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EXISTENCE OF CONTINUOUS POSITIVE SOLUTIONS FOR SOME NONLINEAR POLYHARMONIC SYSTEMS OUTSIDE THE UNIT BALL

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Abstract. We study the existence of continuous positive solutions of the m-polyharmonic nonlinear elliptic system

$$(-\Delta)^m u + \lambda p(x) g(v) = 0,$$
$$(-\Delta)^m v + \mu q(x) f(u) = 0$$

in the complement of the unit closed ball in $\mathbb{R}^n$ ($n > 2m$ and $m \geq 1$). Here the constants λ, μ are nonnegative, the functions f, g are nonnegative, continuous and monotone. We prove two existence results for the above system subject to some boundary conditions, where the nonnegative functions p, q satisfy some appropriate conditions related to a Kato class of functions.

1. Introduction

In this article, we discuss the existence of positive continuous solutions (in the sense of distributions) for the m-polyharmonic nonlinear elliptic system

$$\begin{gathered}
(-\Delta)^m u + \lambda p(x) g(v) = 0, \quad x \in D, \\
(-\Delta)^m v + \mu q(x) f(u) = 0, \quad x \in D, \\
\lim_{x \to \xi \in \partial D} \frac{u(x)}{(|x|^2 - 1)^{m-1}} = a\varphi(\xi), \quad \lim_{x \to \xi \in \partial D} \frac{v(x)}{(|x|^2 - 1)^{m-1}} = b\psi(\xi), \\
\lim_{|x| \to \infty} \frac{u(x)}{(|x|^2 - 1)^{m-1}} = \alpha, \quad \lim_{|x| \to \infty} \frac{v(x)}{(|x|^2 - 1)^{m-1}} = \beta,
\end{gathered} \tag{1.1}$$

where D is the complementary of the unit ball in $\mathbb{R}^n$ ($n > 2m$) and m is a positive integer. The constants λ, μ are nonnegative, $f, g : (0, \infty) \to [0, \infty)$ are monotone and continuous and $p, q : D \to [0, \infty)$ are measurable functions. Also we fix two nontrivial nonnegative continuous functions φ and ψ on ∂D and the constants a, b, α, β are nonnegative and satisfy $a + \alpha > 0$, $b + \beta > 0$.

Since our tools are based on potential theory approach, we denote by $G^B_{m,n}$ the Green function of $(-\Delta)^m$ on the unit ball B in $\mathbb{R}^n$ ($n \geq 2$) with Dirichlet boundary conditions $(\frac{\partial}{\partial \nu})^j u = 0$, $0 \leq j \leq m - 1$ and where $\frac{\partial}{\partial \nu}$ is the outward normal derivative.

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Boggio [5] obtained an explicit expression for $G^B_{m,n}$ given by

$$G^B_{m,n}(x,y) = k_{m,n}|x-y|^{2m-n} \int_1^{\frac{[x,y]}{|x-y|}} \frac{(r^2-1)^{m-1}}{r^{n-1}} dr, \tag{1.2}$$

where $k_{m,n}$ is a positive constant and $[x,y]^2 = |x-y|^2 + (1-|x|^2)(1-|y|^2)$, for $x, y \in B$.

It is obvious that the positivity of $G^B_{m,n}$ holds in B but this does not hold in an arbitrary bounded domain (see for example [8]). For $m = 1$, we do not have this restriction. Putting

$$[x,y]^2 = |x-y|^2 + (|x|^2-1)(|y|^2-1),$$

for $x, y \in D$ and denote by $G^D_{m,n}$ the Green function of $(-\Delta)^m$ in D with Dirichlet boundary conditions $(\frac{\partial}{\partial\nu})^j u = 0$, $0 \le j \le m-1$, then $G^D_{m,n}$ has the same expression defined by (1.2). That is,

$$G^D_{m,n}(x,y) = k_{m,n}|x-y|^{2m-n} \int_1^{\frac{[x,y]}{|x-y|}} \frac{(r^2-1)^{m-1}}{r^{n-1}} dr, \quad \text{for } x, y \in D.$$

In [4], the authors proved some estimates for $G^D_{m,n}$. In particular, they showed that there exists $C_0 > 0$ such that for each $x, y, z \in D$, we have

$$\frac{G^D_{m,n}(x,z)G^D_{m,n}(z,y)}{G^D_{m,n}(x,y)} \le C_0\big[(\frac{\rho(z)}{\rho(x)})^m G^D_{m,n}(x,z) + (\frac{\rho(z)}{\rho(y)})^m G^D_{m,n}(y,z)\big],$$

where throughout this paper, $\rho(x) = 1 - \frac{1}{|x|}$, for all $x \in D$. This form is called the 3G-inequality and has been exploited to introduce the polyharmonic Kato class $K^\infty_{m,n}(D)$ which is defined as follows

Definition 1.1 ([4])**.** A Borel measurable function q in D belongs to the Kato class $K^\infty_{m,n}(D)$ if q satisfies

$$\lim_{\alpha\to 0}\Big(\sup_{x\in D}\int_{D\cap B(x,\alpha)} (\frac{\rho(y)}{\rho(x)})^m G^D_{m,n}(x,y)|q(y)|dy\Big) = 0,$$
$$\lim_{M\to\infty}\Big(\sup_{x\in D}\int_{(|y|\ge M)} (\frac{\rho(y)}{\rho(x)})^m G^D_{m,n}(x,y)|q(y)|dy\Big) = 0.$$

This class is well studied when $m = 1$ in [3]. As a typical example of functions belonging to the class $K^\infty_{m,n}(D)$, we quote an example from [4]: Let $\gamma, \nu \in \mathbb{R}$ and q be the function defined in D by $q(x) = \frac{1}{|y|^{\nu-\gamma}(|y|-1)^\gamma}$. Then

$$q \in K^\infty_{m,n}(D) \Leftrightarrow \gamma < 2m < \nu.$$

Our main purpose in this paper is to study problem (1.1) when p and q satisfy an appropriate condition related to the Kato class $K^\infty_{m,n}(D)$ and to investigate the existence and the asymptotic behavior of such positive solutions. For this aim we shall refer to the bounded continuous solution $H_D\varphi$ of the Dirichlet problem (see [1])

$$\Delta u = 0 \quad \text{in } D,$$
$$\lim_{x\to\xi\in\partial D} u(x) = \varphi(\xi), \quad \lim_{|x|\to\infty} u(x) = 0,$$

where φ is a nonnegative nontrivial continuous function on ∂D. Also, we refer to the potential of a measurable nonnegative function f, defined in D by

$$V_{m,n}f(x) = \int_D G^D_{m,n}(x,y)f(y)dy.$$

The outline of our article is as follows. In Section 2, we recapitulate some properties of functions belonging to $K^\infty_{m,n}(D)$ developed in [4] and adopted to our interest. In Section 3, we aim at proving a first existence result for (1.1). In fact, let a, b, α, β be nonnegative real numbers with $a + \alpha > 0$, $b + \beta > 0$ and φ, ψ are nontrivial nonnegative continuous functions on ∂D. Let h be the harmonic function defined in D by $h(x) = 1 - \frac{1}{|x|^{n-2}}$. Let θ and ω be the functions defined in D by

$$\theta(x) = \gamma(x)(\alpha\, h(x) \,+\, a\, H_D\varphi(x)),$$
$$\omega(x) = \gamma(x)(\beta\, h(x) \,+\, b\, H_D\psi(x)),$$

where $\gamma(x) = (|x|^2 - 1)^{m-1}$.

The functions f, g, p and q are required to satisfy the following hypotheses.

(H1) $f,\, g : (0,\infty) \to [0,\infty)$ are nondecreasing and continuous;

(H2)

$$\lambda_0 := \inf_{x\in D} \frac{\theta(x)}{V_{m,n}(pg(\omega))(x)} > 0,$$
$$\mu_0 := \inf_{x\in D} \frac{\omega(x)}{V_{m,n}(qf(\theta))(x)} > 0;$$

(H3) The functions p and q are measurable nonnegative and satisfy

$$x \to \tilde{p}(x) = \frac{p(x)\, g(\omega(x))}{\gamma(x)} \quad \text{and} \quad x \to \tilde{q}(x) = \frac{q(x)\, f(\theta(x))}{\gamma(x)}$$

belong to the Kato class $K^\infty_{m,n}(D)$.

Then we prove the following result.

Theorem 1.2. *Assume* (H1)–(H3). *Then for each* $\lambda \in [0, \lambda_0)$ *and each* $\mu \in [0, \mu_0)$, *problem* (1.1) *has a positive continuous solution* (u, v) *that for each* $x \in D$ *satisfies*

$$(1 - \frac{\lambda}{\lambda_0})\theta(x) \le u(x) \le \theta(x),$$
$$(1 - \frac{\mu}{\mu_0})\omega(x) \le v(x) \le \omega(x).$$

Next, we establish a second existence result for problem (1.1) where $a = b = \lambda = \mu = 1$. Namely, we study the system

$$\begin{gathered}
(-\Delta)^m u + p(x)g(v) = 0, \quad x \in D \quad \text{(in the sense of distributions)},\\
(-\Delta)^m v + q(x)f(u) = 0, \quad x \in D,\\
\lim_{x\to\xi\in\partial D} \frac{u(x)}{(|x|^2-1)^{m-1}} = \varphi(\xi), \quad \lim_{x\to\xi\in\partial D} \frac{v(x)}{(|x|^2-1)^{m-1}} = \psi(\xi),\\
\lim_{|x|\to\infty} \frac{u(x)}{(|x|^2-1)^{m-1}} = \alpha, \quad \lim_{|x|\to\infty} \frac{v(x)}{(|x|^2-1)^{m-1}} = \beta.
\end{gathered} \tag{1.3}$$

To study this problem, we fix a positive continuous function ϕ on ∂D. We put $\rho_0 = \gamma h_0$, where $h_0 = H_D\phi$ and we assume the following hypotheses:

(H4) The functions $f, g : (0,\infty) \to [0,\infty)$ are nonincreasing and continuous;

(H5) The functions $p_1 := p\frac{g(\rho_0)}{\rho_0}$ and $q_1 := q\frac{f(\rho_0)}{\rho_0}$ belong to the Kato class $K^{\infty}_{m,n}(D)$.

Here, we mention that the method used to prove Theorem 1.3 stated below is different from that in Theorem 1.2. In fact, with loss of λ and μ, the boundary ∂D will play a capital role to construct a positive and continuous solution for (1.3) by means of a fixed point argument.

Our second existence result is the following.

Theorem 1.3. *Assume* (H4)–(H5)*. Then there exists a constant $c > 1$ such that if $\varphi \geq c\phi$ and $\psi \geq c\phi$ on ∂D, then problem* (1.3) *has a positive continuous solution (u, v) that for each $x \in D$ satisfies*

$$(|x|^2 - 1)^{m-1}(\alpha\, h(x) + h_0(x)) \leq u(x) \leq (|x|^2 - 1)^{m-1}(\alpha\, h(x) + H_D\varphi(x)),$$
$$(|x|^2 - 1)^{m-1}(\beta\, h(x) + h_0(x)) \leq v(x) \leq (|x|^2 - 1)^{m-1}(\beta\, h(x) + H_D\psi(x)).$$

This result is a follow up to the one obtained by Athreya [2].

For $m = 1$, the existence of solutions for nonlinear elliptic systems has been extensively studied for both bounded and unbounded $C^{1,1}$-domains in $\mathbb{R}^n$ $(n \geq 3)$ (see for example [6, 7, 9, 10, 11, 12, 13, 14, 15]). The motivation for our study comes from the results proved in [10] and which correspond to the case $m = 1$ in this article. Section 4 gives some examples where hypotheses (H2) and (H3) are satisfied and to illustrate Theorem 1.3.

In the sequel and in order to simplify our statements we denote by C a generic positive constant which may vary from line to line and for two nonnegative functions f and g on a set S, we write $f(x) \asymp g(x)$, for $x \in S$, if there exists a constant $C > 0$ such that $g(x)/C \leq f(x) \leq Cg(x)$ for all $x \in S$. Let

$$C_0(D) := \{f \in C(D) : \lim_{|x|\to 1} f(x) = \lim_{|x|\to\infty} f(x) = 0\}.$$

2. Preliminary results

In this section, we are concerned with some results related to the Kato class $K^{\infty}_{m,n}(D)$ which are useful for the proof of our main results stated in Theorems 1.2 and 1.3.

Proposition 2.1 ([4])**.** *Let q be a function in $K^{\infty}_{m,n}(D)$, then*

$$\|q\|_D := \sup_{x\in D} \int_D \Big(\frac{\rho(y)}{\rho(x)}\Big)^m G^D_{m,n}(x, y)|q(y)|dy < \infty.$$

To present the following Proposition, we need to denote by $\mathcal{H}$ the set of nonnegative harmonic functions h defined in D by

$$h(x) = \int_{\partial D} P(x, \xi)\nu(d\xi),$$

where ν is a nonnegative measure on ∂D and $P(x, \xi) = \frac{|x|^2-1}{|x-\xi|^n}$ is the Poisson kernel in D. From the 3G-inequality, we derive the following result.

Proposition 2.2. *Let q be a nonnegative function in $K^{\infty}_{m,n}(D)$. Then we have*

(i)

$$\alpha_q := \sup_{x,y\in D} \int_D \frac{G^D_{m,n}(x, z)G^D_{m,n}(z, y)}{G^D_{m,n}(x, y)} q(z)dz < \infty;$$

(ii) *For any function $h \in \mathcal{H}$ and each $x \in D$, we have*

$$\int_D G^D_{m,n}(x,z)(|z|^2-1)^{m-1}h(z)q(z)dz \le \alpha_q(|x|^2-1)^{m-1}h(x).$$

Proof. From the 3G-inequality, there exists $C_0 > 0$ such that for each $x, y, z \in D$, we have

$$\frac{G^D_{m,n}(x,z)G^D_{m,n}(z,y)}{G^D_{m,n}(x,y)} \le C_0\big[(\frac{\rho(z)}{\rho(x)})^m G^D_{m,n}(x,z) + (\frac{\rho(z)}{\rho(y)})^m G^D_{m,n}(y,z)\big].$$

This implies that $\alpha_q \le 2C_0\,\|q\|_D$. Then the assertion (i) holds from Proposition 2.1.

Now, we shall prove (ii). Let $h \in \mathcal{H}$, then there exists a nonnegative measure ν on ∂D such that

$$h(x) = \int_{\partial D} P(x,\xi)\nu(d\xi). \tag{2.1}$$

On the other hand, by using the transformation $r^2 = 1 + \frac{\varrho(x,y)}{|x-y|^2}(1-t)$ in (1.2), where $\varrho(x,y) = [x,y]^2 - |x-y|^2 = (|x|^2-1)(|y|^2-1)$, we obtain

$$G^D_{m,n}(x,y) = \frac{k_{m,n}}{2}\frac{(\varrho(x,y))^m}{[x,y]^n}\int_0^1 \frac{(1-t)^{m-1}}{(1-t\frac{\varrho(x,y)}{[x,y]^2})^{n/2}}dt.$$

This implies for each $x, z \in D$ and $\xi \in \partial D$ that

$$\lim_{y\to\xi}\frac{G^D_{m,n}(z,y)}{G^D_{m,n}(x,y)} = \frac{(|z|^2-1)^{m-1}P(z,\xi)}{(|x|^2-1)^{m-1}P(x,\xi)}.$$

So, it follows from Fatou's lemma that

$$\begin{aligned}
&\int_D G^D_{m,n}(x,z)\frac{(|z|^2-1)^{m-1}P(z,\xi)}{(|x|^2-1)^{m-1}P(x,\xi)}q(z)dz \\
&\le \liminf_{y\to\xi}\int_D \frac{G^D_{m,n}(x,z)G^D_{m,n}(z,y)}{G^D_{m,n}(x,y)}q(z)dz \le \alpha_q.
\end{aligned}$$

This, together with (2.1), completes the proof. □

Proposition 2.3 ([4]). *Let $q \in K^\infty_{m,n}(D)$. Then the function $z \to \frac{(|z|-1)^{2m-1}}{|z|^{n-1}}q(z)$ is in $L^1(D)$.*

Proposition 2.4. [4] *Let $q \in K^\infty_{m,n}(D)$ and h be a bounded function in $\mathcal{H}$. Then the function*

$$x \to \int_D (\frac{|y|^2-1}{|x|^2-1})^{m-1} G^D_{m,n}(x,y)\,h(y)\,|q(y)|dy$$

lies in $C_0(D)$.

For a nonnegative function $q \in K^\infty_{m,n}(D)$, we denote

$$\mathcal{F}_q = \{p \in K^\infty_{m,n}(D) : |p| \le q \text{ in } D\}.$$

Proposition 2.5 ([4]). *For any nonnegative function $q \in K^\infty_{m,n}(D)$, the family of functions*

$$\{\int_D (\frac{|y|^2-1}{|x|^2-1})^{m-1} G^D_{m,n}(x,y)h_0(y)p(y)dy,\ p \in \mathcal{F}_q\}$$

is uniformly bounded and equicontinuous in $\overline{D}\cup\{\infty\}$. Consequently it is relatively compact in $C(\overline{D}\cup\{\infty\})$.

3. Proofs of main results

Proof of Theorem 1.2. Let $\lambda \in [0, \lambda_0)$ and $\mu \in [0, \mu_0)$. We define the sequences $(u_k)_{k\geq 0}$ and $(v_k)_{k\geq 0}$ by

$$\begin{aligned} v_0 &= \omega, \\ u_k &= \theta - \lambda V_{m,n}(pg(v_k)), \\ v_{k+1} &= \omega - \mu V_{m,n}(qf(u_k)). \end{aligned}$$

We intend to prove that for all $k \in \mathbb{N}$,

$$0 < (1 - \frac{\lambda}{\lambda_0})\theta \leq u_k \leq u_{k+1} \leq \theta, \tag{3.1}$$

$$0 < (1 - \frac{\mu}{\mu_0})\omega \leq v_{k+1} \leq v_k \leq \omega. \tag{3.2}$$

Note that from the definition of λ_0 and μ_0 we have

$$\lambda_0 V_{m,n}(pg(\omega)) \leq \theta, \tag{3.3}$$

$$\mu_0 V_{m,n}(qf(\theta)) \leq \omega. \tag{3.4}$$

From (3.3) we have

$$u_0 = \theta - \lambda V_{m,n}(pg(v_0)) \geq (1 - \frac{\lambda}{\lambda_0})\theta > 0.$$

Then $v_1 - v_0 = -\mu V_{m,n}(qf(u_0)) \leq 0$. Since g is nondecreasing we obtain

$$u_1 - u_0 = \lambda V_{m,n}(p(g(v_0) - g(v_1))) \geq 0.$$

Now, since v_0 is positive and f is nondecreasing,

$$v_1 \geq \omega - \mu V_{m,n}(q\, f(\theta)).$$

We deduce from (3.4) that

$$v_1 \geq (1 - \frac{\mu}{\mu_0})\omega > 0.$$

This implies that $u_1 \leq \theta$. Finally, we obtain that

$$\begin{aligned} 0 < (1 - \frac{\lambda}{\lambda_0})\theta \leq u_0 \leq u_1 \leq \theta, \\ 0 < (1 - \frac{\mu}{\mu_0})\omega \leq v_1 \leq v_0 \leq \omega. \end{aligned}$$

By induction, we suppose that (3.1) and (3.2) hold for k. Since f is nondecreasing and $u_{k+1} \leq \theta$, we have

$$v_{k+2} - v_{k+1} = \mu V_{m,n}(q(f(u_k) - f(u_{k+1}))) \leq 0,$$

and

$$\begin{aligned} v_{k+2} &= \omega - \mu V_{m,n}(q\, f(u_{k+1})) \\ &\geq \omega - \mu V_{m,n}(qf(\theta)) \\ &\geq (1 - \frac{\mu}{\mu_0})\omega. \end{aligned}$$

To reach the last inequality, we use (3.4). Then
$$0 < (1 - \frac{\mu}{\mu_0})\omega \le v_{k+2} \le v_{k+1} \le \omega.$$
Now, using that g is nondecreasing we have
$$u_{k+2} - u_{k+1} = \lambda V_{m,n}(p(g(v_{k+1}) - g(v_{k+2})) \ge 0.$$
Since $v_{k+2} > 0$, we obtain
$$0 < (1 - \frac{\lambda}{\lambda_0})\theta \le u_{k+1} \le u_{k+2} \le \theta.$$
Therefore, the sequences $(u_k)_{k\ge 0}$ and $(v_k)_{k\ge 0}$ converge respectively to two functions u and v satisfying
$$\begin{gathered}(1 - \frac{\lambda}{\lambda_0})\theta \le u \le \theta,\\ (1 - \frac{\mu}{\mu_0})\omega \le v \le \omega.\end{gathered}$$
We claim that
$$u = \theta - \lambda V_{m,n}(p\, g(v)), \tag{3.5}$$
$$v = \omega - \mu V_{m,n}(q\, f(u)). \tag{3.6}$$
Since $v_k \le \omega$ for all $k \in \mathbb{N}$, using hypothesis (H$_3$) and the fact that g is nondecreasing, there exists $\tilde{p} \in K^{\infty}_{m,n}(D)$ such that
$$pg(v) \le pg(\omega) \le \tilde{p}\,\gamma, \tag{3.7}$$
and so $p|g(v_k) - g(v)| \le 2\tilde{p}\gamma$ for all $k \in \mathbb{N}$. From Proposition 2.4, we obtain
$$V_{m,n}(\tilde{p}\,\gamma) \in C(\overline{D}), \tag{3.8}$$
and by Lebesgue's theorem we deduce that
$$\lim_{k\to\infty} V_{m,n}(pg(v_k)) = V_{m,n}(pg(v)).$$
So, letting $k \to \infty$ in the equation $u_k = \theta - \lambda V_{m,n}(pg(v_k))$, we obtain (3.5). Similarly, we obtain (3.6).

Next, we claim that (u, v) satisfies
$$\begin{gathered}(-\Delta)^m u + \lambda pg(v) = 0,\\ (-\Delta)^m v + \mu qf(u) = 0.\end{gathered} \tag{3.9}$$
Indeed, using (3.7) and Proposition 2.3, we obtain $pg(v) \in L^1_{\rm loc}(D)$. Using again (3.7), it follows from (3.8) that
$$V_{m,n}(pg(v)) \in C(\overline{D}).$$
Which implies that
$$V_{m,n}(pg(v)) \in L^1_{\rm loc}(D).$$
Similarly
$$qf(u),\ V_{m,n}(qf(u)) \in L^1_{\rm loc}(D).$$
Now, applying the operator $(-\Delta)^m$ in both (3.5) and (3.6), we deduce that (u, v) is a positive solution (in the sense of distributions) of (3.9).

On the other hand, using Proposition 2.4 and (3.7), we deduce that

$$x \to \frac{V_{m,n}(pg(v))(x)}{(|x|^2-1)^{m-1}} \in C_0(D)$$

and

$$x \to \frac{V_{m,n}(qf(u))(x)}{(|x|^2-1)^{m-1}} \in C_0(D).$$

Thus, we deduce from (3.5) and (3.6) that

$$\lim_{x\to\xi\in\partial D} \frac{u(x)}{(|x|^2-1)^{m-1}} = a\varphi(\xi), \quad \lim_{x\to\xi\in\partial D} \frac{v(x)}{(|x|^2-1)^{m-1}} = b\psi(\xi),$$
$$\lim_{|x|\to\infty} \frac{u(x)}{(|x|^2-1)^{m-1}} = \alpha, \quad \lim_{|x|\to\infty} \frac{v(x)}{(|x|^2-1)^{m-1}} = \beta.$$

Furthermore, the continuity of $\theta, \omega, V_{m,n}(pg(v))$ and $V_{m,n}(qf(u))$ imply that $(u,v) \in (C(D))^2$. This completes the proof. □

Proof of Theorem 1.3. Put $c = 1 + \alpha_{p_1} + \alpha_{q_1}$, where α_{p_1} and α_{q_1} are the constants given in Proposition 2.2 and associated respectively to the functions p_1 and q_1 given in hypothesis (H_5). Suppose that $\varphi \geq c\phi$ and $\psi \geq c\phi$. Then it follows from the maximum principle that for each $x \in D$, we have

$$H_D\varphi(x) \geq c\, h_0(x), \tag{3.10}$$
$$H_D\psi(x) \geq c\, h_0(x). \tag{3.11}$$

We consider the non-empty closed convex set

$$\Lambda = \{w \in C(\overline{D} \cup \{\infty\}) : h_0 \leq w \leq H_D\varphi\}.$$

We define the operator T defined on Λ as

$$Tw = H_D\varphi - \frac{V_{m,n}(pg[\gamma(\beta h + H_D\psi) - V_{m,n}(qf(\tilde{w}))])}{\gamma},$$

where $\tilde{w}(x) = \gamma(x)(w(x) + \alpha h(x)) = (|x|^2-1)^{m-1}(w(x) + \alpha h(x))$. We need to check that the operator T has a fixed point w in Λ.

First, we prove that $T\Lambda$ is relatively compact in $C(\overline{D} \cup \{\infty\})$. Let $w \in \Lambda$, then we have $w + \alpha h \geq h_0$.

Since f is nonincreasing, it follows from Proposition 2.2 that

$$V_{m,n}(qf(\tilde{w})) \leq V_{m,n}(qf(\gamma\, h_0)) = V_{m,n}(qf(\rho_0)) \leq \alpha_{q_1}\, \rho_0.$$

Which implies

$$\gamma(\beta h + H_D\psi) - V_{m,n}(qf(\tilde{w})) \geq \gamma(\beta h + H_D\psi - \alpha_{q_1}\, h_0). \tag{3.12}$$

According to (3.11), we obtain

$$\gamma(\beta h + H_D\psi) - V_{m,n}(qf(\tilde{w})) \geq \gamma(\beta h + h_0) \geq \rho_0. \tag{3.13}$$

Hence

$$Tw \leq H_D\varphi. \tag{3.14}$$

Also, since g is nonincreasing, we obtain

$$pg(\gamma(\beta h + H_D\psi) - V_{m,n}(qf(\tilde{w}))) \leq pg(\rho_0). \tag{3.15}$$

So it follows that for each $y \in D$, we have

$$\frac{p(y)g[\gamma(y)(\beta h(y) + H_D\psi(y)) - V_{m,n}(qf(\tilde{w}))(y)]}{\gamma(y)} \leq p_1(y)\, h_0(y). \tag{3.16}$$

Therefore, we deduce from (H5) and Proposition 2.5 that the family of functions

$$\{x \to \frac{V_{m,n}(pg[\gamma(\beta h + H_D\psi) - V_{m,n}(qf(\tilde{w}))])(x)}{\gamma(x)},\ w \in \Lambda\}$$

is relatively compact in $C(\overline{D} \cup \{\infty\})$. Moreover, since $H_D\varphi \in C(\overline{D} \cup \{\infty\})$, we have the set $T\Lambda$ is relatively compact in $C(\overline{D} \cup \{\infty\})$.

Next, we claim that $T\Lambda \subset \Lambda$. Indeed, let $\omega \in \Lambda$, by using (3.15), (H_5) and Proposition 2.2, we have

$$\frac{V_{m,n}(pg[\gamma(\beta h + H_D\psi) - V_{m,n}(qf(\tilde{w}))])(x)}{\gamma(x)} \leq \alpha_{p_1} h_0(x),$$

for each $x \in D$. According to (3.10), we obtain

$$Tw(x) \geq (1 + \alpha_{q_1})h_0(x) \geq h_0(x), \quad \text{for each } x \in D.$$

This, together with (3.14), proves that $T\omega \in \Lambda$.

Now, we prove the continuity of the operator T in Λ with respect to the supremum norm. Let $(w_k)_{k\in\mathbb{N}}$ be a sequence in Λ which converges uniformly to a function $w \in \Lambda$. Then, for each $x \in D$, we have

$$|Tw_k(x) - Tw(x)| \leq \frac{V_{m,n}(p|g(s_k) - g(s)|)(x)}{\gamma(x)}, \tag{3.17}$$

where $s_k = \gamma(\beta h + H_D\psi) - V_{m,n}(qf(\gamma(w_k + \alpha h)))$ and $s = \gamma(\beta h + H_D\psi) - V_{m,n}(qf(\gamma(w + \alpha h)))$. Using the fact that g is nonincreasing and (3.12) , we have

$$\begin{aligned} p(g(s_k) + g(s)) &\leq 2pg(\gamma(\beta h + H_D\psi - \alpha_{q_1} h_0)) \\ &\leq 2pg(\rho_0) = 2p_1\rho_0. \end{aligned}$$

To reach the last inequality we use (3.11).

Since from (H5) and Proposition 2.4, the function

$$x \to \int_D (\frac{|y|^2 - 1}{|x|^2 - 1})^{m-1} G^D_{m,n}(x, y)\, h_0(y)\, p_1(y) dy$$

is in $C_0(D)$, also using the fact that

$$p|g(s_k) - g(s)| \leq p(g(s_k) + g(s)),$$

it follows from (3.17) and the dominated convergence theorem that for each $x \in D$, the sequence $(Tw_k(x))$ converges to $Tw(x)$ as $k \to \infty$. Since $T\Lambda$ is relatively compact in $C(\overline{D} \cup \{\infty\})$, we deduce that the pointwise convergence implies the uniform convergence; that is,

$$\|Tw_k - Tw\|_\infty \to 0 \quad \text{as } k \to \infty.$$

This shows that T is a continuous mapping from Λ into itself. Then by using Schauder fixed point theorem, there exists $w \in \Lambda$ such that $Tw = w$. Now, for each $x \in D$, put

$$u(x) = (|x|^2 - 1)^{m-1}(w(x) + \alpha h(x)), \tag{3.18}$$

$$v(x) = (|x|^2 - 1)^{m-1}(\beta h(x) + H_D\psi(x)) - V_{m,n}(qf(u))(x). \tag{3.19}$$

Then

$$u(x) - \alpha(|x|^2 - 1)^{m-1}h(x) = (|x|^2 - 1)^{m-1}H_D\varphi(x) - V_{m,n}(pg(v))(x). \tag{3.20}$$

As the remainder of the proof, we aim to show that (u, v) is the desired solution of problem (1.3). By using respectively (3.18), (3.19) and (3.13), clearly (u, v) satisfies for each $x \in D$,

$$h_0(x) + \alpha h(x) \le \frac{u(x)}{(|x|^2-1)^{m-1}} \le H_D\varphi(x) + \alpha h(x) \tag{3.21}$$

and

$$h_0(x) + \beta h(x) \le \frac{v(x)}{(|x|^2-1)^{m-1}} \le H_D\psi(x) + \beta h(x).$$

On the other hand, from (3.18), we have $u(x) \ge \rho_0(x)$ for each $x \in D$. Since f is nonincreasing, this implies

$$qf(u) \le qf(\rho_0) = q_1\rho_0.$$

Note that from (H5) we have q_1 is in the Kato class $K^\infty_{m,n}(D)$, so it follows from Proposition 2.3 that $qf(u) \in L^1_{\rm loc}(D)$ and from Proposition 2.2 that $V_{m,n}(q\,f(u)) \in L^1_{\rm loc}(D)$.

Similarly, we obtain $pg(v) \in L^1_{\rm loc}(D)$ and $V_{m,n}(pg(v)) \in L^1_{\rm loc}(D)$. Then applying the elliptic operator $(-\Delta)^m$ in both (3.18) and (3.19), we obtain clearly that (u, v) is a positive continuous solution (in the distributional sense) of

$$\begin{gathered}(-\Delta)^m u + p(x)g(v) = 0, \quad x \in D,\\ (-\Delta)^m v + q(x)f(u) = 0, \quad x \in D.\end{gathered}$$

Finally, from (3.20), (3.16), Proposition 2.4 and the fact that $H_D\varphi = \varphi$ on ∂D, we conclude that

$$\lim_{x\to\xi\in\partial D} \frac{u(x)}{(|x|^2-1)^{m-1}} = \varphi(\xi).$$

Also, since $\lim_{|x|\to\infty} H_D\varphi(x) = \lim_{|x|\to\infty} h_0(x) = 0$, it follows from (3.21) that

$$\lim_{|x|\to\infty} \frac{u(x)}{(|x|^2-1)^{m-1}} = \alpha.$$

The proof is complete by using the same arguments for v. □

4. Examples

In this Section, we give some examples where hypotheses (H2) and (H3) are satisfied.

Example 4.1. Let $\alpha = 1$, $a = 0$, $\beta = 1$ and $b = 0$. Let f and g be two nonnegative nondecreasing bounded continuous functions on $(0, \infty)$. Assume that p and q are two nonnegative measurable functions on D satisfying

$$p(x) \le \frac{1}{|x|^{\nu-\kappa}(|x|-1)^\kappa}, \quad q(x) \le \frac{1}{|x|^{\nu-\kappa}(|x|-1)^\kappa},$$

with $\kappa < m$ and $\nu > 2$.

Since $|x| + 1 \asymp |x|$, for each $x \in D$, then we have

$$\begin{gathered}\frac{p(x)\,g(\omega(x))}{(|x|^2-1)^{m-1}} \le \frac{C}{|x|^{\nu-\kappa+m-1}(|x|-1)^{m-1+\kappa}},\\ \frac{q(x)\,f(\theta(x))}{(|x|^2-1)^{m-1}} \le \frac{C}{|x|^{\nu-\kappa+m-1}(|x|-1)^{m-1+\kappa}}.\end{gathered}$$

Using the fact that $\kappa < m$ and $\nu > 2$, it follows that the functions

$$x \to \frac{p(x)\, g(\omega(x))}{(|x|^2-1)^{m-1}} \quad \text{and} \quad x \to \frac{q(x)\, f(\theta(x))}{(|x|^2-1)^{m-1}}$$

are in $K^{\infty}_{m,n}(D)$. Now, since for each $x \in D$, we have

$$\begin{gathered} h(x) = 1 - \frac{1}{|x|^{n-2}} \asymp \frac{|x|-1}{|x|}, \\ \theta(x) = (|x|^2-1)^{m-1} h(x) = \omega(x), \end{gathered} \tag{4.1}$$

then there exists $C > 0$ such that

$$p(x)g(\omega(x)) \leq \frac{C}{|x|^{\nu-\kappa+m-2}(|x|-1)^{m+\kappa}}\omega(x), \quad \text{for each } x \in D.$$

So, we deduce from the choice of ν, κ that there exists $p_0 \in K^{\infty}_{m,n}(D)$ such that

$$p(x)g(\omega(x)) \leq p_0(x)\, \omega(x).$$

Which implies from Proposition 2.2 that $V_{m,n}(p\, g(\omega)) \leq C\, \omega$. Hence $\lambda_0 > 0$. Similarly, we have $\mu_0 > 0$.

Example 4.2. Let $\alpha = 1$, $a = 0$, $\beta = 0$ and $b = 1$. Assume that $\psi \geq c_0 > 0$ on ∂D. Let f and g be two continuous and nondecreasing functions on $(0, \infty)$ satisfying for $t \in (0, \infty)$

$$0 \leq g(t) \leq \eta\, t \text{ and } 0 \leq f(t) \leq \xi\, t, \tag{4.2}$$

where η and ξ are positive constants. Suppose furthermore that p and q are non-negative measurable functions on D such that

$$p(x) \leq \frac{1}{|x|^{\delta-\sigma}(|x|-1)^{\sigma}}, \quad q(x) \leq \frac{1}{|x|^{s-r}(|x|-1)^{r}},$$

where

$$\sigma + 1 < 2m < \delta + n - 2, \tag{4.3}$$

$$r - 1 < 2m < 2 - n + s. \tag{4.4}$$

Here $\theta(x) = (|x|^2-1)^{m-1}h(x)$ and $\omega(x) = (|x|^2-1)^{m-1}H_D\psi(x)$.

Since $\psi \geq c_0 > 0$, it follows that

$$H_D\psi(x) \asymp H_D 1(x) = \frac{1}{|x|^{n-2}}, \quad \text{for each } x \in D. \tag{4.5}$$

Then, from (4.2), we have

$$\frac{p(x)\, g(\omega(x))}{(|x|^2-1)^{m-1}} \leq \eta p(x) H_D\psi(x) \leq \frac{C}{|x|^{n-2+\delta-\sigma}(|x|-1)^{\sigma}}. \tag{4.6}$$

Also, using (4.1), we have

$$\frac{q(x)f(\theta(x))}{(|x|^2-1)^{m-1}} \leq \xi q(x)h(x) \leq \frac{C}{|x|^{1+s-r}(|x|-1)^{r-1}}.$$

This, together with (4.3), (4.4) and (4.6), implies that (H3) is satisfied.

Now, using (4.1), (4.2) and (4.5), for each $x \in D$, we have

$$p(x)g(\omega(x)) \leq \eta p(x)\omega(x) \leq C\, p(x)(|x|^2-1)^{m-1}H_D 1(x) \leq \frac{C(|x|^2-1)^{m-1}\, h(x)}{|x|^{n-3+\delta-\sigma}(|x|-1)^{\sigma+1}}.$$

So it follows from (4.3) that there exists $p_2 \in K^\infty_{m,n}(D)$ such that $p\,g(\omega) \le p_2\,\theta$. Hence, it follows from Proposition 2.2 that $V_{m,n}(p\,g(\omega)) \le C\theta$, which implies that $\lambda_0 > 0$.

Using again (4.1), we obtain, for each $x \in D$,

$$q(x)\,h(x) \le \frac{C}{|x|^{1+s-r}(|x|-1)^{r-1}}.$$

According to (4.2) and (4.4), there exists $q_2 \in K^\infty_{m,n}(D)$ satisfying

$$qf(\theta) \le C\gamma q_2 H_D 1.$$

Finally, we deduce from (4.5) and Proposition 2.2 that $V_{m,n}(q\,f(\theta)) \le C\omega$. This implies that $\mu_0 > 0$.

We end this section by giving an example as an application of Theorem 1.3.

Example 4.3. Let $\tau > 0$, $\varepsilon > 0$, $g(t) = t^{-\tau}$ and $f(t) = t^{-\varepsilon}$. Let p and q be two nonnegative measurable functions in D satisfying

$$p(x) \le \frac{1}{(|x|-1)^{l-(1+\tau)m}|x|^{\vartheta-l+(1+\tau)(n-m)}},$$
$$q(x) \le \frac{1}{(|x|-1)^{k-(1+\varepsilon)m}|x|^{\zeta-k+(1+\varepsilon)(n-m)}},$$

where $l < 2m < \vartheta$ and $k < 2m < \zeta$. Let ϕ be a nonnegative nontrivial continuous function on ∂D and put $\rho_0(x) = (|x|^2-1)^{m-1}H_D\phi(x)$ for $x \in D$.

Since for $x \in D$, we have

$$H_D\phi(x) \ge C\,\frac{|x|-1}{(|x|+1)^{n-1}}.$$

Then we obtain for each $x \in D$ that

$$p_1(x) = p(x)\rho_0^{-\tau-1}(x) \le \frac{C}{(|x|-1)^l|x|^{\vartheta-l}}.$$

Similarly, we have

$$q_1(x) \le \frac{C}{(|x|-1)^k|x|^{\zeta-k}}, \quad x \in D.$$

Hence, hypothesis (H5) is satisfied. So there exists $c > 1$ such that if φ and ψ are two nonnegative nontrivial continuous functions on ∂D satisfying $\varphi \ge c\phi$ and $\psi \ge c\phi$ on ∂D, then for each $\alpha \ge 0$ and $\beta \ge 0$, problem

$$(-\Delta)^m u + p(x)\,v^{-\tau} = 0, \quad x \in D, \quad \text{(in the sense of distributions)},$$
$$(-\Delta)^m v + q(x)\,u^{-\varepsilon} = 0, \quad x \in D,$$
$$\lim_{x\to s\in\partial D}\frac{u(x)}{(|x|^2-1)^{m-1}} = \varphi(s), \quad \lim_{x\to s\in\partial D}\frac{v(x)}{(|x|^2-1)^{m-1}} = \psi(s),$$
$$\lim_{|x|\to\infty}\frac{u(x)}{(|x|^2-1)^{m-1}} = \alpha, \quad \lim_{|x|\to\infty}\frac{v(x)}{(|x|^2-1)^{m-1}} = \beta,$$

has a positive continuous solution (u, v) satisfying for each $x \in D$,

$$(|x|^2-1)^{m-1}(\alpha\,h(x) + h_0(x)) \le u(x) \le (|x|^2-1)^{m-1}(\alpha\,h(x) + H_D\varphi(x)),$$
$$(|x|^2-1)^{m-1}(\beta\,h(x) + h_0(x)) \le v(x) \le (|x|^2-1)^{m-1}(\beta\,h(x) + H_D\psi(x)).$$

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