Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 72, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ALMOST PERIODIC SOLUTIONS OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH STEPANOV-ALMOST PERIODIC TERMS

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Abstract. In this paper we study the existence of almost periodic solutions of an autonomous neutral functional differential equation with Stepanov-almost periodic terms in a Banach space. We use the contraction mapping principle to show the existence and the uniqueness of an almost periodic solution of the equation.

1. Introduction

The theory of almost periodic functions was mainly treated and created by Bohr during 1924-1926. Bohr's theory was substantially developed by Bochner, Weyl, Besicovitch, Farvard, von Neumann, Stepanov, Bogolyubov, and others during the 1920s and 1930s. In 1933, Bochner defined and studied the almost periodic functions with values in Banach spaces. Bohr's theory of almost periodic functions was restricted to the class of uniformly continuous functions. In 1925, Stepanov generalized the class of almost periodic functions in the sense of Bohr without using the hypothesis of continuity. For more details about almost periodic functions and Stepanov-almost periodic functions, see [2, 5]. In recent years, the theory of almost periodic functions has been developed in connection with the problems of differential equations, dynamical systems, stability theory and so on.

Functional differential equations arise as models in several physical phenomena, for example, reaction-diffusion equations, climate models, population ecology, neural networks etc. More recently researchers have given special attentions to the study of equations in which the delay argument occurs in the derivative of the state variable as well as in the independent variable, so-called neutral differential equations. Neutral differential equations have many applications. For example, these equations arises in many phenomena such as in the study of oscillatory systems and also in modelling of several physical problems. Periodicity of solutions of neutral differential equations has been studied by many authors; see [3, 4].

Let $(\mathbb{X}, \|.\|)$ be a complex Banach space. In this paper, we study the existence and the uniqueness of an almost periodic solution to the neutral functional differential

2000 *Mathematics Subject Classification.* 43A60, 34C27, 34K14, 47D06.
Key words and phrases. Stepanov-almost periodic; almost periodic; C_0-semigroup, neutral differential equation; contraction mapping principle.

Submitted May 4, 2011. Published May 31, 2011.

equation

$$\frac{d}{dt}[u(t) - F(t, u(t-g(t)))] = Au(t) + G(t, u(t), u(t-g(t))) \tag{1.1}$$

for $t \in \mathbb{R}$ and $u \in AP(\mathbb{R};\mathbb{X})$, where $AP(\mathbb{R};\mathbb{X})$ be the set of all almost periodic functions from $\mathbb{R}$ to $\mathbb{X}$, A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{0\le t<\infty}$, and $F : \mathbb{R}\times\mathbb{X} \mapsto \mathbb{X}$, $G : \mathbb{R}\times\mathbb{X}\times\mathbb{X} \mapsto \mathbb{X}$ are Stepanov-almost periodic functions.

The existence of almost periodic solutions of abstract differential equations has been considered by many authors; see [2, 6, 8, 9, 10]. Zaidman [8] proved the existence and uniqueness of almost periodic solution to the nonhomogeneous differential equation

$$\frac{d}{dt}u(t) = Au(t) + f(t), \tag{1.2}$$

where A is a linear unbounded operator in $\mathbb{X}$ which is the infinitesimal generator of a C_0-semigroup with exponential decay as $t \to \infty$, and $f : \mathbb{R} \mapsto \mathbb{X}$ is an almost periodic function. Zaidman [9] considered the same equation (1.2) in a Hilbert space $\mathbb{H}$ and proved the existence and the uniqueness of almost periodic solution provided that A is a bounded linear operator in $\mathbb{H}$ such that $\|e^{-tA}\| \le e^{-\omega t}$, for all $t > 0$ and for some $\omega > 0$, and $f : \mathbb{R} \mapsto \mathbb{H}$ is a continuous function which is in $S^2_{ap}(\mathbb{R};\mathbb{H})$, Rao [6] also considered the same equation (1.2) in a Banach space $\mathbb{X}$ and proved the existence and the uniqueness of almost periodic solution provided that A is the infinitesimal generator of a continuous semigroup $\{T(t) : 0 \le t < \infty\}$, with $T(t)$ satisfying $\|T(t)\| \le Me^{-\beta t}$ for some $M > 0$, for some $\beta > 0$ and all $t \ge 0$, and $f : \mathbb{R} \mapsto \mathbb{X}$ is an S^1-almost periodic continuously differentiable function, with f' being S^1-bounded on $\mathbb{R}$.

In this paper, we extend the previous-mentioned results to the equation (1.1). We use the contraction mapping principle to prove the existence and uniqueness of an almost periodic solution of the equation (1.1).

2. Preliminaries

In this section we give some basic definitions, notation, and results. In the rest of this paper, $(\mathbb{X}, \|\cdot\|)$ stands for a complex Banach space.

Definition 2.1. A one parameter family $\{T(t)\}_{0\le t<\infty}$ of bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$ is called a C_0-semigroup of bounded linear operators on $\mathbb{X}$ if

(i) $T(0) = I$, where I is the identity operator on $\mathbb{X}$.
(ii) $T(t+s) = T(t)T(s)$ for every $t, s \ge 0$.
(iii) $\lim_{t\downarrow 0} T(t)x = x$ for every $x \in \mathbb{X}$.

The linear operator A defined by

$$D(A) = \big\{x \in \mathbb{X} : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists } \big\}$$

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} = \frac{d^+T(t)x}{dt}\Big|_{t=0} \quad \text{for } x \in D(A)$$

is the infinitesimal generator of the semigroup $\{T(t)\}_{0\le t<\infty}$, where $D(A)$ is the domain of A.

Theorem 2.2. *Let $\{T(t)\}_{0\le t<\infty}$ be a C_0-semigroup of bounded linear operators on $\mathbb{X}$. Then*

(i) *there exists* $\omega \in \mathbb{R}$ *and* $M \geq 1$ *such that*

$$\|T(t)\| \leq Me^{t\omega}, \quad \forall t \geq 0;$$

(ii) *the mapping* $(t, x) \mapsto T(t)x$ *is jointly continuous from* $[0, \infty) \times \mathbb{X}$ *to* $\mathbb{X}$.

For a detailed proof of the above theorem, see [7, theorem 2.3.1], and [7, corollary 2.3.1].

Definition 2.3. A continuous function $f : \mathbb{R} \mapsto \mathbb{X}$ is said to be almost periodic if for every $\epsilon > 0$ there exists a positive number l such that every interval of length l contains a number τ such that

$$\|f(t + \tau) - f(t)\| < \epsilon \quad \forall t \in \mathbb{R}.$$

Let $AP(\mathbb{R}; \mathbb{X})$ be the set of all almost periodic functions from $\mathbb{R}$ to $\mathbb{X}$. Then $(AP(\mathbb{R}; \mathbb{X}), \|.\|_\infty)$ is a Banach space with supremum norm given by

$$\|u\|_\infty = \sup_{t\in\mathbb{R}} \|u(t)\|.$$

Theorem 2.4. *If* $f \in AP(\mathbb{R}; \mathbb{X})$*, then* f *is uniformly continuous.*

Theorem 2.5 (Bochner's Criterion)**.** *A continuous function* $f : \mathbb{R} \mapsto \mathbb{X}$ *is an almost periodic function if and only if for every sequence of real numbers* (s'_n)*, there exists a subsequence* (s_n) *such that* $(f(t + s_n))$ *converges uniformly for* $t \in \mathbb{R}$.

Lemma 2.6. *If* $u \in AP(\mathbb{R}; \mathbb{X})$ *and* $g \in AP(\mathbb{R}; \mathbb{R})$*, then* $u(. - g(.)) \in AP(\mathbb{R}; \mathbb{X})$.

For a detailed proof of the above lemma see [3, Lemma 2.4]. Let $\mathbb{Y}, \mathbb{W}$ be Banach spaces. We define the set $AP(\mathbb{R} \times \mathbb{X}; \mathbb{Y})$ which consists of all continuous functions $f : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{Y}$ such that $f(., x) \in AP(\mathbb{R}; \mathbb{Y})$ uniformly for each $x \in E$, where E is any compact subset of $\mathbb{X}$.

Proposition 2.7 ([1, Proposition 1])**.** *If* $f \in AP(\mathbb{R} \times \mathbb{X}; \mathbb{Y})$ *and* $h \in AP(\mathbb{R}; \mathbb{X})$*, then the function* $f(., h(.)) \in AP(\mathbb{R}; \mathbb{Y})$.

Throughout the rest of the paper we fix $p, 1 \leq p < \infty$. Denote by $L^p_{\rm loc}(\mathbb{R}; \mathbb{X})$ the space of all functions from $\mathbb{R}$ into $\mathbb{X}$ which are locally p-integrable in Bochner-Lebesgue sense. We say that a function, $f \in L^p_{\rm loc}(\mathbb{R}; \mathbb{X})$ is p-Stepanov bounded (S^p-bounded) if

$$\|f\|_{S^p} = \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(s)\|^p ds \Big)^{1/p} < \infty.$$

We indicate by $L^p_s(\mathbb{R}; \mathbb{X})$ the set of S^p-bounded functions.

Definition 2.8. A function $f \in L^p_s(\mathbb{R}; \mathbb{X})$ is said to be almost periodic in the sense of Stepanov (S^p-almost periodic) if for every $\epsilon > 0$ there exists a positive number l such that every interval of length l contains a number τ such that

$$\sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(s + \tau) - f(s)\|^p ds \Big)^{1/p} < \epsilon.$$

Let $S^p_{ap}(\mathbb{R}; \mathbb{X})$ be the set of all S^p-almost periodic functions.

It is clear that $f(t)$ almost periodic implies $f(t)$ is S^p-almost periodic; that is, $AP(\mathbb{R}; \mathbb{X}) \subset S^p_{ap}(\mathbb{R}; \mathbb{X})$. Moreover, if $1 \leq m < p$, then $f(t)$ is S^p-almost periodic implies $f(t)$ is S^m-almost periodic.

Lemma 2.9 ([2], Bochner). *If $f \in S^p_{ap}(\mathbb{R};\mathbb{X})$ and uniformly continuous, then f is almost periodic.*

Lemma 2.10 (Bochner). *$f \in S^p_{ap}(\mathbb{R};\mathbb{X})$ if and only if $f^b \in AP(\mathbb{R}; L^p([0,1];\mathbb{X}))$, where $f^b(t) = \{f(t+s) : s \in [0,1]\}$, $t \in \mathbb{R}$.*

For a detailed proof of the above lemma, see [2, pp. 78, 79].

We define the set $S^p_{ap}(\mathbb{R}\times\mathbb{X};\mathbb{Y})$ which consists of all functions $f : \mathbb{R}\times\mathbb{X} \mapsto \mathbb{Y}$ such that $f(.,x) \in S^p_{ap}(\mathbb{R};\mathbb{Y})$ uniformly for each $x \in E$, where E is any compact subset of $\mathbb{X}$.

We define the set $S^p_{ap}(\mathbb{R}\times\mathbb{X}\times\mathbb{Y};\mathbb{W})$ which consists of all functions $f : \mathbb{R}\times\mathbb{X}\times\mathbb{Y} \mapsto \mathbb{W}$ such that $f(.,x,y) \in S^p_{ap}(\mathbb{R};\mathbb{W})$ uniformly for each $(x,y) \in E$, where E is any compact subset of $\mathbb{X}\times\mathbb{Y}$.

Example 2.11. $(\mathbb{R}^2, \|\cdot\|)$ is a Banach space, where

$$\|x\| = |x_1| + |x_2|, \quad x = (x_1,x_2) \in \mathbb{R}^2.$$

The functions $F : \mathbb{R}\times\mathbb{R}^2 \mapsto \mathbb{R}^2$, $G : \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2 \mapsto \mathbb{R}^2$ are defined by

$$\begin{gathered} F(t,x) = (f(t), \sin x_1 - \sin x_2), \\ G(t,x,y) = K_2(f(t), e^{-x_1} - e^{-x_2} + \cos y_1 - \cos y_2), \end{gathered}$$

where $K_2 > 0$, $x = (x_1,x_2)$, $y = (y_1,y_2)$, and

$$f(t) = \begin{cases} n & t = n\pi,\ n \in \mathbb{Z} \\ \sin t & \text{otherwise.} \end{cases}$$

Notice that $f \in S^p_{ap}(\mathbb{R};\mathbb{R})$ but $f \notin AP(\mathbb{R};\mathbb{R})$, as it is unbounded and discontinuous. Hence it is easy to see that $F \in S^p_{ap}(\mathbb{R}\times\mathbb{R}^2;\mathbb{R}^2)$, $G \in S^p_{ap}(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2;\mathbb{R}^2)$ but $F \notin AP(\mathbb{R}\times\mathbb{R}^2;\mathbb{R}^2)$, $G \notin AP(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2;\mathbb{R}^2)$.

Throughout the rest of the paper we assume that A is the infinitesimal generator of C_0-semigroup $\{T(t)\}_{0\le t<\infty}$. In the view of theorem 2.2(i), we also assume that there exists constants $\omega > 0$ and $M \ge 1$ such that

$$\|T(t)\| \le Me^{-\omega t} \ \text{ for } \ 0 \le t < \infty. \tag{2.1}$$

Definition 2.12. By an almost periodic mild solution $u : \mathbb{R} \mapsto \mathbb{X}$ of the differential equation (1.1) we mean that $u \in AP(\mathbb{R};\mathbb{X})$, and $u(t)$ satisfies

$$u(t) = F(t, u(t-g(t))) + \int_{-\infty}^{t} T(t-s)G(s,u(s),u(s-g(s)))ds, \quad t \in \mathbb{R}. \tag{2.2}$$

Throughout the rest of the paper we consider the following assumptions.

- (H1) $g \in AP(\mathbb{R};\mathbb{R})$, $F \in S^p_{ap}(\mathbb{R}\times\mathbb{X};\mathbb{X})$ and $G \in S^p_{ap}(\mathbb{R}\times\mathbb{X}\times\mathbb{X};\mathbb{X})$.
- (H2) The functions F, G satisfy the property that there exists $K_1, K_2 > 0$ such that
$$\|F(t,u(t)) - F(s,v(s))\| \le K_1\|u(t) - v(s)\|$$
for all $t, s \in \mathbb{R}$ and for each $u, v \in AP(\mathbb{R};\mathbb{X})$, and
$$\|G(t,x_1,\phi(t)) - G(t,x_2,\varphi(t))\| \le K_2(\|x_1 - x_2\| + \|\phi - \varphi\|_\infty)$$
for all $t \in \mathbb{R}$ and for $(x_1,\phi), (x_2,\varphi) \in \mathbb{X}\times AP(\mathbb{R};\mathbb{X})$.

Example 2.13. Consider the function G defined in example 2.11. For $x, y \in \mathbb{R}^2$ and $u, v \in AP(\mathbb{R};\mathbb{R}^2)$, we observe that

$$\begin{aligned}
&\|G(t,x,u(t)) - G(t,y,v(t))\| \\
&\le K_2|e^{-x_1} - e^{-x_2} + \cos u_1(t) - \cos u_2(t) - e^{-y_1} + e^{-y_2} - \cos v_1(t) - \cos v_2(t)| \\
&\le K_2(|x_1 - y_1| + |x_2 - y_2| + |u_1(t) - v_1(t)| + |u_2(t) - v_2(t)|) \\
&= K_2(\|x - y\| + \|u(t) - v(t)\|) \\
&\le K_2(\|x - y\| + \|u - v\|_\infty) \quad \forall t \in \mathbb{R}.
\end{aligned}$$

Thus G satisfies the assumption (H2).

Define $F : \mathbb{R} \times \mathbb{R}^2 \mapsto \mathbb{R}^2$ by $F(t,x) = K_1(0, \sin x_1 - \sin x_2)$, where $K_1 > 0$. Clearly $F \in S^p_{ap}(\mathbb{R} \times \mathbb{R}^2; \mathbb{R}^2)$.

For $u, v \in AP(\mathbb{R};\mathbb{R}^2)$ and $t, s \in \mathbb{R}$, we observe that

$$\begin{aligned}
\|F(t,u(t)) - F(s,v(s))\| &\le K_1|\sin u_1(t) - \sin u_2(t) - \sin v_1(s) + \sin v_2(s)| \\
&\le K_1(|\sin u_1(t) - \sin v_1(s)| + |\sin u_2(t) - \sin v_2(s)|) \\
&\le K_1(|u_1(t) - v_1(s)| + |u_2(t) - v_2(s)|) \\
&= K_1\|u(t) - v(s)\|.
\end{aligned}$$

Thus F satisfies the assumption (H2).

3. Main results

In this section we prove the existence and uniqueness of almost periodic mild solution for (1.1). We define two mappings Λ and L by

$$(\Lambda u)(t) = F(t, u(t - g(t))) + \int_{-\infty}^{t} T(t-s)G(s, u(s), u(s - g(s)))ds, \tag{3.1}$$

$$(Lf)(t) = \int_{-\infty}^{t} T(t-s)f(s)ds, \quad t \in \mathbb{R}. \tag{3.2}$$

Throughout the rest of the paper we indicate the conjugate index of p by q; that is, $\frac{1}{p} + \frac{1}{q} = 1$. We show the following.

Proposition 3.1. *If $f \in S^p_{ap}(\mathbb{R} \times \mathbb{X}; \mathbb{Y})$ and $g \in AP(\mathbb{R}; \mathbb{X})$, then $f(., g(.)) \in S^p_{ap}(\mathbb{R}; \mathbb{Y})$.*

Proof. From Lemma 2.10, it follows that $f^b \in AP(\mathbb{R} \times \mathbb{X}; L^p([0,1]; \mathbb{Y}))$, where $f^b(t,x) = \{f(t+s,x) : s \in [0,1]\}$, $t \in \mathbb{R}, x \in \mathbb{X}$. From proposition 2.7, it follows that $f^b(., g(.)) \in AP(\mathbb{R}; L^p([0,1]; \mathbb{Y}))$. Again from Lemma 2.10, we get $f(., g(.)) \in S^p_{ap}(\mathbb{R}; \mathbb{Y})$. □

Proposition 3.2. *If $f \in S^p_{ap}(\mathbb{R} \times \mathbb{X} \times \mathbb{Y}; \mathbb{W})$, $g \in AP(\mathbb{R}; \mathbb{X})$ and $h \in AP(\mathbb{R}; \mathbb{Y})$, then $f(., g(.), h(.)) \in S^p_{ap}(\mathbb{R}; \mathbb{W})$.*

Proof. From the Bochner's Criterion, it follows that $(g(.), h(.)) \in AP(\mathbb{R}; \mathbb{X} \times \mathbb{Y})$. Hence from the proposition 3.1, we get $f(., g(.), h(.)) \in S^p_{ap}(\mathbb{R}; \mathbb{W})$. □

Lemma 3.3. *If $f(t)$ is an S^p-almost periodic function, then the function $(Lf)(t)$ is an almost periodic function.*

Proof. We consider

$$(Lf)_k(t) = \int_{t-k}^{t-k+1} T(t-s)f(s)ds, \quad k \in \mathbb{N},\ t \in \mathbb{R}.$$

Then

$$\begin{aligned} \|(Lf)_k(t)\| &\le \int_{t-k}^{t-k+1} \|T(t-s)\| \|f(s)\| ds \\ &\le M \int_{t-k}^{t-k+1} e^{-\omega(t-s)} \|f(s)\| ds. \end{aligned} \tag{3.3}$$

Case 1: $1 < p < \infty$. Then $1 < q < \infty$. Using the Hölder's inequality, we have

$$\begin{aligned} &M \int_{t-k}^{t-k+1} e^{-\omega(t-s)} \|f(s)\| ds \\ &\le M \Big(\int_{t-k}^{t-k+1} e^{-\omega q(t-s)} ds \Big)^{1/q} \Big(\int_{t-k}^{t-k+1} \|f(s)\|^p ds \Big)^{1/p} \\ &\le \frac{M}{\sqrt[q]{q\omega}} \left(e^{-q\omega(k-1)} - e^{-q\omega k} \right)^{1/q} \|f\|_{S^p} \\ &= M \frac{e^{-\omega k}}{\sqrt[q]{q\omega}} (e^{q\omega} - 1)^{1/q} \|f\|_{S^p}. \end{aligned}$$

Since the series $\sum_{k=1}^{\infty} e^{-\omega k}$ is convergent, therefore from the Weierstrass test the sequence of functions $\sum_{k=1}^{n} (Lf)_k(t)$ is uniformly convergent on $\mathbb{R}$. Hence we have

$$(Lf)(t) = \sum_{k=1}^{\infty} (Lf)_k(t).$$

From theorem 2.2(ii), $(Lf)(.)$ is continuous. Let $\epsilon > 0$. Then there exists a positive number l such that every interval of length l contains a number τ such that

$$\sup_{t \in \mathbb{R}} \Big(\int_{t}^{t+1} \|f(s+\tau) - f(s)\|^p ds \Big)^{1/p} < \epsilon_1,$$

where

$$0 < \epsilon_1 < \frac{\sqrt[q]{q\omega}(e^{\omega} - 1)\epsilon}{M(e^{q\omega} - 1)^{1/q}}.$$

Now we consider $\|(Lf)_k(s+\tau) - (Lf)_k(s)\|$

$$\begin{aligned} &= \Big\| \int_{s+\tau-k}^{s+\tau-k+1} T(s+\tau-z)f(z)dz - \int_{s-k}^{s-k+1} T(s-z)f(z)dz \Big\| \\ &\le \int_{s-k}^{s-k+1} \|T(s-z)\| \|f(\tau+z) - f(z)\| dz \\ &\le M \int_{s-k}^{s-k+1} e^{-\omega(s-z)} \|f(\tau+z) - f(z)\| dz \\ &\le M \Big(\int_{s-k}^{s-k+1} e^{-\omega q(s-z)} dz \Big)^{1/q} \Big(\int_{s-k}^{s-k+1} \|f(z+\tau) - f(z)\|^p dz \Big)^{1/p} \\ &< \epsilon_1 M \frac{e^{-\omega k}}{\sqrt[q]{q\omega}} \left(e^{q\omega} - 1 \right)^{1/q}. \end{aligned}$$

Therefore,

$$\sum_{k=1}^{\infty}\|(Lf)_k(s+\tau)-(Lf)_k(s)\| \le \frac{\epsilon_1 M}{\sqrt[q]{q\omega}}\sum_{k=1}^{\infty} e^{-\omega k}\left(e^{q\omega}-1\right)^{1/q}$$
$$= \frac{\epsilon_1 M(e^{q\omega}-1)^{1/q}}{\sqrt[q]{q\omega}(e^{\omega}-1)} < \epsilon.$$

Hence $(Lf)(t)$ is an almost periodic function.

Case 2: $p=1$. Then $q=\infty$ and using the Hölder's inequality, we have

$$M\int_{t-k}^{t-k+1} e^{-\omega(t-s)}\|f(s)\|ds \le M\Big(\sup_{t-k\le s\le t-k+1} e^{-\omega(t-s)}\Big)\Big(\int_{t-k}^{t-k+1}\|f(s)\|ds\Big)$$
$$\le Me^{-\omega(k-1)}\|f\|_{S^1}.$$

Since the series $\sum_{k=1}^{\infty} e^{-\omega(k-1)}$ is convergent, therefore from the Weierstrass test and from (3.3), the sequence of functions $\sum_{k=1}^{n}(Lf)_k(t)$ is uniformly convergent on $\mathbb{R}$. Hence we have

$$(Lf)(t)=\sum_{k=1}^{\infty}(Lf)_k(t).$$

Notice that $(Lf)(.)$ is continuous. Let $\epsilon>0$. Then there exists a positive number l such that every interval of length l contains a number τ such that

$$\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\|f(s+\tau)-f(s)\|ds\Big) < \epsilon_2,$$

where

$$0<\epsilon_2<\frac{(e^{\omega}-1)\epsilon}{Me^{\omega}}.$$

Now we consider

$$\begin{aligned}
&\|(Lf)_k(s+\tau)-(Lf)_k(s)\|\\
&= \Big\|\int_{s+\tau-k}^{s+\tau-k+1} T(s+\tau-z)f(z)dz - \int_{s-k}^{s-k+1} T(s-z)f(z)dz\Big\|\\
&\le \int_{s-k}^{s-k+1}\|T(s-z)\|\|f(\tau+z)-f(z)\|dz\\
&\le M\int_{s-k}^{s-k+1} e^{-\omega(s-z)}\|f(\tau+z)-f(z)\|dz\\
&\le M\Big(\sup_{s-k\le z\le s-k+1} e^{-\omega(s-z)}\Big)\Big(\int_{s-k}^{s-k+1}\|f(z+\tau)-f(z)\|dz\Big)\\
&< \epsilon_2 Me^{-\omega(k-1)}.
\end{aligned}$$

Therefore,

$$\sum_{k=1}^{\infty}\|(Lf)_k(s+\tau)-(Lf)_k(s)\| \le \epsilon_2 M\sum_{k=1}^{\infty} e^{-\omega(k-1)} = \epsilon_2 M\frac{e^{\omega}}{e^{\omega}-1} < \epsilon.$$

Hence $(Lf)(t)$ is an almost periodic function. $\square$

Lemma 3.4. *The operator Λ maps $AP(\mathbb{R};\mathbb{X})$ into itself.*

Proof. Let $u \in AP(\mathbb{R};\mathbb{X})$. From Lemma 2.6, we get $u(.-g(.)) \in AP(\mathbb{R};\mathbb{X})$. Hence from theorem 2.4, $u(t-g(t))$ is uniformly continuous on $\mathbb{R}$. For given $\epsilon > 0$ there exists $\delta > 0$ such that $\|u(t_1 - g(t_1)) - u(t_2 - g(t_2))\| < \epsilon/K_1$ whenever $|t_1 - t_2| < \delta$. From the assumption (H2), we obtain

$$\begin{aligned}\|F(t_1, u(t_1 - g(t_1))) - F(t_2, u(t_2 - g(t_2)))\| &\le K_1\|u(t_1 - g(t_1)) - u(t_2 - g(t_2))\| \\ < \epsilon \quad \text{whenever } |t_1 - t_2| < \delta.\end{aligned}$$

Hence $F(t, u(t-g(t)))$ is uniformly continuous. And also from the proposition 3.1, we have $F(., u(.-g(.))) \in S^p_{ap}(\mathbb{R};\mathbb{X})$. Thus from Lemma 2.9, we obtain $F(., u(.-g(.))) \in AP(\mathbb{R};\mathbb{X})$. From proposition 3.2, we obtain $G(., u(.), u(.-g(.))) \in S^p_{ap}(\mathbb{R};\mathbb{X})$. Hence from Lemma 3.3, we obtain $(LG)(., u(.), u(.-g(.))) \in AP(\mathbb{R};\mathbb{X})$. Thus $(\Lambda u)(.) \in AP(\mathbb{R};\mathbb{X})$. □

Theorem 3.5. *Suppose* $(K_1 + \frac{2M}{\omega}K_2) < 1$. *Then* (1.1) *has unique almost periodic mild solution.*

Proof. Let $u, v \in AP(\mathbb{R};\mathbb{X})$. We observed that

$$\begin{aligned}&\|(\Lambda u)(t) - (\Lambda v)(t)\| \\ &\le \|F(t, u(t-g(t))) - F(t - v(t-g(t)))\| \\ &\quad + \int_{-\infty}^{t} \|T(t-s)\|\|G(s, u(s), u(s-g(s))) - G(s, v(s), v(s-g(s)))\|ds \\ &\le K_1\|u(t-g(t)) - v(t-g(t))\| \\ &\quad + MK_2\int_{-\infty}^{t} e^{-\omega(t-s)}(\|u(s) - v(s)\| + \|u-v\|_\infty)ds \\ &\le K_1\|u-v\|_\infty + 2MK_2\Big(\int_{-\infty}^{t} e^{-\omega(t-s)}ds\Big)\|u-v\|_\infty \\ &\le \Big(K_1 + \frac{2M}{\omega}K_2\Big)\|u-v\|_\infty.\end{aligned}$$

Thus

$$\|\Lambda u - \Lambda v)\|_\infty \le \big(K_1 + \frac{2M}{\omega}K_2\big)\|u-v\|_\infty.$$

Thus Λ is a contraction map on $AP(\mathbb{R};\mathbb{X})$. Therefore, Λ has unique fixed point in $AP(\mathbb{R};\mathbb{X})$, that is, there exist unique $\psi \in AP(\mathbb{R};\mathbb{X})$ such that $\Lambda\psi = \psi$. Therefore the equation (1.1) has unique almost periodic mild solution. □

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