscr{H}_n$, then for every $n \geq 1$ the operator $\mathscr{A}_n$ is so in $\mathscr{H}_n$.*

Remark 4.5. Unfortunately, the converse of the Theorem 4.4 is not true in general case. Indeed, consider the sequence of operators $\mathscr{A}_n u_n = u_n, 0 < \dim \mathscr{H}_n = d_n < \infty$, $n \geq 1$. In this case for every $n \geq 1$ operator $\mathscr{A}_n$ is an operator with discrete spectrum. But an inverse of the direct sum operator $\mathscr{A} = \oplus_{n=1}^{\infty}\mathscr{A}_n$ is not compact operator in $\mathscr{H} = \oplus_{n=1}^{\infty}\mathscr{H}_n$, because $\dim \mathscr{H} = \infty$ and $\mathscr{A}$ is an identity operator in $\mathscr{H}$.

Theorem 4.6. *If $\mathscr{A} = \oplus_{n=1}^{\infty}\mathscr{A}_n, \mathscr{A}_n$ is an operator with discrete spectrum in $\mathscr{H}_n$, $n \geq 1$, $\cap_{n=1}^{\infty}\rho(\mathscr{A}_n) \neq \emptyset$ and for any $\lambda \in \cap_{n=1}^{\infty}\rho(\mathscr{A}_n)$, $\lim_{n\to\infty} \|R_\lambda(\mathscr{A}_n)\| = 0$, then $\mathscr{A}$ is an operator with discrete spectrum in $\mathscr{H}$.*

Proof. For each $\lambda \in \cap_{n=1}^{\infty}\rho(\mathscr{A}_n)$ we have $R_\lambda(\mathscr{A}_n) \in C_\infty(\mathscr{H}_n)$, $n \geq 1$. Now define the operators $\mathscr{K}_m : \mathscr{H} \to \mathscr{H}$, $m \geq 1$, as

$$\mathscr{K}_m := \{R_\lambda(\mathscr{A}_1)u_1, R_\lambda(\mathscr{A}_2)u_2, \dots, R_\lambda(\mathscr{A}_m)u_m, 0, 0, \dots\}, \quad u = (u_n) \in \mathscr{H}.$$

The convergence of the operators $\mathscr{K}_m$ to the operator $\mathscr{K}$ in operator norm will be investigated. For $u = (u_n) \in \mathscr{H}$, we have

$$\begin{aligned} \|\mathscr{K}_m u - \mathscr{K} u\|_{\mathscr{H}}^2 &= \sum_{n=m+1}^{\infty} \|R_\lambda(\mathscr{A}_n)u_n\|_{\mathscr{H}_n}^2 \\ &\leq \sum_{n=m+1}^{\infty} \|R_\lambda(\mathscr{A}_n)\|^2 \|\|u_n\|_{\mathscr{H}_n}^2 \\ &\leq \Big(\sup_{n\geq m+1} \|R_\lambda(\mathscr{A}_n)\|\Big)^2 \sum_{n=1}^{\infty} \|u_n\|_{\mathscr{H}_n}^2 \\ &= \Big(\sup_{n\geq m+1} \|R_\lambda(\mathscr{A}_n)\|\Big)^2 \|u\|_{\mathscr{H}}^2 \end{aligned}$$

thus we get $\|\mathscr{K}_m u - \mathscr{K} u\| \leq \sup_{n\geq m+1} \|R_\lambda(\mathscr{A}_n)\|$, $m \geq 1$. This implies that sequence of operators $(\mathscr{K}_m)$ converges in operator norm to the operator $\mathscr{K}$. Then by the important theorem of the theory of compact operators it implies that $\mathscr{K} \in C_\infty(\mathscr{H})$ [2], because for any $m \geq 1$, $\mathscr{K}_m \in C_\infty(\mathscr{H})$. □

Using the Theorem 3.2 and Theorem 4.6 can be proved the following result.

Theorem 4.7. *If $A_n^{-1} \in C_\infty(H_n)$, $n \geq 1$, $\sup_{n\geq 1}(b_n - a_n) < \infty$ and the sequence of first minimal eigenvalues $\lambda_1(A_n)$ of the operators A_n, $n \geq 1$ satisfy the condition*

$$\lambda_1(A_n) \to \infty \quad \text{as } n \to \infty,$$

then the normal extension $\widetilde{L}$ of the minimal operator L_0 is an operator with discrete spectrum in L^2.

From the definition of the characteristic numbers $\mu(.)$ of any compact operator in any Hilbert space [2] and Theorem 4.1 it is easy to prove that the validity of the following result.

Theorem 4.8. (i) *If $\mathscr{A} = \oplus_{n=1}^{\infty}\mathscr{A}_n$, $\mathscr{H} = \oplus_{n=1}^{\infty}\mathscr{H}_n$ and $\mathscr{A} \in C_\infty(\mathscr{H})$, then for every $n \geq 1$, $\mathscr{A}_n \in C_\infty(\mathscr{H}_n)$ and $\{\mu_m(\mathscr{A}) : m \geq 1\} = \cup_{n=1}^{\infty}\{\mu_k(\mathscr{A}_n) : k \geq 1\}$;*

(ii) *If* $\mathscr{A} = \oplus_{n=1}^{\infty}\mathscr{A}_n$, $\mathscr{H} = \oplus_{n=1}^{\infty}\mathscr{H}_n$ *and* $\mathscr{A} \in C_p(\mathscr{H})$, $1 \le p \le +\infty$, *then for every* $n \ge 1$, $\mathscr{A}_n \in C_p(\mathscr{H}_n)$.

Theorem 4.9. *Let* $\mathscr{H} = \oplus_{n=1}^{\infty}\mathscr{H}_n$, $\mathscr{A} = \oplus_{n=1}^{\infty}\mathscr{A}_n$ *and* $\mathscr{A}_n \in C_p(\mathscr{H}_n)$, $n \ge 1$, $1 \le p < \infty$. *In this case* $\mathscr{A} \in C_p(\mathscr{H})$ *if and only if the series* $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_k^p(\mathscr{A}_n)$ *is convergent.*

Proof. Let us $\mathscr{A} \in C_p(\mathscr{H})$. Then the series $\sum_{m=1}^{\infty}\mu_m^p(\mathscr{A})$ is convergent. In this case by the Theorem 4.8 (i) and important theorem on the convergence of rearrangement series it is obtained that the series $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_k^p(\mathscr{A}_n)$ is convergent.

On the contrary, if the series $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_k^p(\mathscr{A}_n)$ is convergent, then the series $\sum_{m=1}^{\infty}\mu_m^p(\mathscr{A})$ being a rearrangement of the above series, is also convergent. So $\mathscr{A} \in C_p(\mathscr{H})$. □

Now we will present an application the last theorem. For all $n \ge 1$, Let $\mathfrak{H}_n$ be a Hilbert space, $\Delta_n = (a_n, b_n)$, $-\infty < a_n < b_n < a_{n+1} < \dots < \infty$, $A_n : D(A_n) \subset \mathfrak{H}_n \to \mathfrak{H}_n$, $A_n = A_n^* \ge E, U_n : \mathfrak{H}_n \to \mathfrak{H}_n$ is unitary operator, $A_n^{-1}U_n = U_nA_n^{-1}$, $L_{U_n}u_n = u_n' + A_nu_n, A_nW_2^1(\mathfrak{H}_n, \Delta_n) \subset W_2^1(\mathfrak{H}_n, \Delta_n)$,

$$H_n = L^2(\mathfrak{H}_n, \Delta_n), D(L_{U_n}) = \{u_n \in W_2^1(\mathfrak{H}_n, \Delta_n) : u_n(b_n) = U_nu_n(a_n)\},$$

$L_{U_n} : H_n \to H_n$, $U = \oplus_{n=1}^{\infty}U_n$, $L_U = \oplus_{n=1}^{\infty}L_{U_n}$, $H = \oplus_{n=1}^{\infty}H_n$ and $h = \sup_{n\ge1}(b_n - a_n) < \infty$.

Since for all $n \ge 1$, U_n is a unitary operator in $\mathfrak{H}_n$, then L_{U_n} is normal operator in H_n [6]. Also for $L_U : D(L_U) \subset H \to H$, the relation $L_UL_{U^*} = L_{U^*}L_U$ is true; i.e., L_U is a normal operator in H. It is known that if $A_n^{-1} \in C_p(\mathfrak{H}_n)$ for $p > 1$, then $L_{U_n}^{-1} \in C_{2p}(H_n), p > 1$ for all $n \ge 1$[6]. On the other hand, if $A_n^{-1} \in C_\infty(\mathfrak{H}_n), n \ge 1$, then eigenvalues $\lambda_q(L_{U_n})$, $q \ge 1$ of operator L_{U_n} is in the form

$$\lambda_q(L_{U_n}) = \lambda_m(A_n) + \frac{i}{a_n - b_n}(\arg\lambda_m(U_n^*e^{(-A_n(b_n-a_n))}) + 2k\pi),$$

where $m \ge 1$, $k \in \mathbb{Z}$, $n \ge 1$, $q = q(m,k) \in \mathbb{N}$. Therefore, we have the following result.

Theorem 4.10. *If* $A = \oplus_{n=1}^{\infty}A_n, \mathfrak{H} = \oplus_{n=1}^{\infty}\mathfrak{H}_n$ *and* $A^{-1} \in C_{p/2}(\mathfrak{H}), 2 < p < \infty$, *then* $L_U^{-1} \in C_p(H)$.

Proof. The operator L_U is a normal in H. Consequently, for the characteristic numbers of normal operator L_U^{-1} an equality $\mu_q(L_U^{-1}) = |\lambda_q(L_U^{-1})|$, $q \ge 1$ holds [2]. Now we search for convergence of the series $\sum_{q=1}^{\infty}\mu_q^p(L_U^{-1})$, $2 < p < \infty$.

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\mu_q^p(L_{U_n}^{-1})\\
&= \sum_{n=1}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{m=1}^{\infty}\Big(\lambda_m^2(A_n) + \frac{1}{(b_n-a_n)^2}(\delta(m,n) + 2k\pi)^2\Big)^{-p/2}\\
&\le \sum_{n=1}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{m=1}^{\infty}\Big(\lambda_m^2(A_n) + \frac{4k^2\pi^2}{(b_n-a_n)^2}\Big)^{-p/2}\\
&\le \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\lambda_m^2(A_n))^{-p/2} + 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\Big(\lambda_m^2(A_n) + \frac{4k^2\pi^2}{(b_n-a_n)^2}\Big)^{-p/2}
\end{aligned}$$

where $\delta(m,n) = arg\lambda_m(U_n^* e^{(-A_n(b_n-a_n))})$, $n \geq 1$, $m \geq 1$. Then from the inequality $\frac{|ts|}{t^2+s^2} \leq \frac{1}{2}$ for all $t, s \in \mathbb{R} \setminus \{0\}$ and last equation we have the inequality

$$\begin{aligned} &\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \big(\lambda_m^2(A_n) + \frac{4k^2\pi^2}{(b_n-a_n)^2}\big)^{-p/2} \\ &\leq 2^{-p}\pi^{-p/2}h^{p/2}\Big(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\frac{1}{\lambda_m(A_n)}|^{p/2} \sum_{k=1}^{\infty} |\frac{1}{k}|^{p/2}\Big) \end{aligned}$$

Since $A^{-1} \in C_{p/2}(\mathfrak{H})$, then the series $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\lambda_m(A_n)|^{-p/2}$ is convergent. Thus the series

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \Big(\lambda_m^2(A_n) + \frac{4k^2\pi^2}{(b_n-a_n)^2}\Big)^{-p/2}$$

is also convergent. Then from the relation

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\lambda_m(A_n)|^{-p} \leq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\lambda_m(A_n)|^{-p/2}$$

and the convergence of the series $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\lambda_m(A_n)|^{-p/2}$ we get that the series $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\lambda_m(A_n)|^{-p}$ is convergent too. Consequently the series $\sum_{q=1}^{\infty} \mu_q^p(L_U^{-1})$, $2 < p < \infty$ is convergent and thus $L_U^{-1} \in C_p(H), 2 < p < \infty$. $\square$

Theorems 4.8 and 4.9. can be can generalized as follows.

Corollary 4.11. *For $n \geq 1$, let $\mathscr{A}_n \in C_{p_n}(\mathscr{H}_n)$, $1 \leq p_n < \infty$ and $p = \sup_{n\geq 1} p_n < \infty$. Then $\mathscr{A} = \oplus_{n=1}^{\infty}\mathscr{A}_n \in C_p(\mathscr{H})$ if and only if the series $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu_k^p(\mathscr{A}_n)$ converges.*

Proof. In this case for each $n \geq 1$, $\mathscr{A}_n \in C_p(\mathscr{H}_n)$. So, by using Theorem 4.8(i), the validity of the proposition is clear. $\square$

The following result it is obtained from the above claim.

Corollary 4.12. *If for any $n \geq 1$, $A_n^{-1} \in C_{p_n/2}(\mathfrak{H}_\mathfrak{n})$, $2 < p = \sup_{n\geq 1} p_n < \infty$, and $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu_k^{p/2}(A_n^{-1}) < +\infty$,
then $L_U^{-1} \in C_p(H)$.*

Proof. Indeed, by Corollary 4.11 the operator $A^{-1} \in C_{p/2}(\mathfrak{H})$ and $A^{-1} = \oplus_{n=1}^{\infty} A_n^{-1}$, $\mathfrak{H} = \oplus_{n=1}^{\infty}\mathfrak{H}_n$. Consequently, from Theorem 4.10 it follows that $L_U^{-1} \in C_p(H)$. $\square$

Corollary 4.11 can be generalized the following sense.

Theorem 4.13. *Let $\mathscr{H} = \oplus_{n=1}^{\infty}\mathscr{H}_n$, $\mathscr{A} = \oplus_{n=1}^{\infty}\mathscr{A}_n \in L(\mathscr{H})$, for each $n \geq 1, 1 \leq p_n < +\infty$, $\mathscr{A}_n \in C_{p_n}(\mathscr{H}_n)$ and $p = sup_{n\geq 1}p_n < +\infty$. If the series $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu_k^{p_n}(\mathscr{A}_n)$ is convergent, then $\mathscr{A} \in C_p(\mathscr{H})$.*

Proof. If for every $n \geq 1$, $\|\mathscr{A}_n\| \leq 1$, then the validity of the above theorem is clear from Corollary 4.11 and the inequality

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu_k^p(\mathscr{A}_n) \leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu_k^{p_n}(\mathscr{A}_n) < +\infty\,.$$

Now consider the general case. In this case the operator $\mathscr{A}$ can be written in form

$$\mathscr{A} = CB, \quad C = \oplus_{n=1}^{\infty}((1+\|\mathscr{A}_n\|)E_n), \quad B = \oplus_{n=1}^{\infty}(\frac{\mathscr{A}_n}{1+\|\mathscr{A}_n\|}),$$

here $\sup_{n\geq 1}\|(1+\|\mathscr{A}_n\|)E_n\| = 1+\sup_{n\geq 1}\|\mathscr{A}_n\| < +\infty$. Then from Theorem 4.2 it is obtained that $C \in L(\mathscr{H})$.

On the other hand, since $\|B_n\| = \frac{\|\mathscr{A}_n\|}{1+\|\mathscr{A}_n\|} \leq 1$, $n \geq 1$, and

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_k^{p_n}(B_n) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\mu_k^{p_n}(\mathscr{A}_n)}{(1+\|\mathscr{A}_n\|)^{p_n}} \leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_k^{p_n}(\mathscr{A}_n) < +\infty,$$

then from Theorem 4.13, $B \in C_p(\mathscr{H})$ with $p = \sup_{n\geq 1} p_n$. So, on the important result of the operator theory $\mathscr{A} = CB \in C_p(\mathscr{H})$ [2]. $\square$

The following results give some information in case when $\sup_{n\geq 1} p_n = +\infty$.

Theorem 4.14. *Let $\mathscr{H} = \oplus_{n=1}^{\infty}\mathscr{H}_n$, $\mathscr{A} = \oplus_{n=1}^{\infty}\mathscr{A}_n \in L(\mathscr{H})$, for each $n \geq 1$, $\mathscr{A}_n \in C_{p_n}(\mathscr{H}_n)$, $1 \leq p_n < +\infty$, $p_n = \inf\{\alpha \in [1,+\infty) : \mathscr{A}_n \in C_\alpha(\mathscr{H}_n)\}$ and $\sup_{n\geq 1} p_n = +\infty$. Then for every $p, 1 \leq p < +\infty, \mathscr{A} \notin C_p(\mathscr{H})$.*

Proof. Assume that for some $p, 1 \leq p < +\infty$, $\mathscr{A} \in C_p(\mathscr{H})$. Then by the Theorem 4.8 (ii), for every $n \geq 1$, $\mathscr{A}_n \in C_p(\mathscr{H}_n)$, $1 \leq p < +\infty$. Since $\sup_{n\geq 1} p_n = +\infty$, then there exist $p_m, 1 \leq p_m < +\infty$ such that $p < p_m$ and $\mathscr{A}_m \in C_{p_m}(\mathscr{H}_m)$. On the other hand since $C_p \subset C_{p_m}$ and $C_p \neq C_{p_m}$, $p_n = \inf\{\alpha \in [1,+\infty) : \mathscr{A}_n \in C_\alpha(\mathscr{H}_n)\}$, then $\mathscr{A}_m \notin C_p(\mathscr{H}_m)$. This contradiction shows that the claim of the theorem is true. $\square$

Corollary 4.15. *Let $\mathscr{H} = \oplus_{n=1}^{\infty}\mathscr{H}_n$, $\mathscr{A} = \oplus_{n=1}^{\infty}\mathscr{A}_n \in L(\mathscr{H})$, for every $n \geq 1$ $\mathscr{A}_n \in C_{p_n}(\mathscr{H}_n)$, $1 \leq p_n \leq +\infty$, $p_n = inf\{\alpha \in [1,+\infty) : \mathscr{A}_n \in C_\alpha(\mathscr{H}_n)\}$ and $sup_{n\geq 1}p_n = +\infty$. If for some $m \in \mathbb{N}$, $\mathscr{A}_m \in C_\infty(\mathscr{H}_m)$, then for every $p, 1 \leq p < +\infty$, $\mathscr{A} \notin C_p(\mathscr{H})$.*

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