Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 84, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

FUNDAMENTAL SOLUTIONS TO THE p-LAPLACE EQUATION IN A CLASS OF GRUSHIN VECTOR FIELDS

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Abstract. We find the fundamental solution to the p-Laplace equation in a class of Grushin-type spaces. The singularity occurs at the sub-Riemannian points, which naturally corresponds to finding the fundamental solution of a generalized Grushin operator in Euclidean space. We then use this solution to find an infinite harmonic function with specific boundary data and to compute the capacity of annuli centered at the singularity.

1. Motivation

The p-Laplace equation is the model equation for nonlinear potential theory. The Euclidean results of [9] can be extended into a class of sub-Riemannian spaces possessing an algebraic group law, called Carnot groups [8]. Fundamental solutions to the p-Laplace equation in a subclass of Carnot groups called groups of Heisenberg-type have been found in [6, 8]. The exploration of the p-Laplace equation in sub-Riemannian spaces without an algebraic group law is currently a topic of interest. In this paper, we will find the fundamental solution to the p-Laplace equation for $1 < p < \infty$ in a class of Grushin-type spaces. The singularity occurs at the sub-Riemannian points, which naturally corresponds to finding the fundamental solution of a generalized Grushin operator in Euclidean space.

2. Grushin-type spaces

Before presenting the main theorem, we recall the construction of such spaces and their main properties. We begin with $\mathbb{R}^n$, possessing coordinates $(x_1, x_2, \ldots, x_n)$ and vector fields

$$X_i = \rho_i(x_1, x_2, \ldots, x_{i-1}) \frac{\partial}{\partial x_i}$$

for $i = 2, 3, \ldots, n$ where $\rho_i(x_1, x_2, \ldots, x_{i-1})$ is a (possibly constant) real-valued function. We decree that $\rho_1 \equiv 1$ so that

$$X_1 = \frac{\partial}{\partial x_1}\,.$$

2000 *Mathematics Subject Classification.* 35H20, 53C17, 17B70.
Key words and phrases. Grushin-type spaces; p-Laplacian.

Submitted December 10, 2010. Published June 29, 2011.

A quick calculation shows that when $i<j$ and $\rho_j(x_1,x_2,\dots,x_{j-1})$ is differentiable, the Lie bracket is given by

$$X_{ij}\equiv[X_i,X_j]=\rho_i(x_1,x_2,\dots,x_{i-1})\frac{\partial\rho_j(x_1,x_2,\dots,x_{j-1})}{\partial x_i}\frac{\partial}{\partial x_j}. \tag{2.1}$$

If the ρ_i's are polynomials, at each point there is a finite number of iterations of the Lie bracket so that $\frac{\partial}{\partial x_i}$ has a non-zero coefficient. This is easily seen for X_1 and X_2, and the result is obtained inductively for X_i. (It is noted that the number of iterations necessary is a function of the point.) It follows that Hörmander's condition is satisfied by such vector fields.

We may further endow $\mathbb{R}^n$ with an inner product (singular where the polynomials vanish) so that the collection $\{X_i\}$ forms an orthonormal basis. This produces a sub-Riemannian manifold that we shall call g_n, which is also the tangent space to a generalized Grushin-type space G_n. Points in G_n will also be denoted by $x=(x_1,x_2,\dots,x_n)$.

Though G_n is not a Lie group, it is a metric space with the natural metric being the Carnot-Carathéodory distance, which is defined for points x and y as follows:

$$d_C(x,y)=\inf_\Gamma\int_0^1\|\gamma'(t)\|\,dt\,.$$

Here Γ is the set of all curves γ such that $\gamma(0)=x,\gamma(1)=y$ and

$$\gamma'(t)\in\operatorname{span}\{\{X_i(\gamma(t))\}_{i=1}^n\}.$$

In the case of polynomials, Chow's theorem (see, for example, [1]) states any two points can be joined by such a curve. In the non-polynomial case, one can explicitly construct horizontal curves of finite length connecting any two points. This means $d_C(x,y)$ is an honest metric. Using this metric, we can define a Carnot-Carathéodory ball of radius r centered at a point x_0 by

$$B_C(x_0,r)=\{p\in G_n:d_C(x,x_0)<r\}$$

and similarly, we shall denote a bounded domain in G_n by Ω. The Carnot-Carathéodory metric behaves differently at points where the functions ρ_i vanish. Fixing a point x_0 where the ρ_i are sufficiently differentiable, consider the n-tuple $r_{x_0}=(r^1_{x_0},r^2_{x_0},\dots,r^n_{x_0})$ where $r^i_{x_0}$ is the minimal number of Lie bracket iterations required to produce

$$[X_{j_1},[X_{j_2},[\cdots[X_{j_{r^i_{x_0}}},X_i]\cdots](x_0)\neq 0\,.$$

Note that though the minimal length is unique, the iteration used to obtain that minimum is not. Note also that

$$\rho_i(x_0)\neq 0\leftrightarrow r^i_{x_0}=0\,.$$

Using [1, Theorem 7.34] we obtain the local estimate at x_0

$$d_C(x_0,x)\sim\sum_{i=1}^n|x_i-x_i^0|^{\frac{1}{1+r^i_{x_0}}}\,. \tag{2.2}$$

Given a smooth function f on G_n, we define the horizontal gradient of f as

$$\nabla_0 f(x)=(X_1f(x),X_2f(x),\dots,X_nf(x))$$

and the symmetrized second order (horizontal) derivative matrix by

$$((D^2f(x))^\star)_{ij} = \frac{1}{2}(X_iX_jf(x) + X_jX_if(x))$$

for $i, j = 1, 2, \dots n$.

Definition 2.1. The function $f : G_n \to \mathbb{R}$ is said to be $C^1_{\rm sub}$ if X_if is continuous for all $i = 1, 2, \dots, n$. Similarly, the function f is $C^2_{\rm sub}$ if X_iX_jf is continuous for all $i, j = 1, 2, \dots, n$.

Remark 2.2. We note that Euclidean C^1 functions are $C^1_{\rm sub}$ functions, but the class of $C^1_{\rm sub}$ functions is larger than the class of Euclidean C^1 functions. For example, when $n = 2$ and $\rho_2(x_1) = x_1$, we have $\sqrt{x_2}$ is $C^1_{\rm sub}$ at the origin, but is clearly not Euclidean C^1 at the origin. The interested reader is directed to [1] for a more complete discussion.

Using these derivatives, we consider two main operators on $C^2_{\rm sub}$ functions called the p-Laplacian

$$\Delta_pf = \operatorname{div}(\|\nabla_0f\|^{p-2}\nabla_0f) = \sum_{i=1}^{n} X_i(\|\nabla_0f\|^{p-2}X_if)$$

defined for $1 < p < \infty$ and the infinite Laplacian

$$\Delta_\infty f = \sum_{i,j=1}^{n} X_ifX_jfX_iX_jf = \langle\nabla_0f, (D^2f)^\star\nabla_0f\rangle.$$

For a more in-depth study of Grushin-type spaces, the reader is directed to [1, 2, 3] and the references therein.

3. The co-area formula and measure theory

We begin by fixing $m, n \in \mathbb{N}$ and $k, c \in \mathbb{R}$ so that $m < n$, $c \neq 0$, and $k \geq 0$. We also fix a vector $a = (a_1, a_2, \dots, a_m) \in \mathbb{R}^m$ and then consider the following vector fields:

$$\begin{aligned} X_i &= \frac{\partial}{\partial x_i} \quad \text{for } i = 1 \text{ to } m \\ X_j &= c\Big(\sum_{i=1}^{m}(x_i - a_i)^2\Big)^{k/2}\frac{\partial}{\partial x_j} \quad \text{for } j = m+1 \text{ to } n. \end{aligned} \tag{3.1}$$

Note that this choice corresponds to $\rho_i(x_1, x_2, \dots, x_{i-1}) = 1$ for $1 \leq i \leq m$ and $\rho_j(x_1, x_2, \dots, x_{j-1}) = c(\sum_{i=1}^{m}(x_i - a_i)^2)^{k/2}$ for $m+1 \leq j \leq n$. Additionally, if $k = 0$ and $c = 1$, we have the Euclidean space $\mathbb{R}^n$. Note also that in local coordinates, the 2-Laplacian operator is given by

$$\Delta_2 = \sum_{i=1}^{m}\frac{\partial^2}{\partial x_i^2} + \sum_{j=m+1}^{n} c^2\Big(\sum_{i=1}^{m}(x_i - a_i)^2\Big)^k\frac{\partial^2}{\partial x_j^2}.$$

In place of Fubini's Theorem for iterated integrals, we will make use of the following Co-Area Formula in the Euclidean context [7], which was extended to the Grushin case via [10, Theorem 4.2].

Theorem 3.1. *Let $\Omega \subset G_n$ be a bounded domain, and let $\psi \in C^1_{\mathrm{sub}}(\Omega)$ be a smooth, real-valued function which extends continuously to $\partial\Omega$. For convenience, we write ∇ for the Euclidean gradient on $G_n = \mathbb{R}^n$. Then for any function $g \in L^1(\Omega)$*

$$\iint_\Omega g\|\nabla\psi\|\, d\mathcal{L}_n = \int_0^\infty \int_{\psi^{-1}\{r\}} g\, d\mathcal{H}dr, \tag{3.2}$$

where $d\mathcal{L}_n$ denotes Lebesgue n-measure on Ω, and $d\mathcal{H}$ denotes Hausdorff $(n-1)$-measure on $\psi^{-1}(\{r\})$.

Corollary 3.2. *As above, the theorem also holds for continuous functions ψ which are smooth everywhere except at isolated points.*

We now consider a point $x_0 \in G_n$ with coordinates $x_0 = (a_1, \dots, a_m, b_{m+1}, \dots, b_n)$ and a non-negative, continuous radial function $\psi : \mathbb{R}^n \to \mathbb{R}$ that is smooth when $x_0 \neq x$ and with $\psi(x_0) = 0$. The following notation is suggestive for the inverse images of ψ.

$$\begin{aligned} \mathcal{B}_R(x_0) &= \psi^{-1}([0,R)) = \{x \in \Omega : \psi(x) < R\}, \\ \partial\mathcal{B}_R(x_0) &= \psi^{-1}(\{R\}) = \{x \in \Omega : \psi(x) = R\}. \end{aligned}$$

The x_0 is omitted when it is clear from the context. Since $\|\nabla_0\psi\| \lesssim \|\nabla\psi\|$ and $p > 1$, we may apply the Co-Area Formula to the well-defined function

$$g = \begin{cases} (\|\nabla_0\psi\|^p/\|\nabla\psi\|) \cdot \|\nabla\psi\| & \|\nabla\psi\| \neq 0 \\ 0 & \|\nabla\psi\| = 0 \end{cases}$$

to obtain the following result.

Proposition 3.3. *With the hypotheses as above, let $\mathcal{V}$ be an absolutely continuous measure to $\mathcal{L}_n$ with Radon-Nikodym derivative $\|\nabla_0\psi\|^p = [d\mathcal{V}/d\mathcal{L}_n]$. Then for sufficiently small $R > 0$,*

$$\mathcal{V}(\mathcal{B}_R) = \int_{\mathcal{B}_R} d\mathcal{V} = \int_0^R \int_{\partial\mathcal{B}_r} \frac{\|\nabla_0\psi\|^p}{\|\nabla\psi\|}\, d\mathcal{H}dr \tag{3.3}$$

In light of the equality in (3.3), we see that the measure space $(G_n, \mathcal{V})$ is globally Ahlfors Q-regular with respect to balls centered at x_0. In particular, for $R > 0$,

$$\mathcal{V}(\mathcal{B}_R) = \sigma_p R^Q \tag{3.4}$$

where $Q = m + (k+1)(n-m) = k(n-m) + n$ and $\sigma_p = \mathcal{V}(\mathcal{B}_1)$ is a fixed positive constant.

For technical purposes we proceed to study the boundary behavior of precompact domains Ω. This now motivates the following definition.

Definition 3.4. For small values $R > 0$, define a measure $\mathcal{S}$ on $\partial\mathcal{B}_R$ as

$$\mathcal{S}(\partial\mathcal{B}_R) = \int_{\partial\mathcal{B}_R} d\mathcal{S} = \int_{\partial\mathcal{B}_R} \frac{\|\nabla_0\psi\|^p}{\|\nabla\psi\|}\, d\mathcal{H}.$$

In particular, S is absolutely continuous with respect to the Hausdorff $(n-1)$-measure $\mathcal{H}$. Using previous results, in particular, the fact that ψ is smooth away from x_0, we now conclude:

Corollary 3.5. (1) *S is locally Ahlfors $(Q-1)$-regular and*

$$\mathcal{S}(\partial\mathcal{B}_R) = Q\sigma_p R^{Q-1}. \tag{3.5}$$

(2) *Let φ be a continuous and integrable function on $\mathcal{B}_R$. Then as $R \to 0$,*

$$\frac{R^{1-Q}}{Q\sigma_p}\int_{\partial\mathcal{B}_R}\varphi\, d\mathcal{S} \;\to\; \varphi(x_0) \tag{3.6}$$

Sketch of Proof. Equation (3.5) follows immediately from differentiating both Equations (3.3) and (3.4). Since $\mathcal{S}$ is absolutely continuous with respect to Hausdorff $(n-1)$-measure $\mathcal{H}$, it follows that $\mathcal{S}$ is Borel regular. As a result, Equation (3.6) is the analogue of the Lebesgue Density Theorem. □

4. The p-Laplace equation

In this section, we compute the fundamental solution of the p-Laplacian for the previously-defined vector fields (3.1) and for $1 < p < \infty$. We then use these formulas to find the explicit formula for a solution to the Dirichlet problem with specific boundary data. The following theorem generalizes [4] and is the Grushin analog of results in the class of Carnot groups known as groups of Heisenberg-type [6, 8].

Theorem 4.1. *Let $x_0 = (a_1, a_2, \dots, a_m, b_{m+1}, b_{m+2}, \dots, b_n)$ be an arbitrary fixed point. Consider the following quantities, for $1 < p < \infty$:*

$$\begin{gathered}
w = \frac{Q-p}{(2k+2)(1-p)} \quad \alpha = \frac{Q-p}{1-p}\\
h(x_1, x_2, \dots, x_n) = c^2\Big(\sum_{i=1}^{m}(x_i-a_i)^2\Big)^{k+1} + (k+1)^2\sum_{j=m+1}^{n}(x_j-b_j)^2\\
f(x_1, x_2, \dots, x_n) = [h(x_1, x_2, \dots, x_n)]^{w}\\
\psi(x_1, x_2, \dots, x_n) = [h(x_1, x_2, \dots, x_n)]^{\frac{1}{2k+2}}\\
\sigma_p = \int_{\mathcal{B}_1}\|\nabla_0\psi\|^p\, d\mathcal{L}_n\\
C_1 = \alpha^{-1}(Q\sigma_p)^{\frac{1}{1-p}} \quad C_2 = (Q\sigma_Q)^{\frac{1}{1-Q}}.
\end{gathered}$$

Then, for the constants C_1 and C_2 as above,

$$\begin{gathered}
\Delta_p C_1 f(x_1, x_2, \dots, x_n) = \delta_{x_0} \quad \text{when } p \neq Q\\
\Delta_p(C_2\log\psi(x_1, x_2, \dots, x_n)) = \delta_{x_0} \quad \text{when } p = Q
\end{gathered} \tag{4.1}$$

in the sense of distributions.

Proof. We first comment that for the sake of rigor, we should invoke the regularization of h given by

$$h_\varepsilon(x_1, x_2, \dots, x_n) = c^2\Big(\sum_{i=1}^{m}(x_i-a_i)^2 + \varepsilon^2\Big)^{k+1} + (k+1)^2\sum_{j=m+1}^{n}(x_j-b_j)^2$$

for $\varepsilon > 0$ and letting $\varepsilon \to 0$. However, we shall proceed formally. Suppressing the variables $(x_1, x_2, \dots, x_n)$, and letting

$$\Sigma = \sum_{i=1}^{m}(x_i-a_i)^2$$

we compute for $p \neq Q$:

$$X_i f = \alpha h^{w-1}c^2(x_i-a_i)\Sigma^k \text{ for } i = 1, 2, \dots, m$$

$$\begin{gathered} X_j f = c\alpha h^{w-1}\Sigma^{\frac{k}{2}}(k+1)(x_j-b_j) \text{ for } j=m+1,\dots,n \\ \|\nabla_0 f\|^2 = c^2\alpha^2 h^{2w-1}\Sigma^k \\ \|\nabla_0 f\|^{p-2} = |c\alpha|^{p-2}h^{w(p-2)-\frac{p-2}{2}}\Sigma^{\frac{k(p-2)}{2}}. \end{gathered}$$

We then are able to compute, for $i=1,2,\dots,m$,

$$\|\nabla_0 f\|^{p-2}X_i f = \alpha|\alpha|^{p-2}|c|^p h^{w(p-1)-\frac{p}{2}}\Sigma^{\frac{kp}{2}}(x_i-a_i)$$

and for $j=m+1,m+2,\dots,n$,

$$\|\nabla_0 f\|^{p-2}X_j f = \alpha|\alpha|^{p-2}c|c|^{p-2}h^{w(p-1)-\frac{p}{2}}\Sigma^{\frac{kp}{2}-\frac{k}{2}}(k+1)(x_j-b_j).$$

Setting

$$D_p \equiv \frac{\Delta_p f}{\alpha|\alpha|^{p-2}|c|^p} \quad \text{and} \quad \Upsilon = w(p-1)-\frac{p}{2}$$

we can then compute

$$\begin{aligned} D_p &= \sum_{i=1}^m h^{\Upsilon}\Sigma^{\frac{kp}{2}} + \sum_{i=1}^m h^{\Upsilon}(kp)\Sigma^{\frac{kp-2}{2}}(x_i-a_i)^2 \\ &\quad + \sum_{i=1}^m \Upsilon h^{\Upsilon-1}2c^2(k+1)\Sigma^{\frac{kp}{2}+k}(x_i-a_i)^2 \\ &\quad + \sum_{j=m+1}^n h^{\Upsilon}(k+1)\Sigma^{\frac{kp}{2}} + \sum_{j=m+1}^n 2\Upsilon h^{\Upsilon-1}\Sigma^{\frac{kp}{2}}(k+1)^3(x_j-b_j)^2 \\ &= h^{\Upsilon-1}\Sigma^{\frac{kp}{2}}\Big(mh+(kp)h+(k+1)(n-m)h \\ &\quad + \big(\alpha(p-1)-p(k+1)\big)\Big(c^2\Sigma^{k+1}+(k+1)^2\sum_{j=m+1}^n(x_j-b_j)^2\Big)\Big) \\ &= h^{\Upsilon}\Sigma^{\frac{kp}{2}}\Big(\alpha(p-1)-p(k+1)+m+(kp)+(k+1)(n-m)\Big) \\ &= h^{\Upsilon}\Sigma^{\frac{kp}{2}}((p-Q)-p+Q)=0. \end{aligned}$$

Note that these computations are valid wherever the function f is smooth and in particular, these are valid away from the point x_0. We next note that

$$\|\nabla_0 f\|^{p-1} \sim \psi^{1-Q}$$

and so we conclude that $\|\nabla_0 f\|^{p-1}$ is locally integrable on G_n with respect to Lebegue measure. We then consider $\phi \in C_0^\infty$ with compact support in the ball

$$\mathcal{B}_R = \{x : \psi(x) < R\}.$$

Let $0<r<R$ be given so that $\mathcal{B}_r \subset \mathcal{B}_R$. In the annulus $\mathcal{A} := \mathcal{B}_R \setminus \overline{\mathcal{B}_r}$ we have, via the Leibniz rule,

$$\begin{aligned} \operatorname{div}(\phi\|\nabla_0 f\|^{p-2}\nabla_0 f) &= \phi\operatorname{div}(\|\nabla_0 f\|^{p-2}\nabla_0 f) + \|\nabla_0 f\|^{p-2}\langle\nabla_0 f,\nabla_0\phi\rangle \\ &= 0 + \|\nabla_0 f\|^{p-2}\langle\nabla_0 f,\nabla_0\phi\rangle. \end{aligned}$$

Let $\mathcal{L}_n$ and $\mathcal{H}$ be the measures from (3.2) and recall

$$\Sigma \equiv \sum_{j=1}^m (x_i-a_i)^2.$$

Applying Stokes' Theorem,

$$\begin{aligned}
&\int_{\mathcal{A}} \|\nabla_0 f\|^{p-2}\langle \nabla_0 f, \nabla_0 \phi\rangle d\mathcal{L}_n \\
&= \int_{\mathcal{A}} \operatorname{div}(\phi \|\nabla_0 f\|^{p-2}\nabla_0 f) d\mathcal{L}_n \\
&= \int_{\mathcal{A}} \Big(\sum_{i=1}^{m} X_i[\phi\|\nabla_0 f\|^{p-2} X_i f] + c\Sigma^{\frac{k}{2}} \sum_{j=m+1}^{n} \frac{\partial}{\partial x_j}\Big(\phi\|\nabla_0 f\|^{p-2} X_j f\Big)\Big) d\mathcal{L}_n \\
&= \int_{\mathcal{A}} \Big(\sum_{i=1}^{m} X_i[\phi\|\nabla_0 f\|^{p-2} X_i f] + \sum_{j=m+1}^{n} \frac{\partial}{\partial x_j}\Big(c\Sigma^{\frac{k}{2}}\phi\|\nabla_0 f\|^{p-2} X_j f\Big)\Big) d\mathcal{L}_n \\
&= \int_{\mathcal{A}} \operatorname{div}_{\text{eucl}} \begin{bmatrix} \phi\|\nabla_0 f\|^{p-2} X_1 f \\ \vdots \\ \phi\|\nabla_0 f\|^{p-2} X_m f \\ c\Sigma^{\frac{k}{2}}\phi\|\nabla_0 f\|^{p-2} X_{m+1} f \\ \vdots \\ c\Sigma^{\frac{k}{2}}\phi\|\nabla_0 f\|^{p-2} X_n f \end{bmatrix} d\mathcal{L}_n \\
&= \int_{\partial\mathcal{A}} \frac{1}{\|\nu\|}\Big(\phi\|\nabla_0 f\|^{p-2}\sum_{i=1}^{m} X_i f \nu_i + c\Sigma^{\frac{k}{2}}\phi\|\nabla_0 f\|^{p-2}\sum_{j=m+1}^{n} X_j f \nu_j\Big) d\mathcal{H} \\
&= -\int_{\partial\mathcal{B}_r} \frac{1}{\|\nu\|}\Big(\phi\|\nabla_0 f\|^{p-2}\sum_{i=1}^{m} X_i f \nu_i + c\Sigma^{\frac{k}{2}}\phi\|\nabla_0 f\|^{p-2}\sum_{j=m+1}^{n} X_j f \nu_j\Big) d\mathcal{H}
\end{aligned}$$

where ν is the outward Euclidean normal of $\mathcal{A}$. Recalling that

$$\psi(x_1, x_2, \dots, x_n) = [h(x_1, x_2, \dots, x_n)]^{1/(2k+2)},$$

and that $\nu_j = \frac{\partial \psi}{\partial x_j}$, we may proceed with the computation,

$$\begin{aligned}
&\int_{\mathcal{A}} \|\nabla_0 f\|^{p-2}\langle \nabla_0 f, \nabla_0 \phi\rangle d\mathcal{L}_n \\
&= -\int_{\partial\mathcal{B}_r} \frac{\alpha\psi^{\alpha-1}}{\|\nu\|}\phi\|\nabla_0\psi^{\alpha}\|^{p-2}\Big(\sum_{i=1}^{m}(\frac{\partial\psi}{\partial x_i})^2 + c^2\Sigma^k\sum_{j=m+1}^{n}(\frac{\partial\psi}{\partial x_j})^2\Big) d\mathcal{H} \\
&= -\int_{\partial\mathcal{B}_r} \frac{\alpha\psi^{\alpha-1}}{\|\nu\|}\phi\|\nabla_0\psi\|^{p-2}|\alpha|^{p-2}\psi^{(p-2)(\alpha-1)}\Big(\|\nabla_0\psi\|^2\Big) d\mathcal{H} \\
&= -\int_{\partial\mathcal{B}_r} \frac{|\alpha|^{p-2}\alpha\psi^{(p-1)(\alpha-1)}}{\|\nu\|}\phi\|\nabla_0\psi\|^p d\mathcal{H}.
\end{aligned}$$

Recall that by definition, $\psi \equiv r$ on $\partial\mathcal{B}_r$. We then have

$$\int_{\mathcal{A}} \|\nabla_0 f\|^{p-2}\langle \nabla_0 f, \nabla_0 \phi\rangle d\mathcal{L}_n = -|\alpha|^{p-2}\alpha r^{1-Q}\int_{\partial\mathcal{B}_r}\frac{\phi\|\nabla_0\psi\|^p}{\|\nu\|} d\mathcal{H}.$$

Letting $r \to 0$, we apply (3.6) and obtain

$$\int_{\mathcal{A}} \|\nabla_0 f\|^{p-2}\langle \nabla_0 f, \nabla_0 \phi\rangle d\mathcal{L}_n \ \to\ -|\alpha|^{p-2}\alpha(Q\sigma_p)\phi(x_0). \tag{4.2}$$

We then obtain the case for $p \neq Q$. The case of $p = Q$ is similar and left to the reader. $\square$

It was shown in [2] and [3] that in Grushin-type spaces, viscosity infinite harmonic functions are limits of weak p-harmonic functions as p tends to infinity. This motivates the following corollary.

Corollary 4.2. *The function ψ, as defined above, is infinite harmonic in the space $G_n \setminus \{x_0\}$.*

Proof. We use the formula that for a smooth function u,

$$\Delta_\infty u = \frac{1}{2}\nabla_0 u \cdot \nabla_0 \|\nabla_0 u\|^2.$$

Computing as in the proof of the Theorem, we have

$$\|\nabla_0 \psi\|^2 = c^2 \Sigma^k h^{\frac{-2k}{2k+2}}.$$

Thus we obtain for $i = 1, 2, \dots, m$,

$$X_i \|\nabla_0 \psi\|^2 = 2kc^2 h^{\frac{-2k}{2k+2}-1} \Sigma^{k-1}(x_i - a_i)\big(h - c^2\Sigma^{k+1}\big)$$

and for $j = m+1, m+2, \dots, n$,

$$X_j \|\nabla_0 \psi\|^2 = -2kc^3 h^{\frac{-2k}{2k+2}-1} \Sigma^{\frac{3k}{2}}(k+1)(x_j - b_j)$$

so that using the derivatives as in the proof of the Theorem,

$$\begin{aligned}
\Delta_\infty \psi &= \sum_{i=1}^{m} 2kc^4 h^{\frac{-4k-1}{2k+2}-1}\Sigma^{2k-1}(x_i - a_i)^2\big(h - c^2\Sigma^{k+1}\big) \\
&\quad - 2kc^4 h^{\frac{-4k-1}{2k+2}-1}(k+1)^2\Sigma^{2k}\sum_{j=m+1}^{n}(x_j - b_j)^2 \\
&= 2kc^4 h^{\frac{-4k-1}{2k+2}-1}\Sigma^{2k}\Big(h - c^2\Sigma^{k+1} - (k+1)^2\sum_{j=m+1}^{n}(x_j - b_j)^2\Big) \\
&= 2kc^4 h^{\frac{-4k-1}{2k+2}-1}\Sigma^{2k} \times (0).
\end{aligned}$$

□

Using the existence-uniqueness of viscosity infinite harmonic functions [2, 3] and the fact that absolute minimizers in Grushin spaces are viscosity infinite harmonic functions and enjoy comparison with cones [5], we conclude the following corollary.

Corollary 4.3. *Let $0 < s \in \mathbb{R}$. Define the function $\Psi_s : \partial\mathcal{B}_s(x_0) \cup \{x_0\} \to \mathbb{R}$ by*

$$\Psi_s(x_1, x_2, \dots, x_n) = \begin{cases} s & \text{on } \partial\mathcal{B}_s(x_0) \\ 0 & \text{at } x_0 \end{cases}$$

Then $s \cdot \psi$ is the unique absolute minimizer of Ψ into the ball $\mathcal{B}_s(x_0)$. In addition, $s \cdot \psi$ enjoys comparison with Grushin cones.

5. Spherical capacity

In this section, we will use previous results to compute the capacity of spherical rings centered at the point $x_0 = (a_1, a_2, \dots, a_m, b_{m+1}, b_{m+2}, \dots, b_n)$. We first recall the definition of p-capacity.

Definition 5.1. Let $\Omega \subset G_n$ be a bounded, open set, and $K \subset \Omega$ a compact subset. For $1 \le p < \infty$ we define the p-capacity as

$$\operatorname{cap}_p(K,\Omega) = \inf\Big\{ \int_\Omega \|\nabla_0 u\|^p \, d\mathcal{L}_n : u \in C_0^\infty(\Omega),\ u|K = 1\Big\}.$$

We note that although the definition is valid for $p = 1$, we will consider only $1 < p < \infty$, as in the previous sections. Because p-harmonic functions are minimizers to the energy integral

$$\int_{G_n} \|\nabla_0 f\|^p \, d\mathcal{L}_n$$

it is natural to consider p-harmonic functions when computing the capacity. In particular, an easy calculation similar to the previous section shows

$$u(x) = \begin{cases} \frac{\psi(x)^\alpha - R^\alpha}{r^\alpha - R^\alpha} & \text{when } p \neq Q \\ \frac{\log \psi(x) - \log R}{\log r - \log R} & \text{when } p = Q \end{cases}$$

is a smooth solution to the Dirichlet problem

$$\begin{gathered} \Delta_p u = 0 \quad \text{in } \mathcal{B}_R(x_0) \setminus \mathcal{B}_r(x_0) \\ u = 1 \quad \text{on } \partial\mathcal{B}_r(x_0) \\ u = 0 \quad \text{on } \partial\mathcal{B}_R(x_0) \end{gathered}$$

for $1 < p < \infty$.

We state the following theorem, which follows from the computations of the previous section.

Theorem 5.2. *Let $0 < r < R$ and $1 < p < \infty$. Then we have*

$$\operatorname{cap}_p\big(\mathcal{B}_r(x_0), \mathcal{B}_R(x_0)\big) = \begin{cases} \alpha^{p-1} Q \sigma_p \big(r^\alpha - R^\alpha\big)^{1-p} & \textit{when } 1 < p < Q \\ Q \sigma_Q [\log R - \log r]^{1-Q} & \textit{when } p = Q \\ \alpha^{p-1} Q \sigma_p \big(R^\alpha - r^\alpha\big)^{1-p} & \textit{when } p > Q. \end{cases}$$

Acknowledgements. The author wishes to thank the anonymous referee for his/her careful reading of this manuscript and for the helpful suggestions to improve its readability and clarity.

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