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COMPACT DECOUPLING FOR THERMOVISCOELASTICITY IN IRREGULAR DOMAINS

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Abstract. Our goal is to prove the compactness of the difference between the thermoviscoelasticity semigroup and its decoupled semigroup. To show this, we prove the norm continuity of this difference, the compactness of the difference of their resolvents and use [4, Theorem 2.3]. We generalize a result by Liu [5]. An illustrative example of a thermoviscoelastic system with Neumann Laplacian on a Jelly Roll domain is given.

1. Introduction

Consider the abstract thermoviscoelastic model

$$\ddot{w}(t) + A_1 w(t) - \int_{-\infty}^{0} g(s) A_1 w(t+s) ds + Bu(t) = 0, \quad t \geq 0, \tag{1.1}$$

$$\dot{u}(t) + A_2 u(t) - B^* \dot{w}(t) = 0, \quad t \geq 0, \tag{1.2}$$

$$w(0) = w^0, \quad \dot{w}(0) = w^1, \quad u(0) = u^0, \quad w(s) = f_0(s), \quad s \in (-\infty, 0), \tag{1.3}$$

where g is a given function satisfying the following conditions:

$$g \in \mathcal{C}^1(-\infty, 0] \cap L^1(-\infty, 0), \tag{1.4}$$

$$g(t) \geq 0, \quad g'(t) \geq 0 \quad \text{for } t < 0, \tag{1.5}$$

$$\int_{-\infty}^{0} g(s) ds < 1. \tag{1.6}$$

By the decoupling technique, we obtain the system

$$\ddot{\bar{w}}(t) + A_1 \bar{w}(t) - \int_{-\infty}^{0} g(s) A_1 \bar{w}(t+s) ds + BA_2^{-1} B^* \dot{\bar{w}}(t) = 0, \quad t \geq 0, \tag{1.7}$$

$$\dot{\bar{u}}(t) + A_2 \bar{u}(t) - B^* \dot{\bar{w}}(t) = 0, \quad t \geq 0, \tag{1.8}$$

$$\bar{w}(0) = w^0, \quad \dot{\bar{w}}(0) = w^1, \quad \bar{u}(0) = u^0, \quad \bar{w}(s) = f_0(s), \quad s \in (-\infty, 0). \tag{1.9}$$

The operators A_1 and A_2 are positive self adjoint and invertible on two Hilbert spaces H_1 and H_2, and B is an unbounded operator from H_2 to H_1. Liu [5] proved that these two systems are well posed and generate two semigroups $\mathcal{T} := (T(t))_{t\geq 0}$

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and $\mathcal{T}_d := (T_d(t))_{t\geq 0}$. Assuming that $BA_2^{-\gamma}$ is compact for some $0 < \gamma < 1$, he proved that their difference is compact. In this paper, proceeding as in [1], we show that $t \mapsto T(t) - T_d(t)$ and $t \mapsto T(t) - S(t)$ are norm continuous for $t > 0$ where $\mathcal{S} := (S(t))_{t\geq 0}$ is the semigroup generated by the first equation (1.7) in the decoupled system. Consequently, under no compactness assumption, $r_{\rm crit}(T(t)) = r_{\rm crit}(T_d(t)) = r_{\rm crit}(S(t))$ for $t \geq 0$ and $\omega_0(\mathcal{T}) = \max\{\omega_{\rm crit}(\mathcal{S}), s(L)\}$.

Assuming that $A_1^{-1/2}BA_2^{-1}$ is a compact operator (in particular if $BA_2^{-\gamma}$ is compact) and $\int_{-\infty}^0 g(s)s^2 ds < \infty$, we prove the compactness of the difference $R(\lambda, L) - R(\lambda, L_d)$ for every $\lambda \in \rho(L) \cap \rho(L_d)$, where L and L_d are the generators of $\mathcal{T}$ and $\mathcal{T}_d$, respectively. Thus, [4, Theorem 2.3] leads to the compactness of $T(t) - T_d(t)$.

To illustrate this generalization, we consider the thermoviscoelastic system

$$\begin{gathered}
\ddot{w} - \mu\Delta w - (\lambda+\mu)\nabla \operatorname{div} w + \mu g_1 * \Delta w + (\lambda+\mu) g_1 * \nabla \operatorname{div} w + m\nabla u = 0 \\
\text{in } \Omega \times (0,\infty), \\
\dot{u} + \beta u - \Delta u - m \operatorname{div} \dot{w} = 0 \quad \text{in } \Omega \times (0,\infty), \\
w = 0, \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma \times (0,\infty), \\
w(x,0) = w^0(x), \quad \dot{w}(x,0) = w^1(x), \quad u(x,0) = u^0(x) \quad \text{in } \Omega, \\
w(x,0) + w(x,s) = f_0(x,s) \quad \text{in } \Omega \times (-\infty, 0)),
\end{gathered}$$

where μ, λ are positive constants. The set Ω is the Jelley Roll, a bounded open set proposed in [9],

$$\Omega = \{(x,y) \in \mathbb{R}^2 : 1/2 < r < 1\} \setminus \Gamma,$$

where Γ is the curve, in $\mathbb{R}^2$, given in polar coordinates by

$$r(\phi) = \frac{\frac{3\pi}{2} + \arctan(\phi)}{2\pi}, \quad -\infty < \phi < \infty.$$

For this system, we show that $A_1^{-1}BA_2^{-1}$, on the canonical modified energy Hilbert space, is a compact operator but the operator $BA_2^{-\gamma}$ is not compact for every $0 < \gamma < 1$.

2. Well-posedness

Let H_1 and H_2 be two Hilbert spaces. The operators $A_1 : \mathcal{D}(A_1) \subset H_1 \to H_1$ and $A_2 : \mathcal{D}(A_2) \subset H_2 \to H_2$ are self adjoint and positive (with not necessarily compact inverses), while $B : \mathcal{D}(B) \subset H_2 \to H_1$ is a closed operator with adjoint operator B^*. Throughout this paper, we assume the following:

$$\mathcal{D}(A_2^{1/2}) \hookrightarrow \mathcal{D}(B) \quad \text{and} \quad \mathcal{D}(A_1^{1/2}) \hookrightarrow \mathcal{D}(B^*), \tag{2.1}$$

$$A_2^{-1}B^*A_1^{1/2} \text{ extends to a bounded linear operator from } H_1 \text{ to } H_2. \tag{2.2}$$

Note that the operator $-A_2$ generates an analytic strongly continuous semigroup $(e^{-A_2 t})_{t\geq 0}$. Under assumption (2.1), $BA_2^{-1/2}$ is a bounded operator from H_2 to H_1 and $BA_2^{-1/2}(BA_2^{-1/2})^*$ is a bounded self adjoint non negative operator in H_1.

Setting $z(t,s) = w(t) - w(t+s)$, $s \in (-\infty, 0)$, the system (1.1)-(1.3) can be transformed into the system

$$\ddot{w}(t) + kA_1 w(t) + \int_{-\infty}^0 g(s)A_1 z(t,s)ds + Bu(t) = 0, \quad t \geq 0, \tag{2.3}$$

$$z_t - \dot{w} + z_s = 0, \tag{2.4}$$
$$z(t,0) = 0, \quad t \geq 0, \tag{2.5}$$
$$\dot{u}(t) + A_2 u(t) - B^* \dot{w}(t) = 0, \quad t \geq 0, \tag{2.6}$$
$$w(0) = w^0, \quad \dot{w}(0) = w^1, \quad u(0) = u^0, \quad z(0,s) = f_0(s), \quad s \in (-\infty, 0), \tag{2.7}$$

with $k = 1 - \int_{-\infty}^0 g(s)ds$. Set the Hilbert space

$$\mathbb{H} = \mathcal{D}(A_1^{1/2}) \times H_1 \times L^2(g, (-\infty,0), \mathcal{D}(A_1^{1/2})) \times H_2$$

endowed with the norm

$$\|(w,v,z,u)\| = \Big(k\|A_1^{1/2} w\|_{H_1}^2 + \|v\|_{H_1}^2 + \|z\|^2_{L^2(g,(-\infty,0),\mathcal{D}(A_1^{1/2}))} + \|u\|_{H_2}^2 \Big)^{1/2}.$$

Here the space $L^2(g, (-\infty,0), \mathcal{D}(A_1^{1/2}))$ consists of $\mathcal{D}(A_1^{1/2})$-valued functions z on $(-\infty, 0)$ endowed with the norm

$$\|z\|^2_{L^2(g,(-\infty,0),\mathcal{D}(A_1^{1/2}))} = \int_{-\infty}^0 g(s)\|A_1^{1/2} z(s)\|^2 ds.$$

System (2.3)-(2.7) can also be written as a first order system

$$\dot{w} = v, \tag{2.8}$$
$$\dot{v} = -kA_1 w - \int_{-\infty}^0 g(s) A_1 z(t,s) ds - Bu, \tag{2.9}$$
$$\dot{z} = v - z_s \tag{2.10}$$
$$\dot{u} = -A_2 u + B^* v, \tag{2.11}$$
$$(w(0), v(0), u(0), z(0)) = (w^0, w^1, f_0, u^0). \tag{2.12}$$

We associate with the system (2.8)-(2.11) the operator

$$L(w,v,z,u) = \Big(v, -kA_1 w - \int_{-\infty}^0 g(s)A_1 z(t,s)ds - Bu, v - z_s, -A_2 u + B^* v \Big),$$
$$\mathcal{D}(L) = \{(w,v,z,u) \in \mathbb{H} : v \in \mathcal{D}(A_1^{1/2}), u \in \mathcal{D}(C), kw + \int_{-\infty}^0 g(s)z(s)ds \in \mathcal{D}(A),$$
$$z \in H^1(g, (-\infty,0), \mathcal{D}(A_1^{1/2})), z(0) = 0\},$$

where $H^1(g, (-\infty,0), \mathcal{D}(A_1^{1/2}))$ is the set

$$\{z \in L^2(g, (-\infty,0), \mathcal{D}(A_1^{1/2})) : z_s \in L^2(g, (-\infty,0), \mathcal{D}(A_1^{1/2}))\}.$$

The decoupled system (1.7)-(1.9) can also be transformed into

$$\dot{\bar{w}} = \bar{v}, \tag{2.13}$$
$$\dot{\bar{v}} = -kA_1 \bar{w} - \int_{-\infty}^0 g(s)A_1 \bar{z}(t,s)ds - BA_2^{-1}B^*\bar{v}, \tag{2.14}$$
$$\dot{\bar{z}} = \bar{v} - \bar{z}_s, \tag{2.15}$$
$$\dot{\bar{u}} = -A_2 \bar{u} + B^* \bar{v}, \tag{2.16}$$
$$(\bar{w}(0), \bar{v}(0, \bar{z}(0), \bar{u}(0)) = (w^0, w^1, f_0, u^0), \tag{2.17}$$

to which we associate the operator

$$L_d(\bar{w},\bar{v},\bar{z},\bar{u}) = (\bar{v}, -kA_1\bar{w} - \int_{-\infty}^{0} g(s)A_1\bar{z}(t,s)ds - BA_2^{-1}B^*\bar{v}, \bar{v}-\bar{z}_s, -A_2\bar{u}+B^*\bar{v}),$$

with $\mathcal{D}(L_d) = \mathcal{D}(L)$. Also, the decoupled second order equation (1.7) can be written as a first order system

$$\dot{\bar{w}} = \bar{v}, \tag{2.18}$$

$$\dot{\bar{v}} = -kA_1\bar{w} - \int_{-\infty}^{0} g(s)A_1\bar{z}(t,s)ds - BA_2^{-1}B^*\bar{v}, \tag{2.19}$$

$$\dot{\bar{z}} = \bar{v} - \bar{z}_s, \tag{2.20}$$

$$(\bar{w}(0), \bar{v}(0), \bar{z}(0)) = (w^0, w^1, f_0), \tag{2.21}$$

with generating operator defined on $\mathcal{H} := \mathcal{D}(A_1^{1/2}) \times H_1 \times L^2(g, (-\infty, 0), \mathcal{D}(A_1^{1/2}))$ by

$$\begin{gathered} M(\bar{w},\bar{v},\bar{z}) = (\bar{v}, -kA_1\bar{w} - \int_{-\infty}^{0} g(s)A_1\bar{z}(t,s)ds - BA_2^{-1}B^*\bar{v}, \bar{v} - \bar{z}_s), \\ \mathcal{D}(M) = \big\{(\bar{w},\bar{v},\bar{z}) \in \mathcal{H} : \bar{v} \in \mathcal{D}(A_1^{1/2}), k\bar{w} + \int_{-\infty}^{0} g(s)\bar{z}(s)ds \in \mathcal{D}(A), \\ \bar{z} \in H^1(g, (-\infty, 0), \mathcal{D}(A_1^{1/2})), \bar{z}(0) = 0\big\}. \end{gathered}$$

Remark 2.1. L, L_d and M are respectively the parts in $\mathbb{H}$ and $\mathcal{H}$ of the matrix operators

$$L_{-1} = \begin{pmatrix} 0 & I & 0 & 0 \\ -k(A_1)_{-1} & 0 & -G_{-1} & -B \\ 0 & I & -\frac{d}{ds} & 0 \\ 0 & B^* & 0 & -(A_2)_{-1} \end{pmatrix},$$

$$(L_d)_{-1} = \begin{pmatrix} 0 & I & 0 & 0 \\ -k(A_1)_{-1} & -BA_2^{-1/2}(BA_2^{-1/2})^* & -G_{-1} & 0 \\ 0 & I & -\frac{d}{ds} & 0 \\ 0 & B^* & 0 & -(A_2)_{-1} \end{pmatrix},$$

$$M = \begin{pmatrix} 0 & I & 0 \\ -k(A_1)_{-1} & -BA_2^{-1/2}(BA_2^{-1/2})^* & -G_{-1} \\ 0 & I & -\frac{d}{ds} \end{pmatrix},$$

where $G_{-1}z = (A_1)_{-1}^{1/2} \int_{-\infty}^{0} g(s)A_1^{1/2}z(s)ds$.

Using Lumer Phillips theorem, the following result can be proved analogously as in [5, Theorem 2.1].

Theorem 2.2. *The operators L, L_d and M generate contraction strongly continuous semigroups $(T(t))_{t\geq 0}$, $(T_d(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ on $\mathbb{H}, \mathbb{H}$ and $\mathcal{H}$ respectively.*

3. Norm continuity of the difference between the semigroups

To show the norm continuity of the difference between the two semigroups, we recall the following technical lemma.

Lemma 3.1 ([1]). *The map $t \mapsto A_2^\alpha e^{-A_2 t}$ is norm continuous from $(0,\infty)$ to $\mathcal{L}(H_2)$ for every $0 \le \alpha < 1$.*

Then we have the following result.

Theorem 3.2. *The map $t \mapsto T(t) - T_d(t)$ is norm continuous from $(0,\infty)$ to $\mathcal{L}(\mathbb{H})$.*

Proof. Let $t > 0$ and $x^0 = (w^0, v^0, f_0, u^0) \in \mathcal{D}(L)$ such that $\|x^0\| \le 1$.

$$
\begin{aligned}
&T(t)(w^0, v^0, f_0, u^0) - T_d(t)(w^0, v^0, f_0, u^0)\\
&= \begin{pmatrix} w(t) - \bar{w}(t) \\ v(t) - \bar{v}(t) \\ z(t) - \bar{z}(t) \\ u(t) - \bar{u}(t) \end{pmatrix} = \int_0^t T(t-s) \begin{pmatrix} 0 \\ BA_2^{-1}B^*\bar{v}(s) - B\bar{u}(s) \\ 0 \\ 0 \end{pmatrix} ds.
\end{aligned}
$$

Let $0 < h < 1$. Setting $F(s) := B\bar{u}(s) - BA_2^{-1}B^*\bar{v}(s)$, we check that $\|F(s+h) - F(s)\| \to 0$ as $h \to 0$ uniformly in x^0. To this end, we have

$$
\begin{aligned}
F(s) &= Be^{-A_2 s}u^0 + B\int_0^s e^{-A_2(s-\sigma)}B^*\bar{v}(\sigma)d\sigma - BA_2^{-1}B^*\bar{v}(s)\\
&= Be^{-A_2 s}u^0 + BA_2^{-1}\int_0^s A_2 e^{-A_2(s-\sigma)}B^*\bar{v}(\sigma)d\sigma - BA_2^{-1}B^*\bar{v}(s).
\end{aligned}
$$

Using an integration by parts,

$$
\begin{aligned}
F(s) &= Be^{-A_2 s}u^0 + [BA_2^{-1}e^{-A_2(s-\sigma)}B^*\bar{v}(\sigma)]_0^s - BA_2^{-1}\int_0^s e^{-A_2(s-\sigma)}B^*\bar{v}'(\sigma)d\sigma\\
&\quad - BA_2^{-1}B^*\bar{v}(s)\\
&= Be^{-A_2 s}u^0 - BA_2^{-1}e^{-A_2 s}B^*v^0 - BA_2^{-1}\int_0^s e^{-A_2(s-\sigma)}B^*\bar{v}'(\sigma)d\sigma\\
&= Be^{-A_2 s}u^0 - BA_2^{-1/2}e^{-A_2 s}A_2^{-1/2}B^*v^0 + kBA_2^{-1}\int_0^s e^{-A_2(s-\sigma)}B^*A_1\bar{w}(\sigma)d\sigma\\
&\quad + BA_2^{-1}\int_0^s e^{-A_2(s-\sigma)}B^*[-\int_{-\infty}^0 g(\tau)A\bar{z}(s,\tau)d\tau + BA_2^{-1}B^*\bar{v}(\sigma)]d\sigma\\
&= BA_2^{-1/2}A_2^{1/2}e^{-A_2 s}u^0 - BA_2^{-1/2}e^{-A_2 s}A_2^{-1/2}B^*v^0\\
&\quad + kBA_2^{-1/2}\int_0^s A_2^{1/2}e^{-A_2(s-\sigma)}A_2^{-1}B^*A_1^{1/2}A_1^{1/2}\bar{w}(\sigma)d\sigma\\
&\quad + BA_2^{-1/2}\int_0^s e^{-A_2(s-\sigma)}A_2^{-1/2}B^*BA_2^{-1}B^*\bar{v}(\sigma)d\sigma\\
&\quad - BA_2^{-1}\int_0^s e^{-A_2(s-\sigma)}B^*\int_{-\infty}^0 g(\tau)A\bar{z}(s,\tau)d\tau d\sigma\\
&= BA_2^{-1/2}A_2^{1/2}e^{-A_2 s}u^0 - BA_2^{-1/2}e^{-A_2 s}A_2^{-1/2}B^*v^0\\
&\quad + kBA_2^{-1/2}\int_0^s A_2^{1/2}e^{-A_2(s-\sigma)}A_2^{-1}B^*A_1^{1/2}A_1^{1/2}\bar{w}(\sigma)d\sigma\\
&\quad + BA_2^{-1/2}\int_0^s e^{-A_2(s-\sigma)}A_2^{-1/2}B^*BA_2^{-1}B^*\bar{v}(\sigma)d\sigma\\
&\quad - BA_2^{-1/2}\int_0^s A_2^{1/2}e^{-A_2(s-\sigma)}A_2^{-1}B^*A_1^{1/2}\int_{-\infty}^0 g(\tau)A_1^{1/2}\bar{z}(s,\tau)d\tau d\sigma.
\end{aligned}
$$

Since $BA_2^{-1/2}$ is a bounded operator and $s \mapsto e^{-A_2 s}$, $s \mapsto A_2^{1/2} e^{-A_2 s}$ are norm continuous from $(0, \infty)$ to $\mathcal{L}(H_2)$, then the mappings $s \mapsto BA_2^{-1/2} A_2^{1/2} e^{-A_2 s}$ and $s \mapsto BA_2^{-1/2} e^{-A_2 s} A_2^{-1/2} B^*$ are norm continuous from $(0, \infty)$ to $\mathcal{L}(H_1)$.

Under the assumption (2.2) we have $A_2^{-1} B^* A_1^{1/2}$ is a bounded operator ; so there exists a constant $\alpha(s)$ such that, $\|A_2^{-1} B^* A_1^{1/2} A_1^{1/2} \bar{w}(\sigma)\| \leq \alpha(s) \|x^0\|$, for every $\sigma \in [0, s]$. Thus, $s \mapsto \int_0^s A_2^{1/2} e^{-A_2(s-\sigma)} A_2^{-1} B^* A_1^{1/2} A_1^{1/2} \bar{w}(\sigma) d\sigma$ is continuous in $(0, \infty)$ uniformly in $\|x^0\| \leq 1$.
Since $A_2^{-1/2} B^* B A_2^{-1} B^*$ is a bounded operator, using the same argument

$$s \mapsto \int_0^s e^{-A_2(s-\sigma)} A_2^{-1/2} B^* B A_2^{-1} B^* \bar{v}(\sigma) d\sigma$$

is continuous in $(0, \infty)$ uniformly in $\|x^0\| \leq 1$. In the other hand

$$\| \int_{-\infty}^0 g(\tau) A_1^{1/2} \bar{z}(s, \tau) d\tau d\sigma \| \leq \Big(\int_{-\infty}^0 g(\tau) d\tau \Big)^{1/2} \Big(\int_{-\infty}^0 g(\tau) \|A_1^{1/2} \bar{z}(s, \tau)\|^2 d\tau \Big) 1/2.$$

So $A_2^{-1} B^* A_1^{1/2} \int_{-\infty}^0 g(\tau) A_1^{1/2} \bar{z}(s, \tau) d\tau$ is a bounded operator uniformly in $\|x^0\| \leq 1$. As a consequence, $s \mapsto \int_0^s A_2^{1/2} e^{-A_2(s-\sigma)} A_2^{-1} B^* A_1^{1/2} (\int_{-\infty}^0 g(\tau) A_1^{1/2} \bar{z}(s, \tau) d\tau) d\sigma$ is norm continuous in $(0, \infty)$ uniformly in $\|x^0\| \leq 1$. Finally, $\|F(s + h) - F(s)\| \to 0$ as $h \to 0$ uniformly in x^0. We have

$$\begin{aligned}
&\begin{pmatrix} w(t) - \bar{w}(t) \\ v(t) - \bar{v}(t) \\ z(t) - \bar{z}(t) \\ u(t) - \bar{u}(t) \end{pmatrix} \\
&= \int_0^t T(t - s) \begin{pmatrix} 0 \\ BA_2^{-1} B^* \bar{v}(s) - B\bar{u}(s) \\ 0 \\ 0 \end{pmatrix} ds \\
&= \int_0^{t+h} T(t + h - s) \begin{pmatrix} 0 \\ F(s) \\ 0 \\ 0 \end{pmatrix} ds - \int_0^t T(t - s) \begin{pmatrix} 0 \\ F(s) \\ 0 \\ 0 \end{pmatrix} ds \\
&= \int_0^{t+h} T(s) \begin{pmatrix} 0 \\ F(t + h - s) \\ 0 \\ 0 \end{pmatrix} ds - \int_0^t T(s) \begin{pmatrix} 0 \\ F(t - s) \\ 0 \\ 0 \end{pmatrix} ds \\
&= \int_0^t T(s) \begin{pmatrix} 0 \\ F(t + h - s) - F(t - s) \\ 0 \\ 0 \end{pmatrix} ds + \int_t^{t+h} T(t + h - s) \begin{pmatrix} 0 \\ F(s) \\ 0 \\ 0 \end{pmatrix} ds \\
&= \int_0^t T(s) \begin{pmatrix} 0 \\ F(t + h - s) - F(t - s) \\ 0 \\ 0 \end{pmatrix} ds + \int_0^h T(s) \begin{pmatrix} 0 \\ F(t + s) \\ 0 \\ 0 \end{pmatrix} ds.
\end{aligned}$$

Hence,

$$
\begin{aligned}
&\| \begin{pmatrix} w(t+h)-\bar{w}(t+h)\\ v(t+h)-\bar{v}(t+h)\\ z(t+h)-\bar{z}(t+h)\\ u(t+h)-\bar{u}(t+h)\end{pmatrix} - \begin{pmatrix} w(t)-\bar{w}(t)\\ v(t)-\bar{v}(t)\\ z(t)-\bar{z}(t)\\ u(t)-\bar{u}(t)\end{pmatrix} \| \\
&\le \| \int_0^t T(s) \begin{pmatrix} 0\\ F(t+h-s)-F(t-s)\\ 0\\ 0\end{pmatrix} ds\| + \|\int_0^h T(s)\begin{pmatrix}0\\ F(t+s)\\ 0\\ 0\end{pmatrix} ds\| \\
&\le \sup_{\tau\in[0,t]} \|T(\tau)\| \int_0^t \|F(t+h-s)-F(t-s)\|ds + \int_0^h \|T(s)\|\|F(t+s)\|ds.
\end{aligned}
$$

In addition, there exists a constant N such that $\sup_{s\in[0,t+1]}\|F(s)\|\le N$ uniformly in x^0, and thus

$$
\begin{aligned}
&\| \begin{pmatrix} w(t+h)-\bar{w}(t+h)\\ v(t+h)-\bar{v}(t+h)\\ z(t+h)-\bar{z}(t+h)\\ u(t+h)-\bar{u}(t+h)\end{pmatrix} - \begin{pmatrix} w(t)-\bar{w}(t)\\ v(t)-\bar{v}(t)\\ z(t)-\bar{z}(t)\\ u(t)-\bar{u}(t)\end{pmatrix} \| \\
&\le \sup_{\tau\in[0,t+1]} \|T(\tau)\| \int_0^t \|F(t+h-s)-F(t-s)\|ds + c(t)Nh.
\end{aligned}
$$

As $\|F(s+h)-F(s)\|\to 0$ as $h\to 0$ uniformly in x_0, we conclude that

$$
\int_0^t \|F(t+h-s)-F(t-s)\|ds \to 0, h\to 0
$$

uniformly for $x_0\in\mathcal{D}(L)$ verifying $\|x_0\|\le 1$. This achieves the proof. □

Theorem 3.3. *The maps $t\mapsto T_d(t)-S(t)$ and $t\mapsto T(t)-S(t)$ are norm continuous from $(0,\infty)$ to $\mathcal{L}(\mathbb{H})$.*

Proof. Let $x_0=(u^0,v^0,f_0,w^0)\in\mathbb{H}$ such that $\|x_0\|\le 1$ and $t>0$.

$$
\begin{aligned}
&T_d(t)x_0-(S(t)((u^0,v^0,f_0),0)\\
&=(0,0,0,e^{-A_2t}w^0+\int_0^t e^{-A_2(t-s)}B^*\pi_2 S(s)((u^0,v^0,f_0)ds)
\end{aligned}
$$

where $\pi_2:\mathcal{D}(A_1^{1/2})\times H_1\times L^2(g,(-\infty,0),\mathcal{D}(A_1^{1/2}))\to H_1$, $(u,v,z)\mapsto v$. Set $\Delta(t)=T_d(t)x_0-(S(t)(u^0,v^0,f_0),0)$. For $h>0$, one has

$$
\begin{aligned}
\Delta(t+h)-\Delta(t) = \Big(&0,0,e^{-A_2(t+h)}w^0-e^{-A_2t}w^0\\
&+\int_0^{t+h} e^{-A_2(t+h-s)}B^*\pi_2S(s)(u^0,v^0,f_0)ds\\
&-\int_0^t e^{-A_2(t-s)}B^*\pi_2S(s)(u^0,v^0,f_0)ds\Big).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\|\Delta(t+h)-\Delta(t)\|\\
&=\|e^{-A_2(t+h)}w^0-e^{-A_2t}w^0+\int_0^{t+h} e^{-A_2(t+h-s)}B^*\pi_2S(s)(u^0,v^0,f_0)ds
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \int_0^t e^{-A_2(t-s)} B^* \pi_2 S(s)(u^0, v^0, f_0) ds)\| \\
&= \|e^{-A_2(t+h)} w^0 - e^{-A_2 t} w^0 + \int_0^t [e^{-A_2(t+h-s)} - e^{-A_2(t-s)}] B^* \pi_2 S(s)(u^0, v^0, f_0) ds \\
&\quad + \int_t^{t+h} e^{-A_2(t+h-s)} B^* \pi_2 S(s)(u^0, v^0, f_0) ds)\| \\
&\le \|e^{-A_2(t+h)} - e^{-A_2 t}\| \|w^0\| \\
&\quad + \| \int_0^t [e^{-A_2(t+h-s)} - e^{-A_2(t-s)}] B^* \pi_2 S(s)(u^0, v^0, f_0) ds\| \\
&\quad + \| \int_t^{t+h} e^{-A_2(t+h-s)} B^* \pi_2 S(s)(u^0, v^0, f_0) ds)\|.
\end{aligned}
$$

Hence, since $A_2^{-1/2} B^*$ is a bounded operator, we have

$$
\begin{aligned}
&\|\Delta(t+h) - \Delta(t)\| \\
&\le \int_0^t \|[A_2^{1/2} e^{-A_2(t+h-s)} - A_2^{1/2} e^{-A_2(t-s)}] A_2^{-1/2} B^* \pi_2 S(s)(u^0, v^0, f_0)\| ds \\
&\quad + \int_t^{t+h} \|A_2^{1/2} e^{-A_2(t+h-s)} A_2^{-1/2} B^* \pi_2 S(s)(u^0, v^0, f_0))\| ds + \|e^{-A_2(t+h)} - e^{-A_2 t}\| \\
&\le \int_0^t \|[A_2^{1/2} e^{-A_2(t+h-s)} - A_2^{1/2} e^{-A_2(t-s)}] A_2^{-1/2} B^* \pi_2 S(s)(u^0, v^0, f_0)\| ds \\
&\quad + \int_t^{t+h} \|A_2^{1/2} e^{-A_2(t+h-s)} A_2^{-1/2} B^* \pi_2 S(s)(u^0, v^0, f_0))\| ds + \|e^{-A_2(t+h)} - e^{-A_2 t}\|.
\end{aligned}
$$

Since the semigroup $(e^{-A_2 t})_{t \ge 0}$ is analytic, it is immediately norm continuous. Thus $\|e^{-A_2(t+h)} - e^{-A_2 t}\| \to 0$ as $h \to 0$. In the other hand $A_2^{-1/2} B^* = (B A_2^{-1/2})^*$ is a bounded operator, thus there exists a constant $\delta(t)$ such that

$$
\begin{gathered}
\|A_2^{-1/2} B^* \pi_2 S(s)(u^0, v^0, f_0)\| \le \delta(t) \|A_2^{-1/2} B^*\| \|(u^0, v^0, f_0)\| \quad \text{for every } s \in [0, t] \\
\|A_2^{-1/2} B^* \pi_2 S(s)(u^0, v^0, f_0)\| \le \delta(t) \|A_2^{-1/2} B^*\| \|x_0\| \quad \text{for every } s \in [0, t] \\
\|A_2^{-1/2} B^* \pi_2 S(s)(u^0, v^0, f_0)\| \le \delta(t) \|A_2^{-1/2} B^*\| \quad \text{for every } s \in [0, t].
\end{gathered}
$$

Moreover

$$
\begin{aligned}
&\| \int_0^t [A_2^{1/2} e^{-A_2(t+h-s)} - A_2^{1/2} e^{-A_2(t-s)}] A_2^{-1/2} B^* \pi_2 S(s)(u^0, v^0, f_0) ds\| \\
&\le \delta(t) \|A_2^{-1/2} B^*\| \int_0^t \|A_2^{1/2} e^{-A_2(t+h-s)} - A_2^{1/2} e^{-A_2(t-s)}\| ds.
\end{aligned}
$$

It follows from Lemma 3.1 and Lebegue theorem that

$$
\|A_2^{1/2} e^{-A_2(t+h-s)} - A_2^{1/2} e^{-A_2(t-s)}\| \to 0
$$

as $h \to 0$ and $t > s$, and

$$
\| \int_0^t [A_2^{1/2} e^{-A_2(t+h-s)} - A_2^{1/2} e^{-A_2(t-s)}] A_2^{-1/2} B^* \pi_2 S(s)(u^0, v^0, f_0) ds\| \to 0
$$

as $h \to 0$ uniformly in x_0. For the third term, we have

$$\begin{aligned}
&\| \int_t^{t+h} A_2^{1/2} e^{-A_2(t+h-s)} A_2^{-1/2} B^* \pi_2 S(s)(u^0, v^0, f_0) ds) \| \\
&= \| \int_0^h A_2^{1/2} e^{-A_2 s} A_2^{-1/2} B^* \pi_2 S(s)(u^0, v^0, f_0) ds \|.
\end{aligned}$$

Using the similar argument as in the second term, there exists $\beta(t)$ such that

$$\begin{aligned}
\| \int_0^h A_2^{1/2} e^{-A_2 s} A_2^{-1/2} B^* \pi_2 S(s)(u^0, v^0, f_0) ds \| &\le \beta(t) \int_0^h \|A_2^{1/2} e^{-A_2 s}\| ds \|x_0\| \\
&\le \|x_0\| \beta(t) \int_0^h s^{-1/2} ds \\
&\le 2\beta(t)\sqrt{h}
\end{aligned}$$

(here we used $\|A_2^{1/2} e^{-A_2 t}\| = O(t^{-1/2})$ as $t > 0$; see for example [6, Theorem 1.4.3]). Consequently, $\| \int_t^{t+h} A_2^{1/2} e^{-A_2(t+h-s)} A_2^{-1/2} B^* \pi_2 S(s)(u^0, v^0, f_0) ds) \| \to 0$ as $h \to 0$ uniformly in x_0. Finally, $\|\Delta(t+h) - \Delta(t)\| \to 0$ as $h \to 0$ uniformly in x_0. Since, by Theorem 3.2, $t \mapsto T(t) - T_d(t)$ is norm continuous on $(0, \infty)$, $t \mapsto T(t) - S(t)$ is norm continuous. $\square$

Theorem 3.3 leads to the following result.

Corollary 3.4. *$r_{\rm crit}(T(t)) = r_{\rm crit}(T_d(t)) = r_{\rm crit}(S(t))$ for $t \ge 0$ and $\omega_0(\mathcal{T}) = \max\{\omega_{\rm crit}(\mathcal{S}_w), s(L)\}$.*

4. Compactness of the difference between the two semigroups

We have also this main result.

Theorem 4.1. *Assume $A_1^{-1/2} B A_2^{-1}$ is compact in $\mathcal{L}(H_2, H_1)$ and $\int_{-\infty}^0 g(s) s^2 ds < \infty$. Then $R(\lambda, L) - R(\lambda, L_d)$ is compact on $\mathbb{H}$ for every $\lambda \in \rho(L) \cap \rho(L_d)$.*

Proof. We have

$$R(\lambda, L_d) - R(\lambda, L) = LR(\lambda, L)[L^{-1} - L_d^{-1}] L_d R(\lambda, L_d). \tag{4.1}$$

Let $(\varphi, \psi, \eta, \xi) \in \mathbb{H} = \mathcal{D}(A_1^{1/2}) \times H_1 \times L^2(g, (-\infty, 0), \mathcal{D}(A_1^{1/2})) \times H_2$. We look for $(w, v, z, u) \in D(L)$ such that $L(w, v, z, u) = (\varphi, \psi, \eta, \xi)$. Note that the equation $L(w, v, z, u) = (\varphi, \psi, \eta, \xi)$ is safistied is equivalent to the system

$$\begin{gathered}
v = \varphi \\
-kA_1 w - \int_{-\infty}^0 g(s) A_1 z(t, s) ds - Bu = \psi v - z_s = \eta \\
-A_2 u + B^* v = \xi
\end{gathered}$$

which is equivalent to the system

$$\begin{gathered}
v = \varphi \\
-kA_1 w - \int_{-\infty}^0 g(s) A_1 z(t, s) ds - Bu = \psi \\
z_s = \varphi - \eta \\
-A_2 u + B^* v = \xi
\end{gathered}$$

which is equivalent to the system

$$
\begin{gathered}
-kA_1w - \int_{-\infty}^{0} g(s)A_1z(t,s)ds - Bu = \psi \\
v = \varphi \\
z = s\varphi - \int_0^s \eta(\tau)d\tau \\
u = A_2^{-1}B^*\varphi - A_2^{-1}\xi.
\end{gathered}
$$

By assumption, $\int_{-\infty}^{0} g(s)s^2ds < \infty$, and since $\varphi \in \mathcal{D}(A_1^{1/2})$, we have $s\varphi \in L^2(g,(-\infty,0),\mathcal{D}(A_1^{1/2}))$. Using Hölder theorem,

$$
\forall s \in (-\infty,0), \quad \Big(\int_s^0 \|A_1 1/2\eta(\tau)\|d\tau\Big)^2 \le -s\int_s^0 \|A_1 1/2\eta(\tau)\|^2 d\tau.
$$

We can assume that η has compact support in $(0,\infty)$, and then

$$
-\int_{-\infty}^{0} g(s)s\int_s^0 \|A_1 1/2\eta(\tau)\|^2 d\tau ds < \infty.
$$

Thus $z \in L^2(g,(-\infty,0),\mathcal{D}(A_1^{1/2}))$. Note that $L(w,v,z,u) = (\varphi,\psi,\eta,\xi)$ is equivalent to the system

$$
\begin{gathered}
w = -k^{-1}A_1^{-1/2}[(\int_{-\infty}^{0} g(s)sds)A_1^{1/2}\varphi + \int_{-\infty}^{0} g(s)A_1^{1/2}\int_s^0 \eta(\tau)d\tau ds] \\
-(kA_1)^{-1}BA_2^{-1}B^*\varphi + (kA_1)^{-1}BA_2^{-1}\xi - (kA_1)^{-1}\psi \\
v = \varphi \\
z = s\varphi - \int_0^s \eta(\tau)d\tau \\
u = A_2^{-1}B^*\varphi - A_2^{-1}\xi
\end{gathered}
$$

which is equivalent to the system

$$
\begin{gathered}
w = -(k)^{-1}(\int_{-\infty}^{0} g(s)sds)\varphi - (kA_1)^{-1/2}\int_{-\infty}^{0} g(s)A_1^{1/2}\int_s^0 \eta(\tau)d\tau ds \\
-(kA_1)^{-1}BA_2^{-1}B^*\varphi + (kA_1)^{-1}BA_2^{-1}\xi - (kA_1)^{-1}\psi \\
v = \varphi \\
z = s\varphi - \int_0^s \eta(\tau)d\tau \\
u = A_2^{-1}B^*\varphi - A_2^{-1}\xi.
\end{gathered}
$$

Replacing Bu by $BA_2^{-1}B^*\bar{v}$ and repeating the above procedure for L, we can prove that the equation $L_d(\bar{w},\bar{v},\bar{z},\bar{u}) = (\varphi,\psi,\eta,\xi)$ is equivalent to the system

$$
\begin{gathered}
\bar{v} = \varphi \\
-kA_1\bar{w} - \int_{-\infty}^{0} g(s)A_1\bar{z}(t,s)ds - BA_2^{-1}B^*\bar{v} = \psi \\
\bar{z}_s = \varphi - \eta \\
-A_2\bar{u} + B^*\bar{v} = \xi
\end{gathered}
$$

which is equivalent to the system

$$
\begin{gathered}
\bar{w} = (-k)^{-1} A_1^{-1/2}\Big[\Big(\int_{-\infty}^0 g(s)s ds\Big) A_1^{1/2}\varphi \\
- \int_{-\infty}^0 g(s) A_1^{1/2} \int_0^s \eta(\tau) d\tau ds + BA_2^{-1}B^*\varphi + \psi\Big] \\
\bar{v} = \varphi \\
\bar{z} = s\varphi - \int_0^s \eta(\tau) d\tau \\
\bar{u} = A_2^{-1}B^*\varphi - A_2^{-1}\xi
\end{gathered}
$$

which is equivalent to the system

$$
\begin{gathered}
\bar{w} = -k^{-1}(\int_{-\infty}^0 g(s)s ds)\varphi + (kA_1)^{-1/2}\int_{-\infty}^0 g(s) A_1^{1/2}\int_0^s \eta(\tau) d\tau ds \\
- (kA_1)^{-1} BA_2^{-1}B^*\varphi - (kA_1)^{-1}\psi \\
\bar{v} = \varphi \\
\bar{z} = s\varphi - \int_0^s \eta(\tau) d\tau \\
\bar{u} = A_2^{-1}B^*\varphi - A_2^{-1}\xi.
\end{gathered}
$$

Therefore, by an easy computation one obtains

$$
L^{-1} - L_d^{-1} = \begin{pmatrix} 0 & 0 & (kA_1)^{-1}BA_2^{-1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

which is a compact operator by assumption. The claim follows from the equality (4.1). $\square$

Now by the Theorem 4.1, Theorem 3.2 and [4, Theorem 3.2], we obtain the main result of this section.

Theorem 4.2. *Assume that $A_1^{-1/2}BA_2^{-1}$ is compact in $\mathcal{L}(H_2, H_1)$,* (1.4)-(1.6) *and* (2.2) *hold. Then $T(t) - T_d(t)$ is compact on $\mathbb{H}$ for all $t \geq 0$.*

As a consequence of this theorem, we have the following result.

Corollary 4.3. $r_{\rm ess}(T(t)) = r_{\rm ess}(T_d(t))$ *for* $t \geq 0$ *and* $\omega_0(\mathcal{T}) = \max\{\omega_{\rm ess}(\mathcal{T}_d), s(L)\}$.

Remark 4.4. The result of Theorem 4.2 has been shown in [5] directly, assuming the compactness of the operator $BA_2^{-\gamma}$ for some $0 < \gamma < 1$. It is clear that this last assumption implies that $A_1^{-1/2}BA_2^{-1}$ is compact from H_2 to H_1.

5. Application

We consider the following model for a linear viscoelastic body Ω of Boltzmann type with thermal damping

$$
\begin{gathered}
\ddot{w} - \mu\Delta w - (\lambda+\mu)\nabla \operatorname{div} w + \mu g_1 * \Delta w + (\lambda+\mu) g_1 * \nabla \operatorname{div} w + m\nabla u = 0 \\
\text{in } \Omega \times (0,\infty),
\end{gathered} \tag{5.1}
$$

$$
\dot{u} + \beta u - \Delta u - m \operatorname{div} \dot{w} = 0 \quad \text{in } \Omega \times (0,\infty), \tag{5.2}
$$

$$w = 0, \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma \times (0, \infty), \tag{5.3}$$

$$w(x,0) = w^0(x), \quad \dot{w}(x,0) = w^1(x), \quad u(x,0) = u^0(x) \quad \text{in } \Omega, \tag{5.4}$$

$$w(x,0) + w(x,s) = f_0(x,s) \quad \text{in } \Omega \times (-\infty, 0)), \tag{5.5}$$

where $\lambda, \mu > 0$ the Lame's constants and $m > 0$ is the thermal strain parameter, β is a positive constant and g is a given function which satisfies the following conditions

- (C1) $g_1 \in \mathcal{C}^1[0,\infty) \cap L^1(0,\infty)$.
- (C2) $g_1(t) \geq 0$ and $g_1'(t) \leq 0$ for $t > 0$,
- (C3) $\int_0^\infty g_1(s)ds < 1$.

The set Ω is the bounded open Jelly Roll set defined in [9],

$$\Omega = \{(x,y) \in \mathbb{R}^2 : \frac{1}{2} < r < 1\} \setminus \Gamma,$$

where Γ is the curve in $\mathcal{R}^2$ given in polar coordinates by

$$r(\phi) = \frac{\frac{3\pi}{2} + \arctan(\phi)}{2\pi}, \quad -\infty < \phi < \infty.$$

Note that

$$\begin{aligned}
&\int_{-\infty}^{t} g_1(t-s)\Delta w(x,s)ds \\
&= \int_{-\infty}^{0} g_1(-s)\Delta w(x,t+s)ds \\
&= \int_{-\infty}^{0} g_1(-s)\Delta(w(x,t+s) - w(x,t))ds + \int_{-\infty}^{0} g_1(-s)\Delta w(x,t)ds \\
&= -\int_{-\infty}^{0} g_1(-s)\Delta z(x,t,s)ds + \int_{-\infty}^{0} g_1(-s)ds\Delta w(x,t).
\end{aligned}$$

A similar expression can be establish for $g_1 * \nabla div\, w$. In order to fit the system (5.1)-(5.5) into the setting abstract system (1.1)-(1.3), we take

$$H_1 = L_2(\Omega, \mathbb{R}^2), \quad H_2 = L_2(\Omega, \mathbb{R}),$$
$$z(x,t,s) = w(x,t) - w(x,t+s), \quad g(s) = g_1(-s),$$

and define the operators A_1, A_2, B by

$$A_1 w = -\mu\Delta_D w - (\lambda+\mu)\nabla\, divw, \quad \mathcal{D}(A_1) = \mathcal{D}(\Delta_D) = H_0^1(\Omega, \mathbb{R}^2),$$
$$A_2 u = (\beta I - \Delta_N)u, \quad \mathcal{D}(A_2) = \mathcal{D}(\Delta_N) = H^1(\Omega, \mathbb{R}),$$
$$Bu = m\nabla u, \quad \mathcal{D}(B) = H^1(\Omega, \mathbb{R}).$$

Here, as the domain Ω is irregular the Dirichlet Laplacian Δ_D and the Neumann Laplacian Δ_N are defined via quadratic forms. More precisely, Δ_D is the unique positive self adjoint operator associated to the closed quadratic form on $H_0^1(\Omega)$

$$\langle \Delta f, g\rangle = \int_\Omega \nabla f \nabla g\, dx,$$

and Δ_N is the unique non negative self adjoint operator associated to the closed quadratic form on $H^1(\Omega)$

$$\langle \Delta f, g\rangle = \int_\Omega \nabla f \nabla g\, dx.$$

It is clear that the adjoint B^* of B is

$$B^* w = -m\, div\, w, \quad \mathcal{D}(B^*) = \{w \in H^1(\Omega, \mathbb{R}^2) : w \cdot \overrightarrow{n} = 0 \text{ in } \partial\Omega\},$$

where $\overrightarrow{n}$ is the outward unit normal vector on the boundary $\partial\Omega$. Note that

$$\mathcal{D}(A_2^{1/2}) = \mathcal{D}(B) \quad \text{and} \quad \mathcal{D}(A_1^{1/2}) \hookrightarrow \mathcal{D}(B^*).$$

We have the following facts.

(i) A_2^{-1} is not compact on H_2, see [9], but A_1 has a compact resolvent on H_1. Consequently, $A_1^{-1/2}$ and $A_1^{-1/2} B A_2^{-1}$ are compact .

(ii) For every $\gamma \in (0, 1]$, $BA_2^{-\gamma}$ is not compact from H_2 into H_1. In fact, it is enough to show that BA_2^{-1} is not compact from H_2 to H_1. For this, we have

$$A_2^{-1} B^* B A_2^{-1} = m^2 A_2^{-1}(-\Delta_N) A_2^{-1} = m^2 A_2^{-1}(A_2 - \beta I) A_2^{-1} = m^2 (A_2^{-1} - \beta A_2^{-2}).$$

Using the spectral mapping theorem, we have $\sigma(A_2^{-1} B^* B A_2^{-1}) = m^2(\sigma(A_2)^{-1} - \beta\sigma(A_2)^{-2})$. As in [9], $(\beta, \infty) \subset \sigma(A_2)$, so $A_2^{-1} B^* B A_2^{-1}$ is not compact on H_2. Consequently BA_2^{-1} is not compact from H_2 to H_1.

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