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A LINEAR FIRST-ORDER HYPERBOLIC EQUATION WITH A DISCONTINUOUS COEFFICIENT: DISTRIBUTIONAL SHADOWS AND PROPAGATION OF SINGULARITIES

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Abstract. It is well-known that distributional solutions to the Cauchy problem for $u_t + (b(t,x)u)_x = 0$ with $b(t,x) = 2H(x-t)$, where H is the Heaviside function, are non-unique. However, it has a unique generalized solution in the sense of Colombeau. The relationship between its generalized solutions and distributional solutions is established. Moreover, the propagation of singularities is studied.

1. Introduction

In this paper we study generalized solutions of the Cauchy problem for a first-order hyperbolic equation

$$\begin{gathered} u_t + (b(t,x)u)_x = 0, \quad (t,x) \in \mathbb{R}^2, \\ u|_{t=0} = u_0, \quad x \in \mathbb{R} \end{gathered} \tag{1.1}$$

with $b(t,x) = 2H(x-t)$, where H is the Heaviside function, in the framework of generalized functions introduced by Colombeau [2, 3]. We will seek solutions in an algebra $\mathscr{G}(\mathbb{R}^2)$ of generalized functions, which will be defined in Section 2 below. We mention that $\mathscr{G}(\mathbb{R})$ contains the space $\mathscr{D}'(\mathbb{R})$ of distributions so that initial data with strong singularities can be considered in our setup. The formulation of problem (1.1) in $\mathscr{G}$ will be given in Section 3.

Until now, the following three questions for a variety of partial differential equations in Colombeau's algebras have been addressed: (a) existence and uniqueness of generalized solutions; (b) behavior of generalized solutions in the framework of distribution theory (distributional shadow); (c) regularity of generalized solutions.

For linear first-order hyperbolic systems with discontinuous coefficients, the existence and uniqueness were established in one space-dimensional case by Oberguggenberger [14], for symmetric hyperbolic systems in higher space-dimensional case by Lafon and Oberguggenberger [12] and for hyperbolic pseudodifferential systems with generalized symbols by Hörmann [10].

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Almost all previous results on question (b) for differential equations have been obtained under the hypothesis of unique distributional solutions. However, it is well-known [11] that distributional solutions of linear first-order hyperbolic equations with discontinuous coefficients may fail to exist or may be non-unique. Question (b) for the linear hyperbolic equation having no distributional solution given by Hurd and Sattinger [11], namely, for problem (1.1) with $b = -H$ and $u_0 \equiv 1$, where H is the Heaviside function, was answered by Oberguggenberger [13]. Similar equations have been studied by Hörmann and de Hoop [9]. In this paper we are concerned with question (b) for linear hyperbolic equations having non-unique distributional solutions. According to Hurd and Sattinger [11], for $t > 0$, problem (1.1) with $u_0 \equiv 0$ has infinitely many distributional solutions

$$u_c(t,x) := \begin{cases} 0, & \text{if } x < 0 \text{ or } x > 2t, \\ c, & \text{if } 0 < x < t, \\ -c, & \text{if } t < x < 2t \end{cases} \tag{1.2}$$

with $c \in \mathbb{R}$. On the other hand, as stated above, it has been proved in [14] that problem (1.1) has a unique generalized solution $U \in \mathscr{G}(\mathbb{R}^2)$ for any initial data. First, we will investigate how the generalized solutions are related to the distributional solutions u_c. Several results on behavior of generalized solutions in the framework of distribution theory of differential equations having non-unique distributional solutions have been obtained. For ordinary differential equations, see [5], and for parabolic equations, see [6].

Concerning the regularity of generalized solutions of problem (1.1), we focus on the case of initial data given by the delta function at $s \in \mathbb{R}$. As can be seen in Section 2, there exist an abundant variety of elements of $\mathscr{G}(\mathbb{R})$ having the property of the delta function at s, which are called Dirac generalized functions at s. In particular, there exist Dirac generalized functions at s which can be interpreted to have different strengths of singularity at s. Thus we have the following question: how does the strength of the singularity of a Dirac generalized function taken as initial data affect the regularity of the generalized solution of problem (1.1)? Secondly, we will give an answer to this question. The propagation of singularities for linear first-order hyperbolic equations with other particular discontinuous coefficients has been studied by Hörmann and de Hoop [9], Garetto and Hörmann [7] and Oberguggenberger [16].

The rest of this paper is organized as follows: we recall the definition and basic properties of the Colombeau algebra $\mathscr{G}$ in Section 2. In Section 3, we first give our formulation of problem (1.1) and describe a result on existence and uniqueness of its generalized solution $U \in \mathscr{G}(\mathbb{R}^2)$ for any initial data $U_0 \in \mathscr{G}(\mathbb{R})$ which has been obtained by Oberguggenberger [14]. In Section 4, we discuss how the generalized solutions are related to the distributional solutions u_c given by form (1.2) (Theorem 4.2). In Section 5, we look at problem (1.1) with various Dirac generalized functions as initial data. We investigate the behavior of the generalized solutions in the framework of distribution theory, and further the regularity of the generalized solutions (Theorems 5.1, 5.3, 5.4, 5.6, 5.7 and 5.9).

2. Colombeau's theory of generalized functions

We will employ the *special Colombeau algebra* denoted by $\mathscr{G}^s$ in Grosser et al. [8], which was called the *simplified Colombeau algebra* in Biagioni [1]. However, here we will simply use the letter $\mathscr{G}$ instead. Let us briefly recall the definition and basic properties of the algebra $\mathscr{G}$ of generalized functions. For more details, see Grosser et al. [8].

Let Ω be a non-empty open subset of $\mathbb{R}^d$. Let $\mathscr{E}(\Omega)$ be the differential algebra of all maps from the interval $(0,1]$ into $C^\infty(\Omega)$. Thus each element of $\mathscr{E}(\Omega)$ is a family $(u^\varepsilon)_{\varepsilon\in(0,1]}$ of real valued smooth functions on Ω. The subalgebra $\mathscr{E}_M(\Omega)$ is defined by all elements $(u^\varepsilon)_{\varepsilon\in(0,1]}$ of $\mathscr{E}(\Omega)$ with the property that, for all $K \Subset \Omega$ and $\alpha \in \mathbb{N}_0^d$, there exists $p \geq 0$ such that

$$\sup_{x\in K} |\partial_x^\alpha u^\varepsilon(x)| = O(\varepsilon^{-p}) \quad \text{as } \varepsilon \downarrow 0.$$

The ideal $\mathscr{N}(\Omega)$ is defined by all elements $(u^\varepsilon)_{\varepsilon\in(0,1]}$ of $\mathscr{E}(\Omega)$ with the property that, for all $K \Subset \Omega$, $\alpha \in \mathbb{N}_0^d$ and $q \geq 0$,

$$\sup_{x\in K} |\partial_x^\alpha u^\varepsilon(x)| = O(\varepsilon^{q}) \quad \text{as } \varepsilon \downarrow 0.$$

The algebra $\mathscr{G}(\Omega)$ of generalized functions is defined by the quotient space

$$\mathscr{G}(\Omega) = \mathscr{E}_M(\Omega)/\mathscr{N}(\Omega).$$

We use capital letters for elements of $\mathscr{G}(\Omega)$ to distinguish generalized functions from distributions and denote by $(u^\varepsilon)_{\varepsilon\in(0,1]}$ a representative of $U \in \mathscr{G}(\Omega)$. Then for any U, $V \in \mathscr{G}(\Omega)$ and $\alpha \in \mathbb{N}_0^d$, we can define the partial derivative $\partial^\alpha U$ to be the class of $(\partial^\alpha u^\varepsilon)_{\varepsilon\in(0,1]}$ and the product UV to be the class of $(u^\varepsilon v^\varepsilon)_{\varepsilon\in(0,1]}$. Also, for any $U =$ class of $(u^\varepsilon(t,x))_{\varepsilon\in(0,1]} \in \mathscr{G}(\mathbb{R}^2)$, we can define its restriction $U|_{t=0} \in \mathscr{G}(\mathbb{R})$ to the line $\{t = 0\}$ to be the class of $(u^\varepsilon(0,x))_{\varepsilon\in(0,1]}$.

Remark 2.1. The algebra $\mathscr{G}(\Omega)$ contains the space $\mathscr{E}'(\Omega)$ of compactly supported distributions. In fact, the map

$$f \mapsto \text{class of } (f * \rho_\varepsilon\ |_\Omega)_{\varepsilon\in(0,1]}$$

defines an imbedding of $\mathscr{E}'(\Omega)$ into $\mathscr{G}(\Omega)$, where

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho\left(\frac{x}{\varepsilon}\right)$$

and ρ is a fixed element of $\mathscr{S}(\mathbb{R}^d)$ such that $\int \rho(x)\,dx = 1$ and $\int x^\alpha \rho(x)\,dx = 0$ for any $\alpha \in \mathbb{N}_0^d$, $|\alpha| \geq 1$. In this sense, we obtain an inclusion relation $\mathscr{E}'(\Omega) \subset \mathscr{G}(\Omega)$. This can be extended in a unique way to an imbedding of the space $\mathscr{D}'(\Omega)$ of distributions. Moreover, this imbedding turns $C^\infty(\Omega)$ into a subalgebra of $\mathscr{G}(\Omega)$.

Definition 2.2. A generalized function $U \in \mathscr{G}(\Omega)$ is said to be *associated with a distribution* $w \in \mathscr{D}'(\Omega)$ if it has a representative $(u^\varepsilon)_{\varepsilon\in(0,1]} \in \mathscr{E}_M(\Omega)$ such that

$$u^\varepsilon \to w \quad \text{in } \mathscr{D}'(\Omega) \quad \text{as } \varepsilon \downarrow 0.$$

We denote by $U \approx w$ and call w the *distributional shadow* of U if U is associated with w.

Remark 2.3. A subalgebra $\mathscr{G}_{\log}(\Omega)$ of $\mathscr{G}(\Omega)$ is defined similarly as $\mathscr{G}(\Omega)$ by replacing the bound $\sup_{x\in K} |\partial_x^\alpha u^\varepsilon(x)| = O(\varepsilon^{-p})$ in $\mathscr{E}_M(\Omega)$ by the stronger bound $\sup_{x\in K} |\partial_x^\alpha u^\varepsilon(x)| = O((\log(1/\varepsilon))^p)$. For any distribution $f \in \mathscr{D}'(\Omega)$, there exists a generalized function $U \in \mathscr{G}_{\log}(\Omega)$ which is associated with f, see Colombeau and Heibig [4]. Therefore, any distribution on Ω can be interpreted as an element of $\mathscr{G}_{\log}(\Omega)$ in the sense of association.

We next define the notion of generalized functions of Dirac type.

Definition 2.4. *We say that $U \in \mathscr{G}(\mathbb{R})$ is a* Dirac generalized function *at $s \in \mathbb{R}$ if it has a representative $(u^\varepsilon)_{\varepsilon\in(0,1]}$ satisfying*

1. *there exists $a(\varepsilon) > 0$, $a(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$, such that $u^\varepsilon(x) = 0$ if $|x-s| \geq a(\varepsilon)$;*
2. *$\int_{\mathbb{R}} u^\varepsilon(x)\,dx = 1$ for all $\varepsilon \in (0,1]$;*
3. *$\sup_{\varepsilon\in(0,1]} \int_{\mathbb{R}} |u^\varepsilon(x)|\,dx < \infty$.*

Then U admits the delta function δ_s at s as distributional shadow.

Regularity theory for linear equations has been based on the subalgebra $\mathscr{G}^\infty(\Omega)$ of *regular generalized functions* in $\mathscr{G}(\Omega)$ introduced by Oberguggenberger [15]. It is defined by all elements which have a representative $(u^\varepsilon)_{\varepsilon\in(0,1]}$ with the property that, for all $K \Subset \Omega$, there exists $p \geq 0$ such that, for all $\alpha \in \mathbb{N}_0^d$,

$$\sup_{x\in K} |\partial_x^\alpha u^\varepsilon(x)| = O(\varepsilon^{-p}) \quad \text{as } \varepsilon \downarrow 0.$$

We observe that all derivatives of u^ε have locally the same order of growth in $\varepsilon > 0$, unlike elements of $\mathscr{E}_M(\Omega)$. This subalgebra $\mathscr{G}^\infty(\Omega)$ has the property $\mathscr{G}^\infty(\Omega) \cap \mathscr{D}'(\Omega) = C^\infty(\Omega)$, see [15, Theorem 25.2, p. 275]. Hence, for the purpose of describing the regularity of generalized functions, $\mathscr{G}^\infty(\Omega)$ plays the same role for $\mathscr{G}(\Omega)$ as $C^\infty(\Omega)$ does in the setting of distributions. The $\mathscr{G}^\infty$-singular support (denoted by $\operatorname{sing\,supp}_{\mathscr{G}^\infty}$) of a generalized function is defined as the complement of the largest open set on which the generalized function is regular in the above sense. A subalgebra $\mathscr{G}^\infty_{\log}(\Omega)$ of $\mathscr{G}_{\log}(\Omega)$ is defined similarly as $\mathscr{G}^\infty(\Omega)$ by replacing the bound $\sup_{x\in K} |\partial_x^\alpha u^\varepsilon(x)| = O(\varepsilon^{-p})$ by the stronger bound $\sup_{x\in K} |\partial_x^\alpha u^\varepsilon(x)| = O((\log(1/\varepsilon))^p)$. The $\mathscr{G}^\infty_{\log}$-singular support (denoted by $\operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}}$) can be also introduced.

Remark 2.5. Let $s \in \mathbb{R}$ and let χ be a fixed element of $\mathscr{D}(\mathbb{R})$ such that χ is symmetric, non-negative, with $\operatorname{supp}\chi \subset [-1,1]$, $\chi(0) > 0$ and $\int_{\mathbb{R}} \chi(x)\,dx = 1$. Put $\chi_\varepsilon(x) = \chi(x/\varepsilon)/\varepsilon$. Then $U \in \mathscr{G}(\mathbb{R})$ defined by the class of $(\chi_\varepsilon(\cdot - s))_{\varepsilon\in(0,1]}$ is a Dirac generalized function at s and $\operatorname{sing\,supp}_{\mathscr{G}^\infty} U = \{s\}$. However, if $U \in \mathscr{G}(\mathbb{R})$ is defined as the class of $(\chi_{h(\varepsilon)}(\cdot - s))_{\varepsilon\in(0,1]}$ with $h(\varepsilon) = 1/\log(1/\varepsilon)$, then it is a Dirac generalized function at s again, but $\operatorname{sing\,supp}_{\mathscr{G}^\infty} U = \emptyset$. Hence, the speed of convergence of a representative of U to the delta function at s can be interpreted as the strength of the singularity of U at s. Thus, there exist infinitely many Dirac generalized functions with different strengths of singularity at s in $\mathscr{G}(\mathbb{R})$.

3. Existence and uniqueness of generalized solutions

We formulate problem (1.1) in Colombeau's algebra $\mathscr{G}$ in the form

$$\begin{gathered} U_t + (BU)_x = 0 \quad \text{in } \mathscr{G}(\mathbb{R}^2), \\ U|_{t=0} = U_0 \quad \text{in } \mathscr{G}(\mathbb{R}) \end{gathered} \tag{3.1}$$

with the generalized function $B \in \mathscr{G}(\mathbb{R}^2)$ having the representative $(b^\varepsilon)_{\varepsilon\in(0,1]}$ given by

$$b^\varepsilon(t,x) := b * \varphi_{h(\varepsilon)} = 2\int\int_{\mathbb{R}^2} H(x - t - h(\varepsilon)y + h(\varepsilon)s)\varphi(s,y)\,dy\,ds,$$

where $h(\varepsilon) := 1/\log(1/\varepsilon)$ and φ is a fixed element of $\mathscr{D}(\mathbb{R}^2)$ such that φ is symmetric, non-negative, with $\operatorname{supp}\varphi \subset [-1,1]\times[-1,1]$, $\varphi(0,0) > 0$ and $\int\int \varphi(t,x)\,dx\,dt =$

1. We note that B belongs to $\mathscr{G}_{\log}(\mathbb{R}^2)$ and further admits $b(t,x) = 2H(x-t)$ as distributional shadow.

Definition 3.1. We say that $U \in \mathscr{G}(\mathbb{R}^2)$ is a (*generalized*) *solution* of problem (3.1) if it has a representative $(u^\varepsilon)_{\varepsilon\in(0,1]} \in \mathscr{E}_M(\mathbb{R}^2)$ such that

$$u^\varepsilon_t + (b^\varepsilon u^\varepsilon)_x = N^\varepsilon, \quad (t,x) \in \mathbb{R}^2,$$
$$u^\varepsilon|_{t=0} = u^\varepsilon_0 + n^\varepsilon, \quad x \in \mathbb{R}$$

for some $(N^\varepsilon)_{\varepsilon\in(0,1]} \in \mathscr{N}(\mathbb{R}^2)$ and $(n^\varepsilon)_{\varepsilon\in(0,1]} \in \mathscr{N}(\mathbb{R})$, where $(u^\varepsilon_0)_{\varepsilon\in(0,1]}$ and $(b^\varepsilon)_{\varepsilon\in(0,1]}$ are representatives of U_0 and B, respectively.

For any $U_0 \in \mathscr{G}(\mathbb{R})$, problem (3.1) has a unique solution $U \in \mathscr{G}(\mathbb{R}^2)$, see [14], in which a more general existence and uniqueness result has been obtained.

4. Relationship to non-unique distributional solutions

In this section we establish the relationship between the generalized solutions of problem (3.1) and the distributional solutions u_c of problem (1.1). For this purpose, we first prepare the following lemma.

Lemma 4.1. *For $\varepsilon \in (0,1)$, let*

$$G^\varepsilon(x) := \int_{-x/h(\varepsilon)}^{2} \frac{dz}{1-\widetilde{b}^\varepsilon(-h(\varepsilon)z)} \quad \text{on } [-2h(\varepsilon), 0),$$

where $\widetilde{b}^\varepsilon(z) := 2\int\int_{\mathbb{R}^2} H(z-h(\varepsilon)y+h(\varepsilon)s)\varphi(s,y)\,dy\,ds$. Then G^ε has the following four properties:

- (i) *G^ε is a strictly increasing continuous function on $[-2h(\varepsilon),0)$ such that $G^\varepsilon(-2h(\varepsilon)) = 0$ and $\lim_{x\uparrow 0} G^\varepsilon(x) = \infty$;*
- (ii) *there exist two constants C_1, $C_2 > 0$ such that, for any $x \in [-2h(\varepsilon),0)$ and $\varepsilon \in (0,1)$,*
$$C_1 \log\Big(\frac{2h(\varepsilon)}{-x}\Big) \le G^\varepsilon(x) \le C_2 \log\Big(\frac{2h(\varepsilon)}{-x}\Big); \tag{4.1}$$
- (iii) *there exists the inverse function $(G^\varepsilon)^{-1}$ on $[0,\infty)$, which satisfies*
$$\frac{d(G^\varepsilon)^{-1}(x)}{dx} = h(\varepsilon)[1-\widetilde{b}^\varepsilon((G^\varepsilon)^{-1}(x))]; \tag{4.2}$$
- (iv) *for any a, $x \in [0,\infty)$,*
$$2\exp\Big(-\frac{a}{C_1}\Big)h(\varepsilon)\varepsilon^{x/C_1} \le \Big|(G^\varepsilon)^{-1}\Big(\frac{x}{h(\varepsilon)}+a\Big)\Big| \le 2\exp\Big(-\frac{a}{C_2}\Big)h(\varepsilon)\varepsilon^{x/C_2}, \tag{4.3}$$
where C_1, $C_2 > 0$ are the constants given in (ii).

Proof. Property (i) is clear. To show property (ii), rewrite $1 = 2\int\int_{s>y}\varphi(s,y)\,dy\,ds$. Then we find that

$$1-\widetilde{b}^\varepsilon(-h(\varepsilon)z) = 1 - 2\int\int_{s>y+z}\varphi(s,y)\,dy\,ds = 2\int\int_{y<s<y+z}\varphi(s,y)\,ds\,dy.$$

Hence, there exist two constants c_1, $c_2 > 0$ such that $c_1 z \le 1-\widetilde{b}^\varepsilon(-h(\varepsilon)z) \le c_2 z$ for $0 \le z \le 2$. Putting $C_1 = 1/c_2$ and $C_2 = 1/c_1$, the reciprocal of $1-\widetilde{b}^\varepsilon(-h(\varepsilon)z)$ satisfies the inequality

$$\frac{C_1}{z} \le \frac{1}{1-\widetilde{b}^\varepsilon(-h(\varepsilon)z)} \le \frac{C_2}{z}.$$

Integrating this over $[-x/h(\varepsilon), 2)$ gives inequality (4.1).

Next, we prove property (iii). By property (i), there exists $(G^\varepsilon)^{-1}$ on $[0,\infty)$. We differentiate $(G^\varepsilon)^{-1}$ to get $d(G^\varepsilon)^{-1}(x)/dx = 1/(G^\varepsilon)'((G^\varepsilon)^{-1}(x))$. We have $(G^\varepsilon)'(x) = 1/h(\varepsilon)(1-\widetilde{b}^\varepsilon(x))$ and so get formula (4.2).

Finally, we prove property (iv). Put $y = (G^\varepsilon)^{-1}(x/h(\varepsilon)+a) < 0$. By property (ii), there exist two constants C_1, $C_2 > 0$ such that

$$C_1 \log\Big(\frac{2h(\varepsilon)}{-y}\Big) \le G^\varepsilon(y) \le C_2 \log\Big(\frac{2h(\varepsilon)}{-y}\Big).$$

Noting that $G^\varepsilon(y) = x/h(\varepsilon)+a$, we have

$$C_1\left[\log 2h(\varepsilon) - \log(-y)\right] \le \frac{x}{h(\varepsilon)} + a \le C_2\left[\log 2h(\varepsilon) - \log(-y)\right].$$

Therefore, we see that

$$\log 2h(\varepsilon) - \frac{a}{C_1} - \frac{x}{C_1 h(\varepsilon)} \le \log(-y) \le \log 2h(\varepsilon) - \frac{a}{C_2} - \frac{x}{C_2 h(\varepsilon)}.$$

Since $h(\varepsilon) = 1/\log(1/\varepsilon)$, it follows that

$$2\exp\Big(-\frac{a}{C_1}\Big)h(\varepsilon)\varepsilon^{x/C_1} \le -y \le 2\exp\Big(-\frac{a}{C_2}\Big)h(\varepsilon)\varepsilon^{x/C_2}.$$

Thus inequality (4.3) follows. □

We now turn to a comparison between generalized solutions of problem (3.1) and the distributional solutions u_c given by (1.2) of problem (1.1).

Theorem 4.2. *For any $c \in \mathbb{R}$ and $T > 0$, there exists initial data $U_0 \approx 0$ such that the solution $U \in \mathscr{G}(\mathbb{R}^2)$ of problem* (3.1) *admits a distributional shadow on $(-T,T)\times\mathbb{R}$, which is given by*

$$u(t,x) = \begin{cases} u_c(t,x), & \text{if } 0 < t < T,\ x \in \mathbb{R},\\ 0, & \text{if } -T < t \le 0,\ x \in \mathbb{R}, \end{cases}$$

where u_c is the function given by (1.2).

Proof. We consider the Cauchy problem

$$\begin{gathered} V_t + BV_x = 0 \quad \text{in } \mathscr{G}(\mathbb{R}^2),\\ V|_{t=0} = V_0 \quad \text{in } \mathscr{G}(\mathbb{R}). \end{gathered} \tag{4.4}$$

The existence and uniqueness of solutions $V \in \mathscr{G}(\mathbb{R}^2)$ of problem (4.4) are guaranteed for all initial data $V_0 \in \mathscr{G}(\mathbb{R})$ by Oberguggenberger [14]. Clearly, V_x satisfies problem (3.1). Define the function $v(t,x) := \int_0^x u(t,y)\,dy$. In order to prove the assertion, it suffices to show that, for any $c \in \mathbb{R}$ and $T > 0$, there exists initial data V_0 such that $V_0' \approx 0$ on $\mathbb{R}$ and $V \approx v$ on $(-T,T)\times\mathbb{R}$. We will only prove this for the case $c > 0$. The case $c \le 0$ can be argued similarly. The proof is divided into three steps.

Step 1. Fix $c > 0$ and $\varepsilon \in (0,1)$ arbitrarily. Let G^ε be as in Lemma 4.1. Recall that G^ε is a strictly increasing continuous function on $[-2h(\varepsilon), 0)$ with $G^\varepsilon(-2h(\varepsilon)) = 0$ and $\lim_{x\uparrow 0} G^\varepsilon(x) = \infty$. Hence, there exists a point $0 < \eta(\varepsilon) <$

$2h(\varepsilon)$ such that $G^\varepsilon(-\eta(\varepsilon)) = 2$. Define

$$w_0^\varepsilon(x) := \begin{cases} ch(\varepsilon)(G^\varepsilon(x) - 2), & \text{if } -\eta(\varepsilon) \le x < 0, \\ ch(\varepsilon)(G^\varepsilon(-x) - 2), & \text{if } 0 < x \le \eta(\varepsilon), \\ 0, & \text{if } |x| > \eta(\varepsilon), \end{cases} \tag{4.5}$$

and let w^ε be a solution of the problem

$$\begin{gathered} w_t^\varepsilon + b^\varepsilon w_x^\varepsilon = 0, \quad (t,x) \in \mathbb{R}^2,\ t \neq x \\ w^\varepsilon|_{t=0} = w_0^\varepsilon, \quad x \in \mathbb{R},\ x \neq 0. \end{gathered}$$

We will show below that, for $t \ge 0$,

$$w^\varepsilon(t,x) = \begin{cases} 0, & \text{if } x \le -2h(\varepsilon) \text{ or } x \ge 2t + 2h(\varepsilon), \\ cx, & \text{if } 0 \le x \le t - 2h(\varepsilon), \\ -cx + 2ct, & \text{if } t + 2h(\varepsilon) \le x \le 2t. \end{cases}$$

Similarly, we can obtain that, for $t < 0$,

$$w^\varepsilon(t,x) = 0 \quad \text{if } x \le t - 2h(\varepsilon) \text{ or } x \ge t + 2h(\varepsilon).$$

The characteristic curve $\gamma^\varepsilon(t,x,\tau)$ passing through (t,x) at time $\tau = t$ is the solution of the problem

$$\begin{gathered} \gamma_\tau^\varepsilon(t,x,\tau) = b^\varepsilon(\tau, \gamma^\varepsilon(t,x,\tau)), \\ \gamma^\varepsilon|_{\tau=t} = x. \end{gathered} \tag{4.6}$$

Along the characteristic curves, w^ε is easily calculated as

$$w^\varepsilon(t,x) = w_0^\varepsilon(\gamma^\varepsilon(t,x,0)). \tag{4.7}$$

If $x \le -2h(\varepsilon)$ and $t > 0$, then $\gamma^\varepsilon(t,x,0) = x$ and $w_0^\varepsilon(x) = 0$. Hence, by (4.7), we have $w^\varepsilon(t,x) = 0$. If $x \ge 2t + 2h(\varepsilon)$ and $t > 0$, then $\gamma^\varepsilon(t,x,0) = x - 2t$ and $w_0^\varepsilon(x - 2t) = 0$. Hence, by (4.7), we get $w^\varepsilon(t,x) = 0$.

We next prove that $w^\varepsilon(t,x) = cx$ if $0 \le x \le t - 2h(\varepsilon)$. Fix (t,x) arbitrarily so that $0 \le x \le t - 2h(\varepsilon)$. Let $\widetilde{b}^\varepsilon$ be as in Lemma 4.1. Then, from (4.6), we see that $\gamma^\varepsilon(t,x,\tau)$ satisfies the equation $(\gamma^\varepsilon(t,x,\tau) - \tau)_\tau = \widetilde{b}^\varepsilon(\gamma^\varepsilon(t,x,\tau) - \tau) - 1$. We divide both sides by $\widetilde{b}^\varepsilon(\gamma^\varepsilon(t,x,\tau) - \tau) - 1$ and integrate it over $[0,t]$ to get

$$\int_0^t \frac{(\gamma^\varepsilon(t,x,\tau) - \tau)_\tau}{\widetilde{b}^\varepsilon(\gamma^\varepsilon(t,x,\tau) - \tau) - 1}\, d\tau = t.$$

Putting $\gamma = \gamma^\varepsilon(t,x,\tau) - \tau$ and noting that $\gamma^\varepsilon(t,x,t) = x$, we have

$$\int_{\gamma^\varepsilon(t,x,0)}^{x-t} \frac{d\gamma}{\widetilde{b}^\varepsilon(\gamma) - 1} = t. \tag{4.8}$$

Since $x - t \le -2h(\varepsilon)$ and $\widetilde{b}^\varepsilon(\gamma) = 0$ for $\gamma \le -2h(\varepsilon)$, it follows that

$$\int_{\gamma^\varepsilon(t,x,0)}^{-2h(\varepsilon)} \frac{d\gamma}{\widetilde{b}^\varepsilon(\gamma) - 1} = x + 2h(\varepsilon).$$

Put $z = -\gamma/h(\varepsilon)$. Then

$$\int_{-\gamma^\varepsilon(t,x,0)/h(\varepsilon)}^{2} \frac{dz}{1 - \widetilde{b}^\varepsilon(-h(\varepsilon)z)} = \frac{x}{h(\varepsilon)} + 2.$$

The left-hand side is rewritten as $G^\varepsilon(\gamma^\varepsilon(t,x,0))$ and so $\gamma^\varepsilon(t,x,0) = (G^\varepsilon)^{-1}(x/h(\varepsilon)+2)$. Therefore, by (4.5) and (4.7), we get $w^\varepsilon(t,x) = cx$.

We next prove that $w^\varepsilon(t,x) = -cx + 2ct$ if $t + 2h(\varepsilon) \le x \le 2t$. Fix (t,x) arbitrarily so that $t + 2h(\varepsilon) \le x \le 2t$. The same argument as above gives (4.8). Since $x - t \ge 2h(\varepsilon)$ and $\widetilde{b}^\varepsilon(\gamma) = 2$ for $\gamma \ge 2h(\varepsilon)$, we have

$$\int_{\gamma^\varepsilon(t,x,0)}^{2h(\varepsilon)} \frac{d\gamma}{\widetilde{b}^\varepsilon(\gamma) - 1} = -x + 2t + 2h(\varepsilon).$$

Put $z = \gamma/h(\varepsilon)$. Then

$$\int_{\gamma^\varepsilon(t,x,0)/h(\varepsilon)}^{2} \frac{dz}{\widetilde{b}^\varepsilon(h(\varepsilon)z) - 1} = \frac{-x + 2t}{h(\varepsilon)} + 2.$$

The left-hand side is equal to $G^\varepsilon(-\gamma^\varepsilon(t,x,0))$, since $\widetilde{b}^\varepsilon(h(\varepsilon)z) - 1 = 1 - \widetilde{b}^\varepsilon(-h(\varepsilon)z)$ for $z \in \mathbb{R}$. Hence, $\gamma^\varepsilon(t,x,0) = -(G^\varepsilon)^{-1}((-x+2t)/h(\varepsilon) + 2)$. Therefore, by (4.5) and (4.7), we obtain that $w^\varepsilon(t,x) = -cx + 2ct$.

Step 2. Fix $T > 0$ arbitrarily. Then $T/h(\varepsilon) > 2$ for $\varepsilon > 0$ small enough. For such $\varepsilon > 0$, we choose $0 < \lambda(\varepsilon) < \eta(\varepsilon)$ such that $G^\varepsilon(-\lambda(\varepsilon)) = T/h(\varepsilon)$, and put

$$\overline{w_0^\varepsilon}(x) := \begin{cases} w_0^\varepsilon(x), & \text{if } |x| > \lambda(\varepsilon), \\ w_0^\varepsilon(\lambda(\varepsilon)), & \text{if } |x| \le \lambda(\varepsilon). \end{cases}$$

Let $\overline{w^\varepsilon}$ be a solution of the problem

$$\begin{gathered} (\overline{w^\varepsilon})_t + b^\varepsilon(\overline{w^\varepsilon})_x = 0, \quad (t,x) \in \mathbb{R}^2, \\ \overline{w^\varepsilon}|_{t=0} = \overline{w_0^\varepsilon}, \quad x \in \mathbb{R}. \end{gathered}$$

Then it is easy to check that, for $t \ge 0$,

$$\overline{w^\varepsilon}(t,x) = \begin{cases} 0, & \text{if } x \le -2h(\varepsilon) \text{ or } x \ge 2t + 2h(\varepsilon), \\ cx, & \text{if } 0 \le x \le \min\{t - 2h(\varepsilon), \gamma^\varepsilon(0,-\lambda(\varepsilon),t)\}, \\ -cx + 2ct, & \text{if } \max\{t + 2h(\varepsilon), \gamma^\varepsilon(0,\lambda(\varepsilon),t)\} \le x \le 2t, \end{cases}$$

and further that, for $t < 0$,

$$\overline{w^\varepsilon}(t,x) = 0 \quad \text{if } x \le t - 2h(\varepsilon) \text{ or } x \ge t + 2h(\varepsilon).$$

We now prove that $\overline{w^\varepsilon}$ converges to v in $\mathscr{D}'((-T,T) \times \mathbb{R})$ as $\varepsilon \downarrow 0$. Consider the characteristic curve $\gamma^\varepsilon(0,-\lambda(\varepsilon),\tau)$ passing through $(0,-\lambda(\varepsilon))$ at $\tau = 0$. There exists $t_1^\varepsilon > 0$ such that $t_1^\varepsilon = \gamma^\varepsilon(0,-\lambda(\varepsilon),t_1^\varepsilon) + 2h(\varepsilon)$. As in Step 1, we can show that

$$t_1^\varepsilon = h(\varepsilon) \int_{\lambda(\varepsilon)/h(\varepsilon)}^{2} \frac{dz}{1 - \widetilde{b}^\varepsilon(-h(\varepsilon)z)} = h(\varepsilon) G^\varepsilon(-\lambda(\varepsilon)) = T.$$

Similarly, for the characteristic curve $\gamma^\varepsilon(0,\lambda(\varepsilon),\tau)$ passing through $(0,\lambda(\varepsilon))$ at $\tau = 0$, there exists $t_2^\varepsilon > 0$ such that $t_2^\varepsilon = \gamma^\varepsilon(0,\lambda(\varepsilon),t_2^\varepsilon) - 2h(\varepsilon)$. Moreover, $t_2^\varepsilon = T$.

Hence, for any $\psi \in \mathscr{D}((-T,T)\times\mathbb{R})$, we see that

$$\begin{aligned}
&\int_{-T}^{T}\int_{-\infty}^{\infty}(\overline{w^\varepsilon}(t,x)-v(t,x))\psi(t,x)\,dx\,dt\\
&=\int\int_{-2h(\varepsilon)<x<\min\{0,t-2h(\varepsilon)\},\,0<t<T}(\overline{w^\varepsilon}(t,x)-v(t,x))\psi(t,x)\,dx\,dt\\
&\quad+\int\int_{t-2h(\varepsilon)<x<t+2h(\varepsilon),\,-T<t<T}(\overline{w^\varepsilon}(t,x)-v(t,x))\psi(t,x)\,dx\,dt\\
&\quad+\int\int_{\max\{2t,t+2h(\varepsilon)\}<x<2t+2h(\varepsilon),\,0<t<T}(\overline{w^\varepsilon}(t,x)-v(t,x))\psi(t,x)\,dx\,dt.
\end{aligned}\tag{4.9}$$

The area of $\{(t,x)\in\mathbb{R}^2 \mid -2h(\varepsilon)<x<\min\{0,t-2h(\varepsilon)\}\}\cap\operatorname{supp}\psi$ converges to 0 as $\varepsilon\downarrow 0$. Moreover, $(\overline{w^\varepsilon}-v)_{\varepsilon\in(0,1]}$ is uniformly bounded on this intersection. Hence, the first integral on the right-hand side of (4.9) converges to 0 as $\varepsilon\downarrow 0$. We can similarly show that the second and third integrals converge to 0 as $\varepsilon\downarrow 0$. Thus, $\overline{w^\varepsilon}$ converges to v in $\mathscr{D}'((-T,T)\times\mathbb{R})$ as $\varepsilon\downarrow 0$.

Step 3. Finally, we construct $(v_0^\varepsilon)_{\varepsilon\in(0,1]}\in\mathscr{E}_M(\mathbb{R})$ such that $(v_0^\varepsilon)'$ converges to 0 in $\mathscr{D}'(\mathbb{R})$ as $\varepsilon\downarrow 0$, and further that the solution v^ε of the problem

$$\begin{gathered}
v_t^\varepsilon+b^\varepsilon v_x^\varepsilon=0,\quad (t,x)\in\mathbb{R}^2,\\
v^\varepsilon|_{t=0}=v_0^\varepsilon,\quad x\in\mathbb{R}
\end{gathered}\tag{4.10}$$

converges to v in $\mathscr{D}'((-T,T)\times\mathbb{R})$ as $\varepsilon\downarrow 0$. The existence of such $(v_0^\varepsilon)_{\varepsilon\in(0,1]}$ implies the existence of initial data $V_0\in\mathscr{G}(\mathbb{R})$ satisfying the desired properties that $V_0'\approx 0$ on $\mathbb{R}$ and $V\approx v$ on $(-T,T)\times\mathbb{R}$.

Let $\chi\in\mathscr{D}(\mathbb{R})$ be as in Remark 2.5. Define the function $v_0^\varepsilon(x):=(\overline{w_0^\varepsilon}*\chi_{\lambda(\varepsilon)})(x)$. We have $\sup_{x\in\mathbb{R}}|\overline{w_0^\varepsilon}(x)|=|w_0^\varepsilon(\lambda(\varepsilon))|=cT-2ch(\varepsilon)$. Furthermore, by inequality (4.3), there exist two constants $C_1,\ C_2>0$ such that $2h(\varepsilon)\varepsilon^{T/C_1}\le\lambda(\varepsilon)\le 2h(\varepsilon)\varepsilon^{T/C_2}$. Hence, we see that the family of v_0^ε defines an element of $\mathscr{E}_M(\mathbb{R})$, and further that $(v_0^\varepsilon)'$ converges to 0 in $\mathscr{D}'(\mathbb{R})$ as $\varepsilon\downarrow 0$.

To show that the solution v^ε of problem (4.10) converges to v in $\mathscr{D}'((-T,T)\times\mathbb{R})$ as $\varepsilon\downarrow 0$, it suffices to prove that $v_0^\varepsilon-\overline{w_0^\varepsilon}$ converges uniformly to 0 on any compact subset of $\mathbb{R}$ as $\varepsilon\downarrow 0$. The difference $v_0^\varepsilon(x)-\overline{w_0^\varepsilon}(x)$ satisfies the inequality

$$\left|v_0^\varepsilon(x)-\overline{w_0^\varepsilon}(x)\right|\le\int_{-\infty}^{\infty}|\overline{w_0^\varepsilon}(x-\lambda(\varepsilon)y)-\overline{w_0^\varepsilon}(x)|\chi(y)\,dy.$$

Moreover,

$$|\overline{w_0^\varepsilon}(x-\lambda(\varepsilon)y)-\overline{w_0^\varepsilon}(x)|\le\sup_{-\eta(\varepsilon)\le\xi\le-\lambda(\varepsilon)}|(\overline{w_0^\varepsilon})'(\xi)|\lambda(\varepsilon)|y|=\frac{c}{1-\widetilde{b}^\varepsilon(-\lambda(\varepsilon))}\lambda(\varepsilon)|y|.$$

As in the proof of Lemma 4.1, we have $1/(1-\widetilde{b}^\varepsilon(-\lambda(\varepsilon)))\le C_2h(\varepsilon)/\lambda(\varepsilon)$ for some constant $C_2>0$. Thus, we get

$$\left|v_0^\varepsilon(x)-\overline{w_0^\varepsilon}(x)\right|\le cC_2\int_{-\infty}^{\infty}|y|\chi(y)\,dy\cdot h(\varepsilon)\to 0\quad\text{as }\varepsilon\downarrow 0.$$

The proof of Theorem 4.2 is now complete. $\square$

Remark 4.3. In Theorem 4.2, for $t<0$, the solution $U\in\mathscr{G}(\mathbb{R}^2)$ admits 0 as distributional shadow, which is the unique distributional solution of problem (1.1) for negative time with 0 initial data, see Hurd and Sattinger [11].

Remark 4.4. Theorem 4.2 means that, in the setting of Colombeau's theory, all distributional solutions u_c with initial data 0 can be regarded as generalized solutions with different initial data.

Remark 4.5. A similar result to Theorem 4.2 does not necessarily hold for other differential equations having non-unique distributional solutions. In fact, there exists an ordinary differential equation having a classical solution with which none of its generalized solutions is associated. For details, see [5].

5. Propagation of singularities

In this section we study the propagation of singularities for problem (3.1). The coefficient B in problem (3.1) is $\mathscr{G}^\infty$-regular, since B is an element of $\mathscr{G}_{\log}(\mathbb{R}^2) \subset \mathscr{G}^\infty(\mathbb{R}^2)$. Hence, the subalgebra $\mathscr{G}^\infty_{\log} \subset \mathscr{G}_{\log}$ is suitable to study the propagation of singularities for problem (3.1). However, we are also interested in the propagation of singularities in $\mathscr{G}^\infty$, since $U_0 \in \mathscr{G}^\infty(\mathbb{R})$ does not necessarily imply that $U \in \mathscr{G}^\infty(\mathbb{R}^2)$. Thus, we discuss the propagation of singularities in both $\mathscr{G}^\infty_{\log}$ and $\mathscr{G}^\infty$ for problem (3.1).

Let $\chi \in \mathscr{D}(\mathbb{R})$ be as in Remark 2.5. Assume that $U_0 \in \mathscr{G}(\mathbb{R})$ is given by the class of $(\chi_{h(\varepsilon)})_{\varepsilon\in(0,1]}$. Then U_0 is a Dirac generalized function at 0 and belongs to $\mathscr{G}^\infty(\mathbb{R}) \setminus \mathscr{G}^\infty_{\log}(\mathbb{R})$. As may be seen in the following theorem, the singularity in $\mathscr{G}^\infty_{\log}$ of the initial data U_0 splits in two directions at the origin due to the discontinuity of the coefficient.

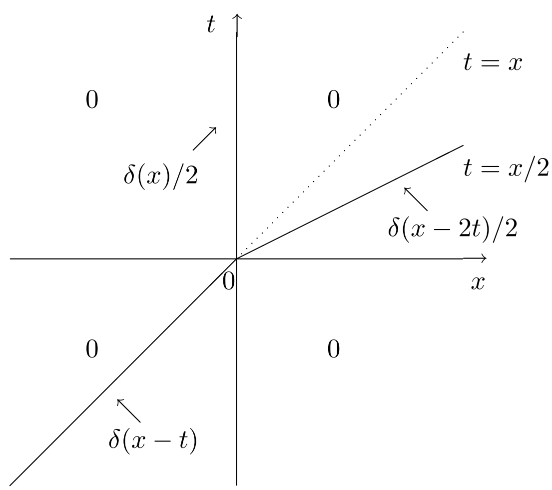

Figure 1. Distributional shadow

Theorem 5.1. *Let $U_0 \in \mathscr{G}(\mathbb{R})$ be as above. Then the solution $U \in \mathscr{G}(\mathbb{R}^2)$ of problem* (3.1) *admits a distributional shadow, which is given by*

$$u(t,x) = \begin{cases} \frac{\delta(x)+\delta(x-2t)}{2}, & \text{if } t \geq 0,\ x \in \mathbb{R}, \\ \delta(x-t), & \text{if } t < 0,\ x \in \mathbb{R}. \end{cases}$$

Furthermore,

$$\operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}} U = \{(t,0) \mid t \geq 0\} \cup \{(t,2t) \mid t \geq 0\} \cup \{(t,t) \mid t \leq 0\} \ (= \operatorname{sing\,supp} u).$$

Proof. Let $v_0^\varepsilon = H * \chi_{h(\varepsilon)}$ and let $V_0 \in \mathscr{G}(\mathbb{R})$ be given by the class of $(v_0^\varepsilon)_{\varepsilon\in(0,1]}$. In order to prove the first assertion, it suffices to show that the solution $V \in \mathscr{G}(\mathbb{R}^2)$ of problem (4.4) admits a distributional shadow, which is given by

$$v(t,x) = \begin{cases} \frac{H(x)+H(x-2t)}{2}, & \text{if } t \geq 0,\ x \in \mathbb{R}, \\ H(x-t), & \text{if } t < 0,\ x \in \mathbb{R}. \end{cases}$$

Let $(v^\varepsilon)_{\varepsilon\in(0,1]}$ be a representative of $V \in \mathscr{G}(\mathbb{R}^2)$ satisfying

$$\begin{gathered} v_t^\varepsilon + b^\varepsilon v_x^\varepsilon = 0, \quad (t,x) \in \mathbb{R}^2, \\ v^\varepsilon|_{t=0} = v_0^\varepsilon, \quad x \in \mathbb{R}. \end{gathered} \tag{5.1}$$

Consider the characteristic curve $\gamma^\varepsilon(0,x,t)$ passing through $(0,x)$ at time $t = 0$. Along the characteristic curves, we have $v^\varepsilon(t,\gamma^\varepsilon(0,x,t)) = v_0^\varepsilon(x)$. We can easily check that v^ε converges to $H(x-t)$ a.e. in $(-\infty,0)\times\mathbb{R}$ as $\varepsilon \downarrow 0$.

We now fix $0 < a \leq 1$ arbitrarily, and put $t_1^\varepsilon := \gamma^\varepsilon(0,-ah(\varepsilon),t_1^\varepsilon) + 2h(\varepsilon)$. As in Step 1 of the proof of Theorem 4.2, we have

$$t_1^\varepsilon = h(\varepsilon)\int_a^2 \frac{dz}{1-\widetilde{b}^\varepsilon(-h(\varepsilon)z)} = h(\varepsilon)\int_a^2 \frac{dz}{1-2\int\int_{s>y+z}\varphi(s,y)\,dy\,ds} \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

Note that, for any $t \geq t_1^\varepsilon$, $\gamma^\varepsilon(0,-ah(\varepsilon),t) = \gamma^\varepsilon(0,-ah(\varepsilon),t_1^\varepsilon)$. Hence, for any $t \geq 0$, we have $\gamma^\varepsilon(0,-ah(\varepsilon),t) \to 0$ as $\varepsilon \downarrow 0$. Moreover,

$$v^\varepsilon(t,\gamma^\varepsilon(0,-ah(\varepsilon),t)) = v_0^\varepsilon(-ah(\varepsilon)) = \int_{-\infty}^{-a}\chi(y)\,dy,$$

so that $v^\varepsilon(t,\gamma^\varepsilon(0,-h(\varepsilon),t)) = 0$ and $v^\varepsilon(t,\gamma^\varepsilon(0,-ah(\varepsilon),t)) \uparrow 1/2$ as $a \downarrow 0$. Similarly, we take $t_2^\varepsilon := \gamma^\varepsilon(0,ah(\varepsilon),t_2^\varepsilon) - 2h(\varepsilon)$ to get

$$t_2^\varepsilon = h(\varepsilon)\int_a^2 \frac{dz}{\widetilde{b}^\varepsilon(h(\varepsilon)z)-1} = h(\varepsilon)\int_a^2 \frac{dz}{2\int\int_{s>y-z}\varphi(s,y)\,dy\,ds - 1} \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

Note that, for any $t \geq t_2^\varepsilon$, $\gamma^\varepsilon(0,ah(\varepsilon),t) = 2t - \gamma^\varepsilon(0,ah(\varepsilon),t_2^\varepsilon) + 4h(\varepsilon)$. Therefore, for any $t \geq 0$, $\gamma^\varepsilon(0,ah(\varepsilon),t) \to 2t$ as $\varepsilon \downarrow 0$. Moreover,

$$v^\varepsilon(t,\gamma^\varepsilon(0,ah(\varepsilon),t)) = v_0^\varepsilon(ah(\varepsilon)) = \int_{-\infty}^{a}\chi(y)\,dy,$$

so that $v^\varepsilon(t,\gamma^\varepsilon(0,h(\varepsilon),t)) = 1$ and $v^\varepsilon(t,\gamma^\varepsilon(0,ah(\varepsilon),t)) \downarrow 1/2$ as $a \downarrow 0$. Hence, taking into account the fact that $v^\varepsilon(t,x)$ is non-decreasing in x, we obtain that v^ε converges to $(H(x)+H(x-2t))/2$ a.e. in $(0,\infty)\times\mathbb{R}$ as $\varepsilon \downarrow 0$. Thus, the first assertion follows.

Next, we prove the second assertion. The proof is divided into four steps.

Step 1. First, we prove that $U \in \mathscr{G}(\mathbb{R}^2)$ is $\mathscr{G}_{\log}^\infty$-regular on $\{(t,x) \in \mathbb{R}^2 \mid x < \min\{0,t\}\} \cup \{(t,x) \in \mathbb{R}^2 \mid x > \max\{t,2t\}\}$.

It is easy to check that $V \in \mathscr{G}(\mathbb{R}^2)$ equals 0 on $\{(t,x) \in \mathbb{R}^2 \mid x < \min\{0,t\}\}$ and further equals 1 on $\{(t,x) \in \mathbb{R}^2 \mid x > \max\{t,2t\}\}$. Hence, $U = V_x \in \mathscr{G}(\mathbb{R}^2)$ is $\mathscr{G}_{\log}^\infty$-regular on the union of these two sets.

Step 2. Secondly, we prove that $\{(t,t) \mid t \leq 0\}$ is contained in $\operatorname{sing\,supp}_{\mathscr{G}_{\log}^\infty} U$.

Fix $t < 0$ arbitrarily. We see that $v^\varepsilon(t,x) = v_0^\varepsilon(\gamma^\varepsilon(t,x,0))$. Since $v_0^\varepsilon = H * \chi_{h(\varepsilon)}$, we get

$$v_x^\varepsilon(t,x) = (v_0^\varepsilon)'(\gamma^\varepsilon(t,x,0))\gamma_x^\varepsilon(t,x,0) = \frac{1}{h(\varepsilon)}\chi\left(\frac{\gamma^\varepsilon(t,x,0)}{h(\varepsilon)}\right)\gamma_x^\varepsilon(t,x,0). \tag{5.2}$$

From problem (4.6) and the definition of $\widetilde{b}^\varepsilon$, we find that $\gamma^\varepsilon(t,x,\tau)$ satisfies the problem

$$\begin{gathered}\gamma^\varepsilon_\tau(t,x,\tau)=\widetilde{b}^\varepsilon(\gamma^\varepsilon(t,x,\tau)-\tau),\\ \gamma^\varepsilon|_{\tau=t}=x.\end{gathered}\tag{5.3}$$

We differentiate these equations in x to get

$$\begin{gathered}\gamma^\varepsilon_{\tau x}(t,x,\tau)=(\widetilde{b}^\varepsilon)'(\gamma^\varepsilon(t,x,\tau)-\tau)\gamma^\varepsilon_x(t,x,\tau),\\ \gamma^\varepsilon_x|_{\tau=t}=1.\end{gathered}$$

We divide the first equation by $\gamma^\varepsilon_x(t,x,\tau)$ and integrate it over $[t,0]$ to see that

$$\int_t^0\frac{\gamma^\varepsilon_{\tau x}(t,x,\tau)}{\gamma^\varepsilon_x(t,x,\tau)}\,d\tau=\int_t^0(\widetilde{b}^\varepsilon)'(\gamma^\varepsilon(t,x,\tau)-\tau)\,d\tau.$$

A simple calculation shows that

$$\gamma^\varepsilon_x(t,x,0)=\exp\Big(\int_t^0(\widetilde{b}^\varepsilon)'(\gamma^\varepsilon(t,x,\tau)-\tau)\,d\tau\Big).$$

Since $\gamma^\varepsilon(t,t,\tau)=\tau$, we see that

$$\gamma^\varepsilon_x(t,t,0)=\exp\Big(\int_t^0(\widetilde{b}^\varepsilon)'(0)\,d\tau\Big)=\exp\Big(-(\widetilde{b}^\varepsilon)'(0)t\Big).$$

By the definition of $\widetilde{b}^\varepsilon$, we have $(\widetilde{b}^\varepsilon)'(0)=2\int_{-1}^1\varphi(s,s)\,ds/h(\varepsilon)$. Hence, noting that $h(\varepsilon)=1/\log(1/\varepsilon)$, we see that

$$\gamma^\varepsilon_x(t,t,0)=\exp\Big(\frac{2}{h(\varepsilon)}\int_{-1}^1\varphi(s,s)\,ds\cdot(-t)\Big)=\big(\frac{1}{\varepsilon}\big)^{2\int_{-1}^1\varphi(s,s)\,ds\cdot(-t)}.\tag{5.4}$$

Combining equations (5.2) and (5.4), we obtain that, for $\varepsilon>0$ small enough,

$$v^\varepsilon_x(t,t)=\frac{1}{h(\varepsilon)}\chi(0)\big(\frac{1}{\varepsilon}\big)^{2\int_{-1}^1\varphi(s,s)\,ds\cdot(-t)}\geq\chi(0)\big(\frac{1}{\varepsilon}\big)^{2\int_{-1}^1\varphi(s,s)\,ds\cdot(-t)}.$$

Since $U=V_x\in\mathscr{G}(\mathbb{R}^2)$, this shows that $\{(t,t)\mid t\leq 0\}\subset\operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}}U$.

Step 3. Thirdly, we prove that $\{(t,0)\mid t\geq 0\}$ and $\{(t,2t)\mid t\geq 0\}$ are contained in $\operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}}U$.

Put $t_1^\varepsilon=\gamma^\varepsilon(0,-ah(\varepsilon),t_1^\varepsilon)+2h(\varepsilon)$. Then as shown above, $t_1^\varepsilon\downarrow 0$ as $\varepsilon\downarrow 0$. For $t\geq t_1^\varepsilon$, consider

$$\frac{v^\varepsilon(t,\gamma^\varepsilon(0,-ah(\varepsilon),t))-v^\varepsilon(t,\gamma^\varepsilon(0,-2h(\varepsilon),t))}{\gamma^\varepsilon(0,-ah(\varepsilon),t)-\gamma^\varepsilon(0,-2h(\varepsilon),t)},$$

where $0<a<1$ is a constant such that $\int_{-\infty}^{-a}\chi(y)\,dy>0$. As shown above, we have

$$\gamma^\varepsilon(0,-ah(\varepsilon),t)=\gamma^\varepsilon(0,-ah(\varepsilon),t_1^\varepsilon)=h(\varepsilon)\int_a^2\frac{dz}{1-\widetilde{b}^\varepsilon(-h(\varepsilon)z)}-2h(\varepsilon).$$

Since $\gamma^\varepsilon(0,-2h(\varepsilon),t)=-2h(\varepsilon)$, we get

$$0<\gamma^\varepsilon(0,-ah(\varepsilon),t)-\gamma^\varepsilon(0,-2h(\varepsilon),t)=h(\varepsilon)\int_a^2\frac{dz}{1-\widetilde{b}^\varepsilon(-h(\varepsilon)z)}.$$

Furthermore,

$$v^\varepsilon(t, \gamma^\varepsilon(0, -ah(\varepsilon), t)) = \int_{-\infty}^{-a} \chi(y)\, dy > 0,$$
$$v^\varepsilon(t, \gamma^\varepsilon(0, -2h(\varepsilon), t)) = 0.$$

Therefore,

$$\frac{v^\varepsilon(t, \gamma^\varepsilon(0, -ah(\varepsilon), t)) - v^\varepsilon(t, \gamma^\varepsilon(0, -2h(\varepsilon), t))}{\gamma^\varepsilon(0, -ah(\varepsilon), t) - \gamma^\varepsilon(0, -2h(\varepsilon), t)} = \frac{\int_{-\infty}^{-a} \chi(y)\, dy}{\int_a^2 \frac{dz}{1-\widetilde{b}^\varepsilon(-h(\varepsilon)z)}} \cdot \frac{1}{h(\varepsilon)}.$$

By the mean value theorem, there exists $x_1^\varepsilon \in (\gamma^\varepsilon(0, -2h(\varepsilon), t), \gamma^\varepsilon(0, -ah(\varepsilon), t))$ such that

$$v_x^\varepsilon(t, x_1^\varepsilon) = \frac{\int_{-\infty}^{-a} \chi(y)\, dy}{\int_a^2 \frac{dz}{1-\widetilde{b}^\varepsilon(-h(\varepsilon)z)}} \cdot \frac{1}{h(\varepsilon)}.$$

Note that $\partial_x^\alpha v^\varepsilon(t, \gamma^\varepsilon(0, -2h(\varepsilon), t)) = 0$ for $\alpha \in \mathbb{N}$. Then, repeating the above process gives us $(x_\alpha^\varepsilon)_{\alpha \geq 2}$ such that $x_\alpha^\varepsilon \in (\gamma^\varepsilon(0, -2h(\varepsilon), t), x_{\alpha-1}^\varepsilon)$ and

$$\begin{aligned}
\partial_x^\alpha v^\varepsilon(t, x_\alpha^\varepsilon) &= \frac{\partial_x^{\alpha-1} v^\varepsilon(t, x_{\alpha-1}^\varepsilon) - \partial_x^{\alpha-1} v^\varepsilon(t, \gamma^\varepsilon(0, -2h(\varepsilon), t))}{x_{\alpha-1}^\varepsilon - \gamma^\varepsilon(0, -2h(\varepsilon), t)} \\
&\geq \frac{\int_{-\infty}^{-a} \chi(y)\, dy}{\big(\int_a^2 \frac{dz}{1-\widetilde{b}^\varepsilon(-h(\varepsilon)z)}\big)^\alpha} \cdot \frac{1}{h(\varepsilon)^\alpha}.
\end{aligned}$$

Since $U = V_x \in \mathscr{G}(\mathbb{R}^2)$, this shows that $\{(t, 0) \mid t \geq 0\} \subset \operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}} U$. In a similar way, we can show that $\{(t, 2t) \mid t \geq 0\} \subset \operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}} U$.

Step 4. Fourthly, we prove that $U \in \mathscr{G}(\mathbb{R}^2)$ is $\mathscr{G}^\infty_{\log}$-regular on $\{(t, x) \in \mathbb{R}^2 \mid 0 < x < 2t\}$.

Step 4-1. To do so, we first estimate $\gamma^\varepsilon(t, x, 0)$ for all (t, x) such that $0 \leq x \leq 2t$.

When $0 \leq x \leq t - 2h(\varepsilon)$, as seen in Step 1 of the proof of Theorem 4.2, $\gamma^\varepsilon(t, x, 0) = (G^\varepsilon)^{-1}(x/h(\varepsilon) + 2)$. By inequality (4.3), there exists a constant $C_2 > 0$ such that

$$0 < -\gamma^\varepsilon(t, x, 0) \leq 2 \exp\big(-\frac{2}{C_2}\big) h(\varepsilon) \varepsilon^{x/C_2}. \tag{5.5}$$

When $t + 2h(\varepsilon) \leq x \leq 2t$, we have $\gamma^\varepsilon(t, x, 0) = -(G^\varepsilon)^{-1}((2t - x)/h(\varepsilon) + 2)$. Hence, by (4.3), we have

$$0 < \gamma^\varepsilon(t, x, 0) \leq 2 \exp\big(-\frac{2}{C_2}\big) h(\varepsilon) \varepsilon^{(2t-x)/C_2}.$$

When $t - 2h(\varepsilon) \leq x < t$, we get

$$\int_{-\gamma^\varepsilon(t,x,0)/h(\varepsilon)}^{(t-x)/h(\varepsilon)} \frac{dz}{1 - \widetilde{b}^\varepsilon(-h(\varepsilon)z)} = \frac{t}{h(\varepsilon)}.$$

As seen from the proof of Lemma 4.1, we have $1/(1 - \widetilde{b}^\varepsilon(-h(\varepsilon)z)) \leq C_2/z$ for $0 \leq z \leq 2$ and so

$$\frac{t}{h(\varepsilon)} \leq \int_{-\gamma^\varepsilon(t,x,0)/h(\varepsilon)}^{(t-x)/h(\varepsilon)} \frac{C_2}{z}\, dz = C_2 \log \frac{t - x}{-\gamma^\varepsilon(t, x, 0)}.$$

Since $h(\varepsilon) = 1/\log(1/\varepsilon)$, it follows that

$$0 < -\gamma^\varepsilon(t, x, 0) \leq (t - x)\varepsilon^{t/C_2}. \tag{5.6}$$

When $t < x \le t + 2h(\varepsilon)$, we get

$$\int_{\gamma^\varepsilon(t,x,0)/h(\varepsilon)}^{(x-t)/h(\varepsilon)} \frac{dz}{1-\widetilde{b}^\varepsilon(-h(\varepsilon)z)} = \frac{t}{h(\varepsilon)},$$

and so

$$0 < \gamma^\varepsilon(t,x,0) \le (x-t)\varepsilon^{t/C_2}.$$

When $t = x$, we have $\gamma^\varepsilon(t,t,0) = 0$.

Step 4-2. We next estimate $\gamma_x^\varepsilon(t,x,0)$. When $0 \le x \le t - 2h(\varepsilon)$, we have $\gamma^\varepsilon(t,x,0) = (G^\varepsilon)^{-1}(x/h(\varepsilon)+2)$. Hence, as in the proof of Lemma 4.1, we get, for some constant $c_2 > 0$,

$$\gamma_x^\varepsilon(t,x,0) = 1 - \widetilde{b}^\varepsilon(\gamma^\varepsilon(t,x,0)) \le c_2\frac{|\gamma^\varepsilon(t,x,0)|}{h(\varepsilon)} \le 2c_2\exp\big(-\frac{2}{C_2}\big)\varepsilon^{x/C_2},$$

where we used formula (4.2) in the first step and inequality (5.5) in the last step.

When $t + 2h(\varepsilon) \le x \le 2t$, $\gamma^\varepsilon(t,x,0) = -(G^\varepsilon)^{-1}((2t-x)/h(\varepsilon)+2)$. Similarly, we get

$$\gamma_x^\varepsilon(t,x,0) = 1 - \widetilde{b}^\varepsilon(-\gamma^\varepsilon(t,x,0)) \le 2c_2\exp\big(-\frac{2}{C_2}\big)\varepsilon^{(2t-x)/C_2}.$$

When $t - 2h(\varepsilon) \le x < t$, we have $\gamma^\varepsilon(t,x,0) = (G^\varepsilon)^{-1}(t/h(\varepsilon) + G^\varepsilon(x-t))$. Differentiating this in x gives

$$\gamma_x^\varepsilon(t,x,0) = \frac{1-\widetilde{b}^\varepsilon(\gamma^\varepsilon(t,x,0))}{1-\widetilde{b}^\varepsilon(x-t)}. \tag{5.7}$$

The numerator of (5.7) can be estimated as follows:

$$1 - \widetilde{b}^\varepsilon(\gamma^\varepsilon(t,x,0)) \le c_2\frac{|\gamma^\varepsilon(t,x,0)|}{h(\varepsilon)} \le c_2\frac{(t-x)\varepsilon^{t/C_2}}{h(\varepsilon)},$$

where we used inequality (5.6) in the last step. Similarly, the denominator of (5.7) is estimated as follows: for some constant $c_1 > 0$, we have $1-\widetilde{b}^\varepsilon(x-t) \ge c_1(t-x)/h(\varepsilon)$. Hence,

$$0 < \gamma_x^\varepsilon(t,x,0) \le \frac{c_2}{c_1}\varepsilon^{t/C_2}.$$

When $t < x \le t + 2h(\varepsilon)$, we have $\gamma^\varepsilon(t,x,0) = -(G^\varepsilon)^{-1}(t/h(\varepsilon) + G^\varepsilon(t-x))$ and so get

$$0 < \gamma_x^\varepsilon(t,x,0) \le \frac{c_2}{c_1}\varepsilon^{t/C_2}.$$

To estimate $\gamma_x^\varepsilon(t,t,0)$, we consider problem (5.3). As in Step 2, we can derive that

$$\gamma_x^\varepsilon(t,x,s) = \exp\Big(-\int_s^t (\widetilde{b}^\varepsilon)'(\gamma^\varepsilon(t,x,\tau)-\tau)\,d\tau\Big). \tag{5.8}$$

Note that $\gamma^\varepsilon(t,t,\tau) = \tau$ and $(\widetilde{b}^\varepsilon)'(0) = 2\int_{-1}^1 \varphi(s,s)\,ds/h(\varepsilon)$. Hence,

$$\gamma_x^\varepsilon(t,t,0) = \varepsilon^{(2\int_{-1}^1 \varphi(s,s)\,ds)t}.$$

Step 4-3. Finally, we prove that, for all $K \Subset \{(t,x) \in \mathbb{R}^2 \mid 0 < x < 2t\}$ and $\alpha \in \mathbb{N}_0^2$,

$$\|\partial^\alpha v_x^\varepsilon(t,x)\|_{L^\infty(K)} \to 0 \quad \text{as } \varepsilon \downarrow 0. \tag{5.9}$$

This implies that $U = V_x \in \mathscr{G}(\mathbb{R}^2)$ is $\mathscr{G}_{\log}^\infty$-regular on $\{(t,x) \in \mathbb{R}^2 \mid 0 < x < 2t\}$.

Note that

$$v_x^\varepsilon(t,x) = \chi\Big(\frac{\gamma^\varepsilon(t,x,0)}{h(\varepsilon)}\Big)\frac{\gamma_x^\varepsilon(t,x,0)}{h(\varepsilon)}.$$

Hence, to prove (5.9), it suffices to show that, for all $K \Subset \{(t,x) \in \mathbb{R}^2 \mid 0 < x < 2t\}$ and $\alpha \in \mathbb{N}^2$,

$$\frac{\|\partial^\alpha \gamma^\varepsilon(t,x,0)\|_{L^\infty(K)}}{h(\varepsilon)} \to 0 \quad \text{as } \varepsilon \downarrow 0. \tag{5.10}$$

Since $(\widetilde{b}^\varepsilon)'(z) \geq 0$ for $z \in \mathbb{R}$, we see from (5.8) that, for $0 \leq s \leq t$,

$$0 < \gamma_x^\varepsilon(t,x,s) \leq 1. \tag{5.11}$$

From (5.8) again, we find that

$$\gamma_{xx}^\varepsilon(t,x,s) = -\gamma_x^\varepsilon(t,x,s)\int_s^t (\widetilde{b}^\varepsilon)''(\gamma^\varepsilon(t,x,\tau)-\tau)\gamma_x^\varepsilon(t,x,\tau)\,d\tau.$$

We see that $\|(\widetilde{b}^\varepsilon)''\|_{L^\infty(\mathbb{R})} \leq C''/h(\varepsilon)^2$ for some constant $C'' > 0$. In view of this inequality and (5.11), we get $|\gamma_{xx}^\varepsilon(t,x,s)| \leq C''(t-s)/h(\varepsilon)^2$ for $0 \leq s \leq t$. Furthermore, if $s = 0$, then from Step 4-2, we see that

$$\frac{|\gamma_{xx}^\varepsilon(t,x,0)|}{h(\varepsilon)} \leq C''\frac{|\gamma_x^\varepsilon(t,x,0)|t}{h(\varepsilon)^3} \to 0 \quad \text{on } K \quad \text{as } \varepsilon \downarrow 0.$$

By repeating this process, we obtain that a similar estimate holds for any derivative of $\gamma^\varepsilon(t,x,0)$ in x. We also find from problem (5.3) that

$$\gamma_t^\varepsilon(t,x,s) = -\widetilde{b}^\varepsilon(x-t)\exp\Big(-\int_s^t (\widetilde{b}^\varepsilon)'(\gamma^\varepsilon(t,x,\tau)-\tau)\,d\tau\Big). \tag{5.12}$$

In view of (5.12), we can similarly show inequality (5.10). The proof of Theorem 5.1 is now complete. ☐

Remark 5.2. We assumed that χ is symmetric. Hence, $\int_{-\infty}^0 \chi(y)\,dy = 1/2$. If, instead of the symmetry of χ, we assume that $\int_{-\infty}^0 \chi(y)\,dy = a$ for $0 \leq a \leq 1$, then the solution $U \in \mathscr{G}(\mathbb{R}^2)$ of problem (3.1) with the initial data U_0 given by the class of $(\chi_{h(\varepsilon)})_{\varepsilon\in(0,1]}$ possesses the distributional shadow

$$u(t,x) = \begin{cases} a\delta(x) + (1-a)\delta(x-2t), & \text{if } t \geq 0,\ x \in \mathbb{R}, \\ \delta(x-t), & \text{if } t < 0,\ x \in \mathbb{R}. \end{cases}$$

Next, we calculate the $\mathscr{G}^\infty$-singular support of the solution $U \in \mathscr{G}(\mathbb{R}^2)$ with the same initial data U_0 as in Theorem 5.1. The following theorem shows that the splitting of the singularity at the origin does not occur in the sense of $\mathscr{G}^\infty$.

Theorem 5.3. *Under the same assumption as in Theorem 5.1, it holds that*

$$\operatorname{sing\,supp}_{\mathscr{G}^\infty} U = \{(t,t) \mid t \leq 0\}.$$

Proof. The proof is divided into two steps.

Step 1. First, we prove that $U \in \mathscr{G}(\mathbb{R}^2)$ is $\mathscr{G}^\infty$-regular on $\mathbb{R}^2 \setminus \{(t,t) \mid t \leq 0\}$.

As can be seen in Step 1 of the proof of Theorem 5.1, the solution $U \in \mathscr{G}(\mathbb{R}^2)$ is $\mathscr{G}^\infty$-regular on $\{(t,x) \in \mathbb{R}^2 \mid x < \min\{0,t\}\} \cup \{(t,x) \in \mathbb{R}^2 \mid x > \max\{t,2t\}\}$. Hence, it suffices to prove that $U \in \mathscr{G}(\mathbb{R}^2)$ is $\mathscr{G}^\infty$-regular on $(0,\infty)\times\mathbb{R}$.

Let $(t,x) \in (0,\infty)\times\mathbb{R}$ and $0 \leq s \leq t$. As in Step 4-3 of the proof of Theorem 5.1, we get $0 < \gamma_x^\varepsilon(t,x,s) \leq 1$ and $|\gamma_{xx}^\varepsilon(t,x,s)| \leq C''(t-s)/h(\varepsilon)^2$. Similarly, we can prove that all derivatives of $\gamma^\varepsilon(t,x,s)$ in x are dominated by a finite sum of terms in the form of $\kappa_i(t-s)^j/h(\varepsilon)^k$ with a constant $\kappa_i>0$. Note by (5.12) that $\gamma^\varepsilon_t(t,x,s)=-\widetilde{b}^\varepsilon(x-t)\gamma^\varepsilon_x(t,x,s)$. Hence, we see that, all derivatives of $\gamma^\varepsilon(t,x,s)$ in t and x are also dominated by a finite sum of terms in the form of $\kappa_i(t-s)^j/h(\varepsilon)^k$. Let us recall that the solution v^ε of problem (5.1) satisfies (5.2). Then, we see that, for all $K\Subset(0,\infty)\times\mathbb{R}$ and $\alpha\in\mathbb{N}_0^2$,

$$\|\partial^\alpha v^\varepsilon_x(t,x)\|_{L^\infty(K)}=O(\varepsilon^{-1})\quad\text{as } \varepsilon\downarrow 0.$$

Since $U=V_x\in\mathscr{G}(\mathbb{R}^2)$, this shows that U is $\mathscr{G}^\infty$-regular on $(0,\infty)\times\mathbb{R}$.

Step 2. Secondly, we prove that $\{(t,t)\mid t\le 0\}$ is contained in $\operatorname{sing\,supp}_{\mathscr{G}^\infty}U$.

For $t<0$, consider

$$\frac{v^\varepsilon(t,\gamma^\varepsilon(0,0,t))-v^\varepsilon(t,\gamma^\varepsilon(0,-2h(\varepsilon),t))}{\gamma^\varepsilon(0,0,t)-\gamma^\varepsilon(0,-2h(\varepsilon),t)}.$$

Clearly, $\gamma^\varepsilon(0,0,t)=t$. As in Step 1 of the proof of Theorem 4.2, we see that $t-\gamma^\varepsilon(0,-2h(\varepsilon),t)=-(G^\varepsilon)^{-1}(-t/h(\varepsilon))$. Hence, by inequality (4.3), we get, for some constant $C_2>0$,

$$0<t-\gamma^\varepsilon(0,-2h(\varepsilon),t)\le 2h(\varepsilon)\varepsilon^{-t/C_2}.$$

Furthermore,

$$v^\varepsilon(t,\gamma^\varepsilon(0,0,t))=\frac{1}{2},$$
$$v^\varepsilon(t,\gamma^\varepsilon(0,-2h(\varepsilon),t))=0.$$

Therefore,

$$\frac{v^\varepsilon(t,\gamma^\varepsilon(0,0,t))-v^\varepsilon(t,\gamma^\varepsilon(0,-2h(\varepsilon),t))}{\gamma^\varepsilon(0,0,t)-\gamma^\varepsilon(0,-2h(\varepsilon),t)}\ge\frac{1}{4h(\varepsilon)}\cdot\frac{1}{\varepsilon^{-t/C_2}}.$$

By the mean value theorem, there exists $x_1^\varepsilon\in(\gamma^\varepsilon(0,-2h(\varepsilon),t),t)$ such that

$$v^\varepsilon_x(t,x_1^\varepsilon)\ge\frac{1}{4h(\varepsilon)}\cdot\frac{1}{\varepsilon^{-t/C_2}}.$$

Note that $\partial_x^\alpha v^\varepsilon(t,\gamma^\varepsilon(0,-2h(\varepsilon),t))=0$ for $\alpha\in\mathbb{N}$. Then we repeat this process to find $(x^\varepsilon_\alpha)_{\alpha\ge2}$ such that $x^\varepsilon_\alpha\in(\gamma^\varepsilon(0,-2h(\varepsilon),t),x^\varepsilon_{\alpha-1})$ and

$$\partial_x^\alpha v^\varepsilon(t,x^\varepsilon_\alpha)=\frac{\partial_x^{\alpha-1}v^\varepsilon(t,x^\varepsilon_{\alpha-1})-\partial_x^{\alpha-1}v^\varepsilon(t,\gamma^\varepsilon(0,-2h(\varepsilon),t))}{x^\varepsilon_{\alpha-1}-\gamma^\varepsilon(0,-2h(\varepsilon),t)}\ge\frac{1}{2^{\alpha+1}h(\varepsilon)^\alpha}\cdot\frac{1}{\varepsilon^{-\alpha t/C_2}}.$$

Since $U=V_x\in\mathscr{G}(\mathbb{R}^2)$, this shows that $\{(t,t)\mid t\le0\}\subset\operatorname{sing\,supp}_{\mathscr{G}^\infty}U$. The proof of Theorem 5.3 is now complete. □

Next, we discuss the case of initial data U_0 given by other Dirac generalized functions at 0. As stated in Remark 2.5, there exist infinitely many Dirac generalized functions with different strengths of singularity at 0. For any constant $t_0>0$, we define $c(\varepsilon):=-(G^\varepsilon)^{-1}(t_0/h(\varepsilon))/2h(\varepsilon)$. We find from inequality (4.3) that there exist two constants C_1, $C_2>0$ independent of t_0 such that $\varepsilon^{t_0/C_1}\le c(\varepsilon)\le\varepsilon^{t_0/C_2}$ for $\varepsilon\in(0,1]$. Hence, we have $(\chi_{c(\varepsilon)})_{\varepsilon\in(0,1]}\in\mathscr{E}_M(\mathbb{R})$, which allows us to define $U_0\in\mathscr{G}(\mathbb{R})$ as the class of $(\chi_{c(\varepsilon)})_{\varepsilon\in(0,1]}$. Then U_0 is a Dirac generalized function at 0 and does not belong to $\mathscr{G}^\infty(\mathbb{R})$. Furthermore, the singularity of U_0 at 0 can be interpreted to become stronger as t_0 becomes large. As may be seen in the following theorem, the stronger the singularity of the initial data U_0 at 0 becomes, the longer the singularity in $\mathscr{G}^\infty_{\log}$ propagates along the line $\{t=x\}$, and it splits at time t_0.

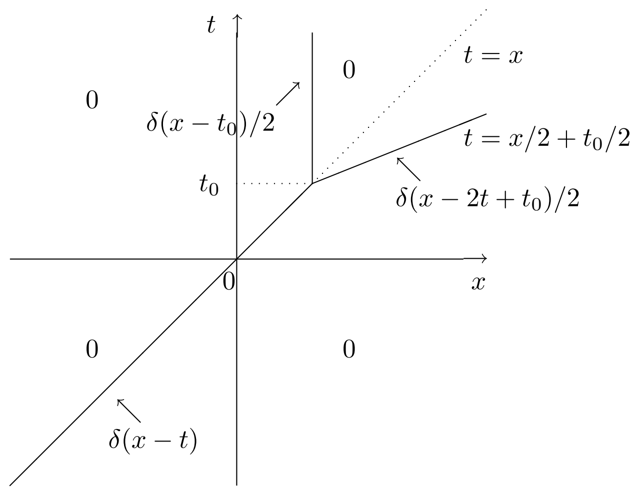

FIGURE 2. Distributional shadow

Theorem 5.4. *Let t_0 and $U_0 \in \mathscr{G}(\mathbb{R})$ be as above. Then the solution $U \in \mathscr{G}(\mathbb{R}^2)$ of problem* (3.1) *admits a distributional shadow, which is given by*

$$u(t,x) = \begin{cases} \frac{\delta(x-t_0)+\delta(x-2t+t_0)}{2}, & \text{if } t \geq t_0,\ x \in \mathbb{R}, \\ \delta(x-t), & \text{if } t < t_0,\ x \in \mathbb{R}. \end{cases}$$

Furthermore,

$$\begin{aligned} &\{(t,t_0) \mid t \geq t_0\} \cup \{(t, 2t-t_0) \mid t \geq t_0\} \cup \{(t,t) \mid t \leq t_0\} \\ &\subset \operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}} U \subset \{(t,t_0) \mid t \geq t_0\} \cup \{(t,2t-t_0) \mid t \geq t_0\} \cup \{(t,t) \mid t \in \mathbb{R}\}. \end{aligned}$$

Proof. Let $v_0^\varepsilon = H * \chi_{c(\varepsilon)}$ and let $V_0 \in \mathscr{G}(\mathbb{R})$ be given by the class of $(v_0^\varepsilon)_{\varepsilon\in(0,1]}$. In order to prove the first assertion, it suffices to show that the solution $V \in \mathscr{G}(\mathbb{R}^2)$ of problem (4.4) admits a distributional shadow, which is given by

$$v(t,x) = \begin{cases} \frac{H(x-t_0)+H(x-2t+t_0)}{2}, & \text{if } t \geq t_0,\ x \in \mathbb{R}, \\ H(x-t), & \text{if } t < t_0,\ x \in \mathbb{R}. \end{cases}$$

Let $(v^\varepsilon)_{\varepsilon\in(0,1]}$ be a representative of $V \in \mathscr{G}(\mathbb{R}^2)$ satisfying problem (5.1). Then we have $v^\varepsilon(t, \gamma^\varepsilon(0,x,t)) = v_0^\varepsilon(x)$ and see that v^ε converges to $H(x-t)$ a.e. in $(-\infty, 0) \times \mathbb{R}$ as $\varepsilon \downarrow 0$.

We now fix $0 < a \leq 1$ arbitrarily, and put $t_1^\varepsilon := \gamma^\varepsilon(0, -ac(\varepsilon), t_1^\varepsilon) + 2h(\varepsilon)$. As in Step 1 of the proof of Theorem 4.2, we have

$$t_1^\varepsilon = h(\varepsilon) \int_{ac(\varepsilon)/h(\varepsilon)}^2 \frac{dz}{1 - \widetilde{b}^\varepsilon(-h(\varepsilon)z)}.$$

By the definition of $c(\varepsilon)$, we get

$$t_1^\varepsilon = t_0 - h(\varepsilon) \int_{2c(\varepsilon)}^{ac(\varepsilon)/h(\varepsilon)} \frac{dz}{1 - \widetilde{b}^\varepsilon(-h(\varepsilon)z)}.$$

As in the proof of Lemma 4.1, we get, for some constant $C_2 > 0$ and $\varepsilon > 0$ small enough,

$$h(\varepsilon)\int_{2c(\varepsilon)}^{ac(\varepsilon)/h(\varepsilon)} \frac{dz}{1-\widetilde{b}^{\varepsilon}(-h(\varepsilon)z)} \le h(\varepsilon)\int_{2c(\varepsilon)}^{ac(\varepsilon)/h(\varepsilon)} \frac{C_2}{z}\,dz.$$

We have

$$h(\varepsilon)\int_{2c(\varepsilon)}^{ac(\varepsilon)/h(\varepsilon)} \frac{C_2}{z}\,dz = -C_2 h(\varepsilon)\log\frac{2h(\varepsilon)}{a} \to 0 \quad \text{as } \varepsilon \downarrow 0,$$

so that $t_1^\varepsilon \to t_0$ as $\varepsilon \downarrow 0$. Note that for any $t \ge t_1^\varepsilon$, $\gamma^\varepsilon(0,-ac(\varepsilon),t) = \gamma^\varepsilon(0,-ac(\varepsilon),t_1^\varepsilon)$. Therefore, for any $0 \le t \le t_0$, $\gamma^\varepsilon(0,-ac(\varepsilon),t) \to t$ as $\varepsilon \downarrow 0$, and for any $t \ge t_0$, $\gamma^\varepsilon(0,-ac(\varepsilon),t) \to t_0$ as $\varepsilon \downarrow 0$. Furthermore,

$$v^\varepsilon(t,\gamma^\varepsilon(0,-ac(\varepsilon),t)) = v_0^\varepsilon(-ac(\varepsilon)) = \int_{-\infty}^{-a} \chi(y)\,dy,$$

and so $v^\varepsilon(t,\gamma^\varepsilon(0,-c(\varepsilon),t)) = 0$ and $v^\varepsilon(t,\gamma^\varepsilon(0,-ac(\varepsilon),t)) \uparrow 1/2$ as $a \downarrow 0$. Similarly, we take $t_2^\varepsilon := \gamma^\varepsilon(0,ac(\varepsilon),t_2^\varepsilon) - 2h(\varepsilon)$ to get

$$t_2^\varepsilon = h(\varepsilon)\int_{ac(\varepsilon)/h(\varepsilon)}^{2} \frac{dz}{\widetilde{b}^{\varepsilon}(h(\varepsilon)z)-1} \to t_0 \quad \text{as } \varepsilon \downarrow 0.$$

Note that, for any $t \ge t_2^\varepsilon$, $\gamma^\varepsilon(0,ac(\varepsilon),t) = 2t - \gamma^\varepsilon(0,ac(\varepsilon),t_2^\varepsilon) + 4h(\varepsilon)$. Hence, for any $0 \le t \le t_0$, $\gamma^\varepsilon(0,ac(\varepsilon),t) \to t$ as $\varepsilon \downarrow 0$, and for any $t \ge t_0$, $\gamma^\varepsilon(0,ac(\varepsilon),t) \to 2t - t_0$ as $\varepsilon \downarrow 0$. Furthermore,

$$v^\varepsilon(t,\gamma^\varepsilon(0,ac(\varepsilon),t)) = v_0^\varepsilon(ac(\varepsilon)) = \int_{-\infty}^{a} \chi(y)\,dy,$$

and so $v^\varepsilon(t,\gamma^\varepsilon(0,c(\varepsilon),t)) = 1$ and $v^\varepsilon(t,\gamma^\varepsilon(0,ac(\varepsilon),t)) \downarrow 1/2$ as $a \downarrow 0$. Therefore, in view of the fact that $v^\varepsilon(t,x)$ is non-decreasing in x, we obtain that v^ε converges to $H(x-t)$ a.e. in $(0,t_0)\times\mathbb{R}$ and to $(H(x-t_0)+H(x-2t+t_0))/2$ a.e. in $(t_0,\infty)\times\mathbb{R}$ as $\varepsilon \downarrow 0$. Thus, the first assertion follows.

Next, we prove the second assertion. We will do so in four steps.

Step 1. First, we prove that $U \in \mathscr{G}(\mathbb{R}^2)$ is $\mathscr{G}^\infty_{\log}$-regular on $\{(t,x) \in \mathbb{R}^2 \mid x < \min\{t_0,t\}\} \cup \{(t,x) \in \mathbb{R}^2 \mid x > \max\{t, 2t-t_0\}\}$.

It is easy to check that $V \in \mathscr{G}(\mathbb{R}^2)$ equals 0 on $\{(t,x) \in \mathbb{R}^2 \mid x < \min\{t_0,t\}\}$ and further equals 1 on $\{(t,x) \in \mathbb{R}^2 \mid x > \max\{t,2t-t_0\}\}$. Hence, $U = V_x \in \mathscr{G}(\mathbb{R}^2)$ is $\mathscr{G}^\infty_{\log}$-regular on the union of these two sets.

Step 2. Secondly, we prove that $\{(t,t) \mid t \le t_0\}$ is contained in $\operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}} U$.

Put $t_1^\varepsilon = \gamma^\varepsilon(0,-c(\varepsilon),t_1^\varepsilon) + 2h(\varepsilon)$. Then, as shown above, $t_1^\varepsilon \uparrow t_0$ as $\varepsilon \downarrow 0$. For $t < t_1^\varepsilon$, consider

$$\frac{v^\varepsilon(t,\gamma^\varepsilon(0,0,t)) - v^\varepsilon(t,\gamma^\varepsilon(0,-c(\varepsilon),t))}{\gamma^\varepsilon(0,0,t) - \gamma^\varepsilon(0,-c(\varepsilon),t)}.$$

As in Step 1 of the proof of Theorem 4.2, we get

$$G^\varepsilon(\gamma^\varepsilon(0,-c(\varepsilon),t) - t) = \frac{t_1^\varepsilon - t}{h(\varepsilon)}.$$

Using inequality (4.3), we have, for some constant $C_2 > 0$,

$$0 < t - \gamma^\varepsilon(0,-c(\varepsilon),t) \le 2h(\varepsilon)\varepsilon^{(t_1^\varepsilon - t)/C_2}.$$

Since $\gamma^\varepsilon(0,0,t) = t$, it follows that

$$0 < \gamma^\varepsilon(0,0,t) - \gamma^\varepsilon(0,-c(\varepsilon),t) \le 2h(\varepsilon)\varepsilon^{(t_1^\varepsilon - t)/C_2}. \tag{5.13}$$

We use $v^\varepsilon(t,\gamma^\varepsilon(0,0,t)) = 1/2$, $v^\varepsilon(t,\gamma^\varepsilon(0,-c(\varepsilon),t)) = 0$ and (5.13) to see that

$$\frac{v^\varepsilon(t,\gamma^\varepsilon(0,0,t)) - v^\varepsilon(t,\gamma^\varepsilon(0,-c(\varepsilon),t))}{\gamma^\varepsilon(0,0,t) - \gamma^\varepsilon(0,-c(\varepsilon),t)} \geq \frac{1}{4h(\varepsilon)} \cdot \frac{1}{\varepsilon^{(t_1^\varepsilon - t)/C_2}}.$$

By the mean value theorem, there exists $x^\varepsilon \in (\gamma^\varepsilon(0,-c(\varepsilon),t), \gamma^\varepsilon(0,0,t))$ such that

$$v_x^\varepsilon(t,x^\varepsilon) \geq \frac{1}{4h(\varepsilon)} \cdot \frac{1}{\varepsilon^{(t_1^\varepsilon - t)/C_2}}.$$

Since $U = V_x \in \mathscr{G}(\mathbb{R}^2)$, this means that $\{(t,t) \mid t \leq t_0\} \subset \operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}} U$.

Step 3. Thirdly, we prove that $\{(t,t_0) \mid t \geq t_0\}$ and $\{(t,2t-t_0) \mid t \geq t_0\}$ are contained in $\operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}} U$.

For $t > t_0$, consider

$$\frac{v^\varepsilon(t,\gamma^\varepsilon(0,-ac(\varepsilon),t)) - v^\varepsilon(t,\gamma^\varepsilon(0,-c(\varepsilon),t))}{\gamma^\varepsilon(0,-ac(\varepsilon),t) - \gamma^\varepsilon(0,-c(\varepsilon),t)}$$

for $0 < a \leq 1$. As shown above, if we put $t_1^\varepsilon = \gamma^\varepsilon(0,-ac(\varepsilon),t_1^\varepsilon) + 2h(\varepsilon)$, then we have $t_1^\varepsilon \uparrow t_0$ as $\varepsilon \downarrow 0$ and

$$\gamma^\varepsilon(0,-ac(\varepsilon),t) = \gamma^\varepsilon(0,-ac(\varepsilon),t_1^\varepsilon) = h(\varepsilon)\int_{ac(\varepsilon)/h(\varepsilon)}^{2} \frac{dz}{1-\widetilde{b}^\varepsilon(-h(\varepsilon)z)} - 2h(\varepsilon).$$

Hence,

$$\gamma^\varepsilon(0,-ac(\varepsilon),t) - \gamma^\varepsilon(0,-c(\varepsilon),t) = h(\varepsilon)\int_{ac(\varepsilon)/h(\varepsilon)}^{c(\varepsilon)/h(\varepsilon)} \frac{dz}{1-\widetilde{b}^\varepsilon(-h(\varepsilon)z)}.$$

We now take $0 < a < 1$ so that $\int_{-\infty}^{-a} \chi(y)\,dy > 0$. Then, as in the proof of Lemma 4.1, we get, for some constant $C_2 > 0$,

$$0 < \gamma^\varepsilon(0,-ac(\varepsilon),t) - \gamma^\varepsilon(0,-c(\varepsilon),t) \leq h(\varepsilon)\int_{ac(\varepsilon)/h(\varepsilon)}^{c(\varepsilon)/h(\varepsilon)} \frac{C_2}{z}\,dz = C_2 h(\varepsilon)\log\frac{1}{a}.$$

Furthermore,

$$v^\varepsilon(t,\gamma^\varepsilon(0,-ac(\varepsilon),t)) = \int_{-\infty}^{-a} \chi(y)\,dy > 0,$$
$$v^\varepsilon(t,\gamma^\varepsilon(0,-c(\varepsilon),t)) = 0.$$

Hence,

$$\frac{v^\varepsilon(t,\gamma^\varepsilon(0,-ac(\varepsilon),t)) - v^\varepsilon(t,\gamma^\varepsilon(0,-c(\varepsilon),t))}{\gamma^\varepsilon(0,-ac(\varepsilon),t) - \gamma^\varepsilon(0,-c(\varepsilon),t)} \geq \frac{\int_{-\infty}^{-a}\chi(y)\,dy}{C_2\log 1/a} \cdot \frac{1}{h(\varepsilon)}.$$

By the mean value theorem, there exists $x_1^\varepsilon \in (\gamma^\varepsilon(0,-c(\varepsilon),t), \gamma^\varepsilon(0,-ac(\varepsilon),t))$ such that

$$v_x^\varepsilon(t,x_1^\varepsilon) \geq \frac{\int_{-\infty}^{-a}\chi(y)\,dy}{C_2\log 1/a} \cdot \frac{1}{h(\varepsilon)}.$$

Note that $\partial_x^\alpha v^\varepsilon(t,\gamma^\varepsilon(0,-c(\varepsilon),t)) = 0$ for $\alpha \in \mathbb{N}$. Hence, we repeat this process to get $(x_\alpha^\varepsilon)_{\alpha\geq 2}$ such that $x_\alpha^\varepsilon \in (\gamma^\varepsilon(0,-c(\varepsilon),t), x_{\alpha-1}^\varepsilon)$ and

$$\partial_x^\alpha v^\varepsilon(t,x_\alpha^\varepsilon) = \frac{\partial_x^{\alpha-1}v^\varepsilon(t,x_{\alpha-1}^\varepsilon) - \partial_x^{\alpha-1}v^\varepsilon(t,\gamma^\varepsilon(0,-c(\varepsilon),t))}{x_{\alpha-1}^\varepsilon - \gamma^\varepsilon(0,-c(\varepsilon),t)} \geq \frac{\int_{-\infty}^{-a}\chi(y)\,dy}{(C_2\log 1/a)^\alpha} \cdot \frac{1}{h(\varepsilon)^\alpha}.$$

Since $U = V_x \in \mathscr{G}(\mathbb{R}^2)$, this shows that $\{(t, t_0) \mid t \ge t_0\} \subset \operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}} U$. In a similar way, we can show that $\{(t, 2t - t_0) \mid t \ge t_0\} \subset \operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}} U$.

Step 4. Fourthly, we prove that $U \in \mathscr{G}(\mathbb{R}^2)$ is $\mathscr{G}^\infty_{\log}$-regular on $\{(t, x) \in \mathbb{R}^2 \mid t_0 < x < t\}$ and $\{(t, x) \in \mathbb{R}^2 \mid t < x < 2t - t_0\}$.

Step 4-1. To do so, we first estimate $\gamma^\varepsilon(t, x, 0)$ for all (t, x) such that $t_0 \le x \le t - 2h(\varepsilon)$ or $t + 2h(\varepsilon) \le x \le 2t - t_0$.

When $t_0 \le x \le t - 2h(\varepsilon)$, as seen in Step 1 of the proof of Theorem 4.2, $\gamma^\varepsilon(t, x, 0) = (G^\varepsilon)^{-1}(x/h(\varepsilon) + 2)$. Hence, $G^\varepsilon(\gamma^\varepsilon(t, x, 0)) = x/h(\varepsilon) + 2$. By the definition of $c(\varepsilon)$, we have $G^\varepsilon(-2h(\varepsilon)c(\varepsilon)) = t_0/h(\varepsilon)$. Taking the difference gives

$$G^\varepsilon(\gamma^\varepsilon(t, x, 0)) - G^\varepsilon(-2h(\varepsilon)c(\varepsilon)) = \frac{x - t_0}{h(\varepsilon)} + 2.$$

By the definition of G^ε, we have

$$\int_{-\gamma^\varepsilon(t,x,0)/h(\varepsilon)}^{2c(\varepsilon)} \frac{dz}{1 - \widetilde{b}^\varepsilon(-h(\varepsilon)z)} = \frac{x - t_0}{h(\varepsilon)} + 2.$$

As seen from the proof of Lemma 4.1, we have $1/(1 - \widetilde{b}^\varepsilon(-h(\varepsilon)z)) \le C_2/z$ for $0 \le z \le 2$ and so

$$\frac{x - t_0}{h(\varepsilon)} + 2 \le \int_{-\gamma^\varepsilon(t,x,0)/h(\varepsilon)}^{2c(\varepsilon)} \frac{C_2}{z}\, dz = C_2 \log \frac{2h(\varepsilon)c(\varepsilon)}{-\gamma^\varepsilon(t, x, 0)}.$$

Since $h(\varepsilon) = 1/\log(1/\varepsilon)$, we have

$$\big(\frac{1}{\varepsilon}\big)^{x - t_0} e^2 \le \Big(\frac{2h(\varepsilon)c(\varepsilon)}{-\gamma^\varepsilon(t, x, 0)}\Big)^{C_2}.$$

Hence,

$$0 < -\gamma^\varepsilon(t, x, 0) \le 2 \exp\big(-\frac{2}{C_2}\big) h(\varepsilon)c(\varepsilon)\varepsilon^{(x - t_0)/C_2}. \tag{5.14}$$

When $t + 2h(\varepsilon) \le x \le 2t - t_0$, we have $\gamma^\varepsilon(t, x, 0) = -(G^\varepsilon)^{-1}((2t - x)/h(\varepsilon) + 2)$. A similar argument to the one above gives

$$0 < \gamma^\varepsilon(t, x, 0) \le 2 \exp\big(-\frac{2}{C_2}\big) h(\varepsilon)c(\varepsilon)\varepsilon^{(2t - x - t_0)/C_2}.$$

Step 4-2. We next estimate $\gamma^\varepsilon_x(t, x, 0)$. When $t_0 \le x \le t - 2h(\varepsilon)$, we have $\gamma^\varepsilon(t, x, 0) = (G^\varepsilon)^{-1}(x/h(\varepsilon) + 2)$. Hence, as in the proof of Lemma 4.1, we get, for some constant $c_2 > 0$,

$$\gamma^\varepsilon_x(t, x, 0) = 1 - \widetilde{b}^\varepsilon(\gamma^\varepsilon(t, x, 0)) \le c_2 \frac{|\gamma^\varepsilon(t, x, 0)|}{h(\varepsilon)} \le 2c_2 \exp\big(-\frac{2}{C_2}\big) c(\varepsilon)\varepsilon^{(x - t_0)/C_2},$$

where we used formula (4.2) in the first step and inequality (5.14) in the last step.

When $t+2h(\varepsilon) \le x \le 2t-t_0$, $\gamma^\varepsilon(t, x, 0) = -(G^\varepsilon)^{-1}((2t-x)/h(\varepsilon)+2)$. Similarly, we get

$$\gamma^\varepsilon_x(t, x, 0) = 1 - \widetilde{b}^\varepsilon(-\gamma^\varepsilon(t, x, 0)) \le 2c_2 \exp\big(-\frac{2}{C_2}\big) c(\varepsilon)\varepsilon^{(2t - x - t_0)/C_2}.$$

Step 4-3. Finally, we prove that, for all $K \Subset \{(t, x) \in \mathbb{R}^2 \mid t_0 < x < t\} \cup \{(t, x) \in \mathbb{R}^2 \mid t < x < 2t - t_0\}$ and $\alpha \in \mathbb{N}_0^2$,

$$\|\partial^\alpha v^\varepsilon_x(t, x)\|_{L^\infty(K)} \to 0 \quad \text{as } \varepsilon \downarrow 0. \tag{5.15}$$

This implies that $U = V_x \in \mathscr{G}(\mathbb{R}^2)$ is $\mathscr{G}^\infty_{\log}$-regular on $\{(t,x) \in \mathbb{R}^2 \mid t_0 < x < t\} \cup \{(t,x) \in \mathbb{R}^2 \mid t < x < 2t - t_0\}$.

Note that

$$v_x^\varepsilon(t,x) = \chi\Big(\frac{\gamma^\varepsilon(t,x,0)}{c(\varepsilon)}\Big)\frac{\gamma_x^\varepsilon(t,x,0)}{c(\varepsilon)}.$$

Hence, to prove (5.15), it suffices to show that, for all $K \Subset \{(t,x) \in \mathbb{R}^2 \mid t_0 < x < t\} \cup \{(t,x) \in \mathbb{R}^2 \mid t < x < 2t - t_0\}$ and $\alpha \in \mathbb{N}^2$,

$$\frac{\|\partial^\alpha \gamma^\varepsilon(t,x,0)\|_{L^\infty(K)}}{c(\varepsilon)} \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

This can be done similarly to Step 4-3 of the proof of Theorem 5.1. The proof of Theorem 5.4 is now complete. $\square$

Remark 5.5. It can be conjectured in Theorem 5.4 that

$$\begin{aligned}&\operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}} U\\ &= \{(t,t_0) \mid t \ge t_0\} \cup \{(t, 2t - t_0) \mid t \ge t_0\} \cup \{(t,t) \mid t \le t_0\} \quad (= \operatorname{sing\,supp} u),\end{aligned}$$

but this is open.

As in Step 2 of the proof of Theorem 5.3, we can apply the mean value theorem repeatedly in Step 2 of the proof of Theorem 5.4 to show the following inclusion relation on the $\mathscr{G}^\infty$-singular support of the solution U. However, it is open whether equality holds.

Theorem 5.6. *Under the same assumption as in Theorem 5.4, it holds that*

$$\operatorname{sing\,supp}_{\mathscr{G}^\infty} U \supset \{(t,t) \mid t \le t_0\}.$$

Finally, we discuss the case that $U_0 \in \mathscr{G}(\mathbb{R})$ is defined as the class of $(\kappa_1\chi_{a_1(\varepsilon)}(\cdot + s_1) + \kappa_2\chi_{a_2(\varepsilon)}(\cdot - s_2))_{\varepsilon\in(0,1]}$, where κ_1, $\kappa_2 \in \mathbb{R}$, s_1, $s_2 > 0$, $a_1(\varepsilon)$, $a_2(\varepsilon) \le h(\varepsilon)$. Then $U_0 \approx \kappa_1\delta_{-s_1} + \kappa_2\delta_{s_2}$, where δ_{-s_1} and δ_{s_2} are the delta functions at $-s_1$ and s_2, respectively. As may be seen in the following theorem, the $\mathscr{G}^\infty_{\log}$-singular support of the corresponding solution $U \in \mathscr{G}(\mathbb{R}^2)$ and the singular support of its distributional shadow do not necessarily coincide.

Theorem 5.7. *Let $U_0 \in \mathscr{G}(\mathbb{R})$ be as above. Then the solution $U \in \mathscr{G}(\mathbb{R}^2)$ of problem* (3.1) *admits a distributional shadow, which is given by*

$$u(t,x) = \begin{cases} \kappa_1\delta(x+s_1) + \kappa_2\delta(x - 2t - s_2), \\ \quad \text{if } t \ge \max\{-s_1,-s_2\}, \\ \kappa_1\delta(x+s_1) + \kappa_2\delta(x-t), \\ \quad \text{if } \min\{-s_1,-s_2\} < t < \max\{-s_1,-s_2\} \text{ and } s_1 > s_2, \\ \kappa_1\delta(x-t) + \kappa_2\delta(x-2t-s_2), \\ \quad \text{if } \min\{-s_1,-s_2\} < t < \max\{-s_1,-s_2\} \text{ and } s_1 < s_2, \\ (\kappa_1+\kappa_2)\delta(x-t), \\ \quad \text{if } t \le \min\{-s_1,-s_2\}. \end{cases}$$

Furthermore,

$$\begin{aligned}&\operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}} U\\ &= \{(t,-s_1) \mid t \ge -s_1\} \cup \{(t, 2t+s_2) \mid t \ge -s_2\} \cup \{(t,t) \mid t \le \max\{-s_1,-s_2\}\}.\end{aligned} \tag{5.16}$$

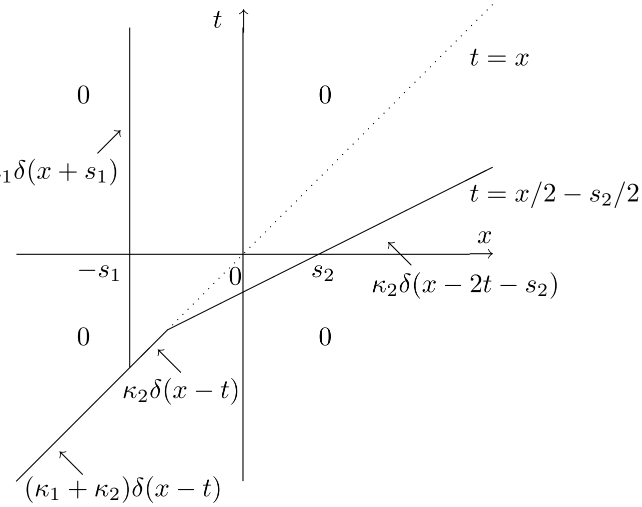

FIGURE 3. Distributional shadow for the case $s_1 > s_2$

Thus, if $\kappa_1 = -\kappa_2 \neq 0$, then

$$\operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}} U \neq \operatorname{sing\,supp} u.$$

Proof. The first assertion can be proved similarly to the proof of Theorem 5.1. We will only prove the second assertion for the case $\kappa_1\kappa_2 \neq 0$. The case $\kappa_1\kappa_2 = 0$ can be argued similarly.

Let $v_0^\varepsilon = H*(\kappa_1\chi_{a_1(\varepsilon)}(\cdot+s_1)+\kappa_2\chi_{a_2(\varepsilon)}(\cdot-s_2))$ and let $V_0 \in \mathscr{G}(\mathbb{R})$ be given by the class of $(v_0^\varepsilon)_{\varepsilon\in(0,1]}$. In order to prove the second assertion, it suffices to investigate the behavior of the solution $V \in \mathscr{G}(\mathbb{R}^2)$ of problem (4.4).

Let $(v^\varepsilon)_{\varepsilon\in(0,1]}$ be a representative of $V \in \mathscr{G}(\mathbb{R}^2)$ satisfying problem (5.1). As in the proof of Theorems 5.1 and 5.4, we can apply the method of characteristic curves to see that V_x identically equals 0 on the complement of the set given by (5.16). Since $U = V_x \in \mathscr{G}(\mathbb{R}^2)$, it follows that U is $\mathscr{G}^\infty_{\log}$-regular on the complement of the set given by (5.16).

Now, we show that $\{(t,-s_1) \mid t \geq -s_1\} \cup \{(t, 2t+s_2) \mid t \geq -s_2\} \subset \operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}} U$. We have $v^\varepsilon(t,x) = v_0^\varepsilon(\gamma^\varepsilon(t,x,0))$. If $-s_1 - a_1(\varepsilon) \leq x \leq -s_1 + a_1(\varepsilon)$ and $t \geq -s_1 + a_1(\varepsilon) + 2h(\varepsilon)$, then $\gamma^\varepsilon(t,x,0) = x$, so that $v^\varepsilon(t,x) = v_0^\varepsilon(x)$. We see that $\partial_x^\alpha v^\varepsilon(t,-s_1) = \kappa_1\chi^{(\alpha-1)}(0)/a_1(\varepsilon)^\alpha$ for $\alpha \in \mathbb{N}$. Hence, $\{(t,-s_1) \mid t \geq -s_1\} \subset \operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}} U$. Similarly, if $2t + s_2 - a_2(\varepsilon) \leq x \leq 2t + s_2 + a_2(\varepsilon)$ and $t \geq -s_2 + a_2(\varepsilon) + 2h(\varepsilon)$, then $\gamma^\varepsilon(t,x,0) = x - 2t$, so that $v^\varepsilon(t,x) = v_0^\varepsilon(x-2t)$. We see that $\partial_x^\alpha v^\varepsilon(t, 2t+s_2) = \kappa_2\chi^{(\alpha-1)}(0)/a_2(\varepsilon)^\alpha$ for $\alpha \in \mathbb{N}$. Hence, $\{(t, 2t+s_2) \mid t \geq -s_2\} \subset \operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}} U$.

Finally, we prove that $\{(t,t) \mid t \leq \max\{-s_1,-s_2\}\} \subset \operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}} U$. For $t < -s_1$, consider

$$\frac{v^\varepsilon(t,\gamma^\varepsilon(0,-s_1,t)) - v^\varepsilon(t,\gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t))}{\gamma^\varepsilon(0,-s_1,t) - \gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t)}.$$

As in Step 1 of the proof of Theorem 4.2, we get, for $\varepsilon > 0$ small enough,

$$G^\varepsilon(\gamma^\varepsilon(0,-s_1,t)-t) = -\frac{t+s_1}{h(\varepsilon)} + 2, \tag{5.17}$$

$$G^\varepsilon(\gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t)-t) = -\frac{t+s_1+a_1(\varepsilon)}{h(\varepsilon)} + 2. \tag{5.18}$$

By (4.3) and (5.18), we have, for some constant $C_2 > 0$,

$$t-\gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t) \le 2\exp\big(-\frac{2}{C_2}\big)h(\varepsilon)\varepsilon^{-(t+s_1+a_1(\varepsilon))/C_2}. \tag{5.19}$$

Subtracting (5.18) from (5.17) gives

$$G^\varepsilon(\gamma^\varepsilon(0,-s_1,t)-t) - G^\varepsilon(\gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t)-t) = \frac{a_1(\varepsilon)}{h(\varepsilon)}. \tag{5.20}$$

On the other hand, by the definition of G^ε, we get

$$\begin{aligned} 0 &< G^\varepsilon(\gamma^\varepsilon(0,-s_1,t)-t) - G^\varepsilon(\gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t)-t) \\ &= \int_{(t-\gamma^\varepsilon(0,-s_1,t))/h(\varepsilon)}^{(t-\gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t))/h(\varepsilon)} \frac{dz}{1-\widetilde{b}^\varepsilon(-h(\varepsilon)z)}. \end{aligned}$$

As in the proof of Lemma 4.1, we have $1/(1-\widetilde{b}^\varepsilon(-h(\varepsilon)z)) \ge C_1/z$ for some constant $C_1 > 0$ and so

$$\begin{aligned} &G^\varepsilon(\gamma^\varepsilon(0,-s_1,t)-t) - G^\varepsilon(\gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t)-t) \\ &\ge \int_{(t-\gamma^\varepsilon(0,-s_1,t))/h(\varepsilon)}^{(t-\gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t))/h(\varepsilon)} \frac{C_1}{z}\,dz \\ &\ge C_1\big[\frac{t-\gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t)}{h(\varepsilon)} - \frac{t-\gamma^\varepsilon(0,-s_1,t)}{h(\varepsilon)}\big]\big[\frac{h(\varepsilon)}{t-\gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t)}\big] \\ &= \frac{C_1[\gamma^\varepsilon(0,-s_1,t)-\gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t)]}{t-\gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t)}. \end{aligned}$$

Hence, by (5.20), we have

$$\gamma^\varepsilon(0,-s_1,t)-\gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t) \le \frac{t-\gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t)}{C_1}\cdot\frac{a_1(\varepsilon)}{h(\varepsilon)}. \tag{5.21}$$

We combine (5.19) and (5.21) to see that

$$\gamma^\varepsilon(0,-s_1,t)-\gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t) \le \frac{2}{C_1}\exp\big(-\frac{2}{C_2}\big)a_1(\varepsilon)\varepsilon^{-(t+s_1+a_1(\varepsilon))/C_2}. \tag{5.22}$$

We use $v^\varepsilon(t,\gamma^\varepsilon(0,-s_1,t)) = \kappa_1/2$, $v^\varepsilon(t,\gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t)) = 0$ and (5.22) to get

$$\begin{aligned} &\frac{v^\varepsilon(t,\gamma^\varepsilon(0,-s_1,t)) - v^\varepsilon(t,\gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t))}{\gamma^\varepsilon(0,-s_1,t)-\gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t)} \\ &\ge \frac{\kappa_1 C_1}{4\exp(-2/C_2)a_1(\varepsilon)}\cdot\frac{1}{\varepsilon^{-(t+s_1+a_1(\varepsilon))/C_2}}. \end{aligned}$$

Then by the mean value theorem, we find $x^\varepsilon \in (\gamma^\varepsilon(0,-s_1-a_1(\varepsilon),t), \gamma^\varepsilon(0,-s_1,t))$ such that

$$v_x^\varepsilon(t,x^\varepsilon) \ge \frac{\kappa_1 C_1}{4\exp(-2/C_2)a_1(\varepsilon)}\cdot\frac{1}{\varepsilon^{-(t+s_1+a_1(\varepsilon))/C_2}}.$$

Since $U = V_x \in \mathscr{G}(\mathbb{R}^2)$, this means that $\{(t,t) \mid t \le -s_1\} \subset \operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}} U$. Similarly, we can see that $\{(t,t) \mid t \le -s_2\} \subset \operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}} U$. Therefore, $\{(t,t) \mid t \le \max\{-s_1,-s_2\}\} \subset \operatorname{sing\,supp}_{\mathscr{G}^\infty_{\log}} U$. Thus, (5.16) follows. $\square$

Remark 5.8. In Theorem 5.7, when $s_1 = s_2$ and $\kappa_1 = -\kappa_2 \neq 0$, the solution $U \in \mathscr{G}(\mathbb{R}^2)$ of problem (3.1) admits a distributional shadow on $(-s_1,\infty)\times\mathbb{R}$, which is given by $u(t,x) = \kappa_1\delta(x+s_1) + \kappa_2\delta(x-2t-s_1)$. Since, for any κ_1, κ_2 such that $\kappa_1 = -\kappa_2 \neq 0$, this distribution u satisfies problem (1.1) with initial data 0 at $t = -s_1$, it follows that there exist infinitely many different distributional solutions with initial data 0 at $t = -s_1$. Thus, Theorem 5.7 means that, in the setting of Colombeau's theory, these distributional solutions with initial data 0 at $t = -s_1$ can be regarded as generalized solutions with different initial data, as in Theorem 4.2.

As in Step 2 of the proof of Theorem 5.3, we can use the mean value theorem repeatedly in the last part of the proof of Theorem 5.7 to get the following equality on the $\mathscr{G}^\infty$-singular support of the solution U. Hence, we see that, even if $U_0 \in \mathscr{G}^\infty(\mathbb{R})$, the singularity in $\mathscr{G}^\infty$ occurs suddenly when the propagation of singularities is observed backward in time.

Theorem 5.9. *Under the same assumption as in Theorem 5.7, if $U_0 \in \mathscr{G}^\infty(\mathbb{R})$, then*

$$\operatorname{sing\,supp}_{\mathscr{G}^\infty} U = \{(t,t) \mid t \le \max\{-s_1,-s_2\}\}.$$

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