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WEIGHTED EIGENVALUE PROBLEMS FOR THE p-LAPLACIAN WITH WEIGHTS IN WEAK LEBESGUE SPACES

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ABSTRACT. We consider the nonlinear eigenvalue problem
$$-\Delta_p u = \lambda g|u|^{p-2}u, \quad u \in \mathcal{D}_0^{1,p}(\Omega)$$
where Δ_p is the p-Laplacian operator, Ω is a connected domain in $\mathbb{R}^N$ with $N > p$ and the weight function g is locally integrable. We obtain the existence of a unique positive principal eigenvalue for g such that g^+ lies in certain subspace of weak-$L^{N/p}(\Omega)$. The radial symmetry of the first eigenfunctions are obtained for radial g, when Ω is a ball centered at the origin or $\mathbb{R}^N$. The existence of an infinite set of eigenvalues is proved using the Ljusternik-Schnirelmann theory on $\mathcal{C}^1$ manifolds.

1. Introduction

For given $N \geq 2$, $1 < p < N$, Ω a non-empty open connected subset of $\mathbb{R}^N$ and $g \in L^1_{\rm loc}$, we discuss the sufficient conditions on g for the existence of positive solutions for the nonlinear eigenvalue problem
$$\begin{gathered} -\Delta_p u = \lambda g|u|^{p-2}u \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{gathered} \tag{1.1}$$
for a suitable value of the parameter λ, where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator.

For $p = 2$, the 2-Laplacian is the usual Laplace operator. For $p \neq 2$ the p-Laplace operator arises in various contexts, for example, in the study of non-Newtonian fluids like dilatant fluids ($p < 2$) and pseudo plastic ($p \geq 2$), torsional creep problem ($p \geq 2$), glaciology ($p \in (1, 4/3]$) etc. The exponent appearing in $\lambda g|u|^{p-2}u$ makes (1.1) to be a natural generalization of the linear weighted eigenvalue problem for the Laplacian.

Here, we look for the weak solutions of (1.1) in the space $\mathcal{D}_0^{1,p}(\Omega)$, which is the completion of $\mathcal{C}_c^\infty(\Omega)$ with respect to the norm
$$\|\nabla u\|_p := \Big(\int_\Omega |\nabla u|^p\Big)^{1/p}.$$

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By an eigenvalue of (1.1) we mean $\lambda \in \mathbb{R}$ such that, (1.1) admits a non-zero weak solution in $\mathcal{D}_0^{1,p}(\Omega)$; i.e., there exists $u \in \mathcal{D}_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v = \lambda \int_\Omega g\,|u|^{p-2}u\,v, \quad \forall\, v \in \mathcal{D}_0^{1,p}(\Omega). \tag{1.2}$$

In this case, we say that u is an eigenfunction associated of the eigenvalue λ. If one of the eigenfunctions corresponding to λ is of constant sign, then we say that λ is a principal eigenvalue. If all the eigenfunctions corresponding to λ are unique up to constant multiples then we say that λ is simple.

In the classical linear case; i.e, when $p = 2, g \equiv 1$ and Ω is a bounded domain, it is well known that (1.1) admits a unique positive principle eigenvalue and it is simple. Furthermore, the set of all eigenvalues can be arranged into a sequence

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots \to +\infty$$

and the corresponding normalized eigenfunctions form an orthonormal basis for the Sobolev space $H_0^1(\Omega)$. Using the Courant-Weinstein variational principle [13, Theorem 6.3.14] the eigenvalues can be expressed as

$$\lambda_k = \inf_{u \perp \{u_1, \dots, u_{k-1}\},\, \|u\|_2 = 1} \int_\Omega |\nabla u|^2, \quad k = 1, 2, \dots \tag{1.3}$$

Lindqvist [28] proved existence, uniqueness and simplicity of a principal eigenvalue for $p > 1$, when $g \equiv 1$ and the domain Ω bounded. Later, Azorero and Alonso [7] identified infinitely many eigenvalues of (1.1), for $p \neq 2$, using the Ljusternik-Schnirelmann type minmax theorem.

Many authors have given sufficient conditions on g for the existence of a positive principal eigenvalue for (1.1), when $\Omega = \mathbb{R}^N$, for example Brown et. al. [10] and Allegretto [2] for $p = 2$, Huang [9], Allegretto and Huang [3] for the respective generalization to $p \neq 2$. Fleckinger et al. [15], studied the problem (1.1) for general p. All these earlier results assume that either g or g^+ should be in $L^{N/p}(\mathbb{R}^N)$. In [24], Willem and Szulkin enlarged the class of weight functions beyond the Lebesgue space $L^{N/p}(\mathbb{R}^N)$. They obtained the existence of positive principal eigenvalue, even for the weights whose positive part has a faster decay than $1/|x|^p$ at infinity and at all the points in the domain (see (3.6)).

For $p = 2$, there are some results available for the weights in Lorentz spaces, for example, Visciglia in [31] looked at (1.1) in the context of generalized Hardy-Sobolev inequality for the positive weights in certain Lorentz spaces. Following this direction, Mythily and Marcello in [23] showed the existence of a unique positive principal eigenvalue for (1.1), when g is in certain Lorentz spaces. Anoop, Lucia and Ramaswamy [6] unified the sufficient conditions given in [2, 10, 23, 24] by showing the existence of a positive principal eigenvalue for (1.1), when g^+ lies in a suitable subspace of weak-$L^{\frac{N}{2}}(\Omega)$. In this paper we obtain an analogous result that unify the sufficient conditions given in [3, 9, 15, 24] for the existence of a positive eigenvalue for (1.1) by considering weights in a suitable subspace of the weak- $L^{N/p}(\Omega)$.

For $p = 2$, the existence of a positive principal eigenvalue for more general positive weights is obtained in [26] using certain capacity conditions of Maz'ja [22] and in [30] using the concentration compactness lemma. However, their eigenfunctions are only a distributional solutions of (1.2) and the first eigenvalue lacks certain qualitative properties. Indeed, here we obtain a unique positive principal eigenvalue and an infinite set of eigenvalues for (1.1) for the weights in a suitable subspace of the Lorentz space $L(\frac{N}{p},\infty)$.

Here we fix the solution space as $\mathcal{D}_0^{1,p}(\Omega)$, which fits very well with the weak formulation of boundary value problems in the unbounded domains. Furthermore, when $1<p<N$, the space $\mathcal{D}_0^{1,p}(\Omega)$ is continuously embedded in the Lebesgue space $L^{p^*}(\Omega)$, where $p^*=\frac{Np}{N-p}$. However, when $p\geq N$, for a general unbounded domain Ω, the space $\mathcal{D}_0^{1,p}(\Omega)$ is not continuously embedded in $L^1_{\rm loc}(\Omega)$ (see [29, Remark 2.2]). The main novelty of our results rely on the embedding of the space $\mathcal{D}_0^{1,p}(\Omega)$ in the Lorentz space $L(p^*,p)$, see [5].

We use a direct variational method for the existence of an eigenvalue. For that we consider the following Rayleigh quotient

$$R(u):=\frac{\int_\Omega|\nabla u|^p}{\int_\Omega g|u|^p} \tag{1.4}$$

with the domain of definition

$$\mathcal{D}^+(g):=\{u\in\mathcal{D}_0^{1,p}(\Omega):\int_\Omega g|u|^p>0\}. \tag{1.5}$$

Let

$$M:=\{u\in\mathcal{D}_0^{1,p}(\Omega):\int_\Omega g|u|^p=1\}, \tag{1.6}$$

$$J(u):=\frac{1}{p}\int_\Omega|\nabla u|^p \tag{1.7}$$

If R is $\mathcal{C}^1$, then we arrive at (1.1) as the Euler-Lagrange equation corresponding to the critical points of R on $\mathcal{D}^+(g)$, with the critical values as the eigenvalues of (1.1). Moreover, there is a one to one correspondence between the critical points of R over $\mathcal{D}^+(g)$ and the critical points of J over M. Thus we look for the sufficient conditions on g^+ for the existence of a critical points of J on M. As in [6], here we consider the space

$$\mathcal{F}_{N/p}:=\text{closure of } \mathcal{C}_c^\infty(\Omega) \text{ in } L(N/p,\infty)$$

Now we state one of our main results.

Theorem 1.1. *Let Ω be an open connected subset of $\mathbb{R}^N$ with $p\in(1,N)$. Let $g\in L^1_{\rm loc}(\Omega)$ be such that $g^+\in\mathcal{F}_{N/p}\setminus\{0\}$. Then*

$$\lambda_1=\inf\{J(u):u\in M\} \tag{1.8}$$

is the unique positive principal eigenvalue of (1.1). *Furthermore, all the eigenfunctions corresponding to λ_1 are of the constant sign and λ_1 is simple.*

Note that g^- is only locally integrable and hence the map G defined as

$$G(u)=\int_\Omega g|u|^p$$

may not even be continuous and hence M may not even be closed in $\mathcal{D}_0^{1,p}(\Omega)$. Nevertheless, we show that the weak limit of a minimizing sequence of J on M lies in M.

In general the eigenfunctions are only in $W^{1,p}_{\rm loc}(\Omega)$ and hence the classical tools for proving the qualitative properties of λ_1 are not applicable, as they require more regularity for the eigenfunctions. However, Kawohl, Lucia and Prashanth [18] developed a weaker version of strong maximum principle for quasilinear operator analogous to the result in [11].

Further, we discuss the sufficient conditions on g for the radial symmetry of the eigenfunctions corresponding λ_1, when Ω is a ball centered at origin or $\mathbb{R}^N$. This generalizes the result of Bhattacharya [8], who proved the radial symmetry of the first eigenfunctions of (1.1), when Ω is a ball centered at origin and $g \equiv 1$.

Theorem 1.2. *Let Ω be a ball centered at origin or $\mathbb{R}^N$. Let g be nonnegative, radial and radially decreasing measurable function. If λ_1 is an eigenvalue of* (1.1)*, then any positive eigenfunction corresponding to λ_1 is radial and radially decreasing.*

A sufficient condition on g, for the existence of infinitely many eigenvalues of (1.1) is also discussed here. Let us point out that a complete description of the set of all eigenvalues of p-Laplacian is widely open for $p \neq 2$. The question of discreteness, countability of the set of all eigenvalues of p-Laplacian is not known, even in the simplest case: $g \equiv 1$ and Ω is a ball. However there are several methods that exhibit infinite number of eigenvalues goes to infinity. For $p \neq 2$, the existence of infinitely many eigenvalues is obtained in [3, 9, 24], using the Ljusternik-Schnirelmann minimax theorem. In this direction we have the following result under certain weaker assumptions on g^+.

Theorem 1.3. *Let Ω be an open connected subset of $\mathbb{R}^N$ with $p \in (1, N)$. Let $g \in L^1_{\rm loc}(\Omega)$ be such that $g^+ \in \mathcal{F}_{N/p} \setminus \{0\}$. Then* (1.1) *admits a sequence of positive eigenvalues going to ∞.*

The classical Ljusternik-Schnirelmann minimax theorem requires a deformation homotopy that is available when M is at least a $\mathcal{C}^{1,1}$ manifold(i.e, transition maps are $\mathcal{C}^1$ and its derivative is locally Lipschitz). The set M that we are considering here is $\mathcal{C}^1$ but generally not $\mathcal{C}^{1,1}$. Szulkin [27] developed the Ljusternik-Schnirelmann theorem on $\mathcal{C}^1$ manifold using the Ekeland variational principle. We use Szulkin's result to obtain an increasing sequence of positive eigenvalues of (1.1) that going to infinity.

This paper is organized as follows. In Section 2, we recall certain basic properties of the symmetric rearrangement of a function and the Lorentz spaces. Section 3 deals with several characterizations of the spaces $\mathcal{F}_d, d > 1$. The examples of functions belonging to $\mathcal{F}_{N/p}$ are also given in Section 3. In Section 4, we present a proof of the existence and other qualitative properties of the first eigenvalue like, simplicity, uniqueness. The radial symmetry of the eigenfunctions corresponding to λ_1 is discussed in Section 4. In section 5, we discuss the Ljusternik-Schirelmann theory on $\mathcal{C}^1$ Banach manifold and give a proof for the existence of infinitely many eigenvalues of (1.1). Further extensions and the applications of weighted eigenvalue problems for the p-Laplacian are indicated in Section 6.

2. Prerequisites

2.1. **Symmetrization.** First, we recall the definition of the symmetrization of a function and its properties. Then we state certain rearrangement inequalities needed for the subsequent sections, for more details on symmetrization we refer to [20, 19, 14].

Let Ω be a domain in $\mathbb{R}^N$. Given a measurable function f on Ω, we define distribution function α_f and decreasing rearrangement f^* of f as below

$$\alpha_f(s) := \big|\{x \in \Omega : |f(x)| > s\}\big|, \quad f^*(t) := \inf\{s > 0 : \alpha_f(s) \le t\}. \tag{2.1}$$

In the following proposition we summarize some useful properties of distribution and rearrangements.

Porposition 2.1. *Let Ω be a domain and f be a measurable function on Ω. Then*

(i) *α_f, f^* are nonnegative, decreasing and right continuous.*
(ii) *$f^*(\alpha_f(s_0)) \leq s_0$, $\alpha_f(f^*(t_0)) \leq t_0$;*
(iii) *$f^*(t) \leq s$ if and only if $\alpha_f(s) \leq t$,*
(iv) *f and f^* are equimeasurable; i.e, $\alpha_f(s) = \alpha_{f^*}(s)$ for all $s > 0$.*
(v) *Let $c, s, t > 0$ such that $c = st^{1/p}$. Then*

$$t^{1/p} f^*(t) \leq c \quad \textit{if and only if} \quad s(\alpha_f(s))^{1/p} \leq c. \tag{2.2}$$

Proof. For a proof of (i), (ii) and (iii), see [14, Propositions 3.2.2 and 3.2.3]. Item (iv) follows from (iii) as follows

$$\alpha_{f^*}(s) = |\{t : f^*(t) > s\}| = |\{t : t < \alpha_f(s)\}| = \alpha_f(s).$$

(v) Taking $s = ct^{\frac{-1}{p}}$ in (iii) one deduces that

$$t^{1/p} f^*(t) \leq c \quad \text{if and only if} \quad \alpha_f(s) \leq t.$$

Now as $t = (c/s)^p$, we obtain

$$\alpha_f(s) \leq t \quad \text{if and only if} \quad s(\alpha_f(s))^{1/p} \leq c.$$

□

Next we define Schwarz symmetrization of measurable sets and functions, see [20] for more details.

Definition 2.2. Let $A \subset \mathbb{R}^N$ be a Borel measurable set of finite measure. We define A_*, the symmetric rearrangement of the set A, to be the open ball centered at origin having the same measure that of A. Thus

$$A_* = \{x : |x| < r\}, \quad \text{with } \omega_N r^N = |A|,$$

where ω_n is the measure of unit ball in $\mathbb{R}^N$.

Let f be a measurable function on $\Omega \subset \mathbb{R}^N$ such that $\alpha_f(s) < \infty$ for each $s > 0$. Then we define the *symmetric decreasing rearrangement* f_* of f on Ω_* as

$$f_*(x) = \int_0^\infty \chi_{\{|f|>s\}_*}(x) ds$$

Next we list a few inequalities concerning f_* that we use for proving the radial symmetry of the eigenfunctions corresponding to the first eigenvalue. For a proof see [20, Section 3.3].

Porposition 2.3. *Let Ω be a ball centered at origin or $\mathbb{R}^N$. Let f be a nonnegative measurable function on Ω such that $\alpha_f(s) < \infty$ for each $s > 0$.*

(a) *If f is radial and radially decreasing then $f = f_*$ a.e.*
(a) *Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a nonnegative Borel measurable function. Then*

$$\int_{\mathbb{R}^N} F(f_*(x)) dx = \int_{\mathbb{R}^N} F(f(x)) dx.$$

(b) *If $\Phi : \mathbb{R}^+ \to \mathbb{R}$ is nonnegative and nondecreasing then*

$$(\Phi \circ f)_* = \Phi \circ f_* \quad \textit{a.e.}$$

2.2. **Lorentz Spaces.** In this section, we recall the definition and the main properties of the Lorentz spaces. For more details on Lorentz spaces see [1, 14, 16].

Given a measurable function f and $p,q \in [1,\infty]$, we set

$$\|f\|_{(p,q)} := \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{q;(0,\infty)}$$

and the Lorentz spaces are defined by $L(p,q) := \{f : \|f\|_{(p,q)} < \infty\}$. In particular for $q=\infty$, we obtain

$$\|f\|_{(p,\infty)} = \sup_{t>0} t^{1/p} f^*(t).$$

For $p>1$, the weak-L^p space is defined as

$$\text{weak-}L^p := \{f : \sup_{s>0} s(\alpha_f(s))^{1/p} < \infty\}.$$

The following lemma identifies the Lorentz space $L(p,\infty)$ with the weak-L^p space.

Lemma 2.4. *Let Ω be a domain in $\mathbb{R}^N$ and f be a measurable function on Ω. For each $p>1$, we have*

$$\sup_{t>0} t^{1/p} f^*(t) = \sup_{s>0} s(\alpha_f(s))^{1/p}.$$

Proof. Let

$$c_1 = \sup_{t>0} t^{1/p} f^*(t), \quad c_2 = \sup_{s>0} s(\alpha_f(s))^{1/p}. \tag{2.3}$$

Without loss of generality we may assume that c_1 is finite. Now for $s>0$, take $t = (\frac{c_1}{s})^p$. Thus $t^{1/p} f^*(t) \le c_1$. Now by taking $c = c_1$ in (2.2), with $c_1 = st^{\frac{1}{p}}$, one can deduce that $s(\alpha_f(s))^{1/p} \le c_1$, for all $s>0$. Hence $c_2 \le c_1$. The other way inequality follows in a similar way. $\square$

The functional $\|\cdot\|_{(p,q)}$ is not a norm on $L(p,q)$. To obtain a norm, we set $f^{**}(t) := \frac{1}{t}\int_0^t f^*(r)dr$ and define

$$\|f\|^*_{(p,q)} := \|t^{\frac{1}{p}-\frac{1}{q}} f^{**}(t)\|_{q\,;(0,\infty)}, \quad \text{for } 1 \le p,q \le \infty.$$

For $p>1$, the functional $\|\cdot\|^*_{(p,q)}$ defines a norm in $L(p,q)$ equivalent to $\|.\|_{(p,q)}$ (see [14, Lemma 3.4.6]). Endowed with this norm $L(p,q)$ is a Banach space, for $p,q \ge 1$.

In the following proposition we summarize some of the properties of $L(p,q)$ spaces, see [14, 16] for the proofs.

Porposition 2.5.
- (i) *If $p>0$ and $q_2 \ge q_1 \ge 1$, then $L(p,q_1) \hookrightarrow L(p,q_2)$*
- (ii) *If $p_2 > p_1 \ge 1$ and q_1, $q_2 \ge 1$, then $L(p_2,q_2) \hookrightarrow L_{\rm loc}(p_1,q_1)$.*
- (iii) *Hölder inequality: Given $(f,g) \in L(p_1,q_1) \times L(p_2,q_2)$ and $(p,q) \in (1,\infty) \times [1,\infty]$ such that $1/p = 1/p_1 + 1/p_2$, $1/q \le 1/q_1 + 1/q_2$, then*
$$\|fg\|_{(p,q)} \le C\|f\|_{(p_1,q_1)} \ \|g\|_{(p_2,q_2)}, \tag{2.4}$$
where C depends only on p.
- (iv) *Let $(p,q) \in (1,\infty)\times(1,\infty)$. Then the dual space of $L(p,q)$ is isomorphic to $L(p',q')$ where $1/p+1/p'=1$ and $1/q+1/q'=1$.*
- (v) *Let $\gamma>0$. Then*
$$\big\||f|^\gamma\big\|_{(p,q)} = \|f\|^\gamma_{(\frac{p}{\gamma},\frac{q}{\gamma})} \tag{2.5}$$

As mentioned before the main interest of considering the Lorentz spaces is that the usual Sobolev embedding, the embedding of $\mathcal{D}^{1,p}_0(\Omega)$ in to $L^{p^*}(\Omega)$, can be improved as below (see for example, appendix in [5]):

Porposition 2.6 (Lorentz-Sobolev embedding). *We have $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L(p^*, p)$; i.e., there exists $C > 0$ such that*

$$\|u\|_{(p^*,\,p)} \le C\|\nabla u\|_p, \quad \forall\, u \in \mathcal{D}_0^{1,p}(\Omega).$$

3. The function space $\mathcal{F}_d$

For $(d, q) \in [1, \infty) \times [1, \infty)$, $C_c^\infty(\Omega)$ is dense in the Banach space $L(d, q)$. However, the closure of $C_c^\infty(\Omega)$ in $L(d, \infty)$ is a closed proper sub space of $L(d, \infty)$ that will henceforth be denoted by

$$\mathcal{F}_d := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{(d,\infty)}} \subset L(d, \infty).$$

Next we list some of the properties of the space $\mathcal{F}_d$, see [6, Proposition 3.1] for a proof.

Porposition 3.1. (i) *For each $d > 1$, $L(d, q) \subset \mathcal{F}_d$ when $1 \le q < \infty$.*
(ii) *For each $a \in \Omega$, the Hardy potential $x \mapsto |x - a|^{\frac{-N}{d}}$ does not belong to $\mathcal{F}_d$.*

Recall that $L(d, d) = L^d(\Omega)$, hence from (i) it follows that $L^{N/p}(\Omega)$ is contained in $F_{N/p}$. Thus Theorem 1.1 readily extends the results in [3, 15], since $g \in L^{N/p}(\Omega)$ is a part of their assumptions. Similarly the result in [9] follows as the positive part of weights he considered is bounded and compactly supported. Note that (ii) shows that $\mathcal{F}_d$ is a proper subspace of the Lorentz space $L(d, \infty)$.

Now we state a few useful characterizations of the space $\mathcal{F}_d$.

Porposition 3.2. *The following statements are equivalent*
(i) *$f \in \mathcal{F}_d$,*
(ii) *$f^*(t) = o(t^{-1/d})$ at 0 and ∞; i.e.,*

$$\lim_{t\to 0_+} t^{1/d} f^*(t) = 0 = \lim_{t\to\infty} t^{1/d} f^*(t). \tag{3.1}$$

(iii) *$\alpha_f(s) = o(s^{-d})$ at 0 and ∞; i.e.,*

$$\lim_{s\to 0_+} s(\alpha_f(s))^{1/d} = 0 = \lim_{s\to\infty} s(\alpha_f(s))^{1/d}. \tag{3.2}$$

Proof. (i)⇒(ii): See the first part of [6, Theorem 3.3].
(ii)⇒(iii): Let (ii) hold. Thus for given $\varepsilon > 0$, there exist $t_1, t_2 > 0$ such that

$$t^{1/d} f^*(t) < \varepsilon, \quad \forall\, t \in (0, t_1) \cup (t_2, \infty). \tag{3.3}$$

Let $s_1 = \varepsilon\,(t_1)^{-1/d}$ and $s_2 = \varepsilon\,(t_2)^{-1/d}$. Note that

$$\text{If } s \in (0, s_2) \cup (s_1, \infty), \text{ then } t = (\frac{\varepsilon}{s})^d \in (0, t_1) \cup (t_2, \infty).$$

Now using (3.3) and (2.2) with $c = \varepsilon$, we obtain

$$s(\alpha_f(s))^{1/d} < \varepsilon, \quad \forall s \in (0, s_2) \cup (s_1, \infty).$$

This shows that $\alpha_f(s) = o(s^{-d})$ at 0 and ∞.
(iii)⇒ (i): Assume (iii). Then for a given $\varepsilon > 0$, there exist s_1, s_2 such that

$$s(\alpha_f(s))^{1/d} < \varepsilon, \quad \forall s \in (0, s_1] \cup [s_2, \infty). \tag{3.4}$$

We use [6, Proposition 3.2] to show that f is in $\mathcal{F}_d$. Let

$$A_\varepsilon := \{x : s_1 \le f(x) < s_2\}, \quad f_\varepsilon := f\chi_{A_\varepsilon}.$$

Note that $|A_\varepsilon| \le \alpha_f(s_1) < \infty$ and $f_\varepsilon \in L^\infty(\Omega)$. Let $g = f\chi_{A^c_\varepsilon}$. Thus it is enough to prove

$$\|f - f_\varepsilon\|_{(d,\infty)} = \|g\|_{(d,\infty)} < \varepsilon.$$

Observe that, for $s \in (s_1, s_2)$, $\alpha_g(s) = \alpha_f(s_2)$ and hence

$$s(\alpha_g(s))^{1/d} < s_2(\alpha_f(s_2))^{1/d} < \varepsilon, \ \forall s \in (s_1, s_2). \tag{3.5}$$

Since $|g| \le |f|$, we have $\alpha_g(s) \le \alpha_f(s)$, for all $s > 0$. Now by combining (3.4) and (3.5) we obtain

$$s(\alpha_g(s))^{1/d} < \varepsilon, \quad \forall s > 0.$$

Hence by lemma 2.4 we obtain $\|g\|_{(d,\infty)} < \varepsilon$. $\square$

Next we give another sufficient condition similar to a condition of Rozenblum, see [26, (2.19)], for a function to be in $\mathcal{F}_d$.

Lemma 3.3. *Let $h \in L(d,\infty)$ and $h > 0$. If f is such that $\int_\Omega h^{d-q}|f|^q < \infty$ for some $q \ge d$. Then $f \in L(d,q)$ and hence in $\mathcal{F}_d$.*

Proof. The result is obvious when $q = d$. For $q > d$, let $g = h^{\frac{d}{q}-1}f$. Then the above integrability condition yields $g \in L^q(\Omega)$. Using property (2.5) we obtain $h^{1-\frac{d}{q}} \in L(\frac{dq}{q-d}, \infty)$. Now by Hölder inequality (2.4) we obtain $f \in L(d,q)$ and hence in $\mathcal{F}_d$ as $L(d,q) \subset \mathcal{F}_d$. $\square$

Remark 3.4. Let $g \in L^q(\mathbb{R}^N)$ with $q \ge d$ and let

$$f(x) = |x|^{(\frac{1}{q}-\frac{1}{d})N} g.$$

Then using the above lemma one can easily verify that $f \in L(d,q)$. In general for any $h \in L(d,\infty)$ with $h > 0$, $f = gh^{1-\frac{d}{q}} \in L(d,q)$. Thus we can obtain Lorentz spaces by interpolating Lebesgue and weak-Lebesgue spaces suitably.

Another class of functions contained in $\mathcal{F}_{N/p}$ is provided by the work of Szulkin and Willem [24]. More specifically they consider the weights g defined by the conditions:

$$\begin{gathered} g \in L^1_{\rm loc}(\Omega), \quad g^+ = g_1 + g_2 \not\equiv 0, \quad g_1 \in L^{N/p}(\Omega), \\ \lim_{|x|\to\infty,\, x\in\Omega} |x|^p g_2(x) = 0, \qquad \lim_{x\to a,\, x\in\overline{\Omega}} |x-a|^p g_2(x) = 0 \quad \forall a \in \overline{\Omega}. \end{gathered} \tag{3.6}$$

The following lemma can be proved using similar arguments as in [6, Lemma 4.1].

Lemma 3.5. *Let $g : \Omega \to \mathbb{R}$ be a measurable function such that*

$$(i) \lim_{|x|\to\infty,\, x\in\Omega} |x|^p g(x) = 0, \quad (ii) \lim_{x\to a,\, x\in\overline{\Omega}} |x-a|^p g(x) = 0, \quad \forall a \in \overline{\Omega}. \tag{3.7}$$

Then there exist finite number of points $a_1, \dots, a_m \in \overline{\Omega}$ with the following property: For every $\varepsilon > 0$ there exists $R := R(\varepsilon) > 0$ such that

$$|g(x)| < \frac{\varepsilon}{|x|^p} \quad a.e.\ x \in \Omega \setminus B(0,R) \tag{3.8}$$

$$|g(x)| < \frac{\varepsilon}{|x-a_i|^p} \quad a.e.\ x \in \Omega \cap B(a_i, R^{-1})\,,\ i = 1, \dots, m, \tag{3.9}$$

$$g \in L^\infty(\Omega \setminus A_\varepsilon), \tag{3.10}$$

where $A_\varepsilon := \bigcup_{i=1}^m B(a_i, R^{-1}) \cap \Omega$.

Theorem 3.6. *Let $g : \Omega \to \mathbb{R}$ be as in the previous lemma. Then $g \in \mathcal{F}_{N/p}$.*

Proof. We use Proposition 3.2(iii) to show that $g \in \mathcal{F}_{N/p}$. For $\varepsilon > 0$, let R be given as in the previous lemma. Let $s_1 := \varepsilon R^{-p}$. We first show that

$$s(\alpha_g(s))^{p/N} < \varepsilon, \quad \forall s < s_1.$$

Using (3.8), for each $s \in (0, s_1)$, we have

$$B(0,R) \subset B(0,(\frac{\varepsilon}{s})^{1/p}) \quad |g(x)| < s, \quad \forall x \in \Omega \setminus B(0,(\frac{\varepsilon}{s})^{1/p}). \tag{3.11}$$

Therefore, for each $s \in (0, s_1)$, the distribution function $\alpha_g(s)$ can be estimated as follows:

$$\alpha_g(s) = \big|\{x \in \Omega \cap B(0,(\frac{\varepsilon}{s})^{1/p}) \colon |f(x)| > s\}\big| \le \omega_N (\frac{\varepsilon}{s})^{N/p},$$

where ω_N is the volume of unit ball in $\mathbb{R}^N$. Thus

$$s(\alpha_g(s))^{p/N} < C_1\varepsilon, \quad \forall\, s < s_1. \tag{3.12}$$

where the constant C_1 is independent of ε.

Next we consider the set $A_\varepsilon = \bigcup_{i=1}^m B(a_i, R^{-1}) \cap \Omega$ and let $s_2 := \|g\|_{L^\infty(\Omega\setminus A_\varepsilon)}$. For $s > s_2$, using (3.9) the distribution function can be estimated as follows:

$$\begin{aligned}
\alpha_g(s) &= \big|\{x \in \Omega : |g(x)| > s\}\big| = \big|\{x \in A_\varepsilon : |g(x)| > s\}\big| \\
&\le \sum_{i=1}^m \big|\{x \in B(a_i, R^{-1}) \cap \Omega : |g(x)| > s\}\big| \\
&\le \sum_{i=1}^m \big|\{x \in B(a_i, R^{-1}) : \varepsilon|x-a_i|^{-p} > s\}\big| \\
&= \sum_{i=1}^m \omega_N (\frac{\varepsilon}{s})^{N/p}.
\end{aligned}$$

Therefore,

$$s(\alpha_g(s))^{\frac{p}{N}} \le C_2\varepsilon \quad \forall s > s_2, \tag{3.13}$$

where C_2 is independent of ε. Now proof follows using condition (iii) of proposition 3.2 together with (3.12) and (3.13). $\square$

As an immediate consequence we have the following remark.

Remark 3.7. The positive part of any function satisfying (3.6) belongs to the space $\mathcal{F}_{N/p}$. In particular Theorem 1.1 summarizes the result by Willem and Szulkin [24].

3.1. **Examples.** Now we consider examples of weights that admit a positive principal eigenvalue for (1.1) to understand how the conditions (3.6) and the properties that define the space $\mathcal{F}_{N/p}$ are related to one another. First, we consider the following functions:

$$g_1(x) = \frac{1}{\big(\log(2+|x|^2)\big)^{p/N}(1+|x|^2)^{p/2}}, \tag{3.14}$$

$$g_2(x) = \frac{1}{|x|^p(1+|x|^2)^{p/2}\big(\log(2+\frac{1}{|x|^2})\big)^{p/N}}. \tag{3.15}$$

One can verify that g_1, g_2 satisfy (3.6) and hence belong to $\mathcal{F}_{N/p}$ and none of them lies in $L^{N/p}(\mathbb{R}^N)$.

Next we give an example of a weight which is in $\mathcal{F}_{N/p}$ but does not satisfy the condition (3.6).

Example 3.8. In the cube $\Omega = \{(x_1, \dots, x_N) \in \mathbb{R}^N : |x_i| < R\}$ with $0 < R < 1$ consider the function defined by

$$g_3(x) = \big|x_1 \log(|x_1|)\big|^{-p/N}, \quad x_1 \neq 0. \tag{3.16}$$

Using the condition (3.3), one can verify that $g_3 \in L(\frac{N}{p}, q)$, for $q > \frac{N}{p}$. But g_3 does not satisfy (3.6). Indeed along the curve $x_2 = (x_1)^{\frac{1}{2N}}$, the limit of $|x|^p g_3(x)$ is infinity as x tends to 0 and this limit is zero as x tends to 0 along the x_1 axis. Thus g_3 does not satisfy the condition (3.6).

4. Existence of an eigenvalue and its properties

In this section we prove the existence and the uniqueness of the positive principal eigenvalue for (1.1) for g for which $g^+ \in \mathcal{F}_{N/p} \setminus \{0\}$. Moreover we prove a few qualitative properties of that positive principal eigenvalue.

4.1. **The existence of a minimizer.** We prove the existence using a direct variational principle. First, we recall the following sets and functional:

$$\mathcal{D}^+(g) = \{u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega g|u|^p > 0\}, \quad M = \{u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega g|u|^p = 1\},$$
$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p, \quad G(u) = \frac{1}{p}\int_\Omega g|u|^p.$$

From the definition of the space $\mathcal{D}_0^{1,p}(\Omega)$, it is obvious that J is coercive and weakly lower semi-continuous. Due to the weak assumption on g^-, the map G may not be even continuous. However the map

$$G^+(u) := \frac{1}{p}\int_\Omega g^+|u|^p$$

is continuous and compact on $\mathcal{D}_0^{1,p}(\Omega)$.

Lemma 4.1. *Let $g^+ \in F_{N/p} \setminus \{0\}$. Then G^+ is compact.*

Proof. Let $\{u_n\}$ converge weakly to u in X. We show that $G^+(u_n) \to G^+(u)$, up to a subsequence. For $\phi \in \mathcal{C}_c^\infty(\Omega)$, we have

$$p(G^+(u_n) - G^+(u)) = \int_\Omega \phi\,(|u_n|^p - |u|^p) + \int_\Omega (g^+ - \phi)\,(|u_n|^p - |u|^p). \tag{4.1}$$

We estimate the second integral using the Lorentz-Sobolev embedding and the Hölder inequality as below

$$\int_\Omega |(g^+ - \phi)|\,\big|(|u_n|^p - |u|^p)\big| \le C\|g^+ - \phi\|_{(N/p,\infty)}\big(\|u_n\|^p_{(p^*,p)} + \|u\|^p_{(p^*,p)}\big) \tag{4.2}$$

where C is a constant which depends only on N, p. Clearly $\{u_n\}$ is a bounded sequence in $L(p^*, p)$. Let

$$m := \sup_n \{\|u_n\|^p_{(p^*,p)} + \|u\|^p_{(p^*,p)}\}.$$

Now using the definition of the space $F_{N/p}$, for a given $\varepsilon > 0$, we choose $g_\varepsilon \in \mathcal{C}_c^\infty(\Omega)$ so that

$$\|g^+ - g_\varepsilon\|_{(N/p,\infty)} < \frac{p\,\varepsilon}{2mC}.$$

Thus by taking $\phi = g_\varepsilon$ in (4.2) we obtain

$$\int_\Omega |(g^+ - g_\varepsilon)| \left|(|u_n|^p - |u|^p)\right| < \frac{p\,\varepsilon}{2}$$

Since $X \hookrightarrow L^p_{\rm loc}(\Omega)$ compactly, the first integral in (4.1) can be made arbitrary small for large n. Thus we choose $n_0 \in \mathbb{N}$ so that

$$\int_\Omega g_\varepsilon(|u_n|^p - |u|^p) < \frac{p\varepsilon}{2}, \quad \forall n > n_0.$$

Hence $|G^+(u_n) - G^+(u)| < \varepsilon$, for $n > n_0$. $\square$

Now we are in a position to prove the existence of a minimizer for J on M.

Theorem 4.2. *Let Ω be a domain in $\mathbb{R}^N$ with $N > p$. Let $g \in L^1_{\rm loc}(\Omega)$ and $g^+ \in \mathcal{F}_{N/p} \setminus \{0\}$. Then J admits a minimizer on M.*

Proof. Since $g \in L^1_{\rm loc}(\Omega)$ and $g^+ \neq 0$, there exists $\varphi \in \mathcal{C}^\infty_c(\Omega)$ such that $\int_\Omega g|\varphi|^p > 0$ (see for example, [18, Proposition 4.2]) and hence $M \neq \emptyset$. Let $\{u_n\}$ be a minimizing sequence of J on M; i.e.,

$$\lim_{n\to\infty} J(u_n) = \lambda_1 := \inf_{u\in M} J(u).$$

By the coercivity of $J, \{u_n\}$ is bounded in $\mathcal{D}^{1,p}_0(\Omega)$ and hence using the reflexivity of $\mathcal{D}^{1,p}_0(\Omega)$ we obtain a subsequence of $\{u_n\}$ that converges weakly. We denote the weak limit by u and the subsequence by $\{u_n\}$ itself. Now using the compactness of G^+, we obtain

$$\lim_{n\to\infty} \int_\Omega g^+|u_n|^p = \int_\Omega g^+|u|^p.$$

Now as $u_n \in M$ we write,

$$\int_\Omega g^-|u_n|^p = \int_\Omega g^+|u_n|^p - 1$$

Since the embedding $\mathcal{D}^{1,p}_0(\Omega) \hookrightarrow L^p_{\rm loc}(\Omega)$ is compact, up to a subsequence $u_n \to u$ a.e. in Ω. Hence by applying Fatou's lemma,

$$\int_\Omega g^-|u|^p \le \int_\Omega g^+|u|^p - 1,$$

which shows that $\int_\Omega g|u|^p \ge 1$. Setting $\widetilde{u} := u/(\int_\Omega g|u|^p)^{1/p}$, the weak lower semi continuity of J yields

$$\lambda_1 \le J(\widetilde{u}) = \frac{J(u)}{\int_\Omega g|u|^p} \le J(u) \le \liminf_n J(u_n) = \lambda_1$$

Thus the equality must hold at each step and hence $\int_\Omega g|u|^p = 1$, which shows that $u \in M$ and $J(u) = \lambda_1$. $\square$

Note that R is not sufficiently regular to conclude that u is an eigenfunction of (1.2) corresponding to λ_1, using critical point theory.

Porposition 4.3. *Let u be a minimizer of R on $\mathcal{D}^+(g)$. Then u is an eigenfunction of* (1.1)

Proof. For each $\phi \in \mathcal{C}_c^\infty(\Omega)$, using dominated convergence theorem one can verify that R admits directional derivative along ϕ. Now since u is a minimizer of J on $\mathcal{D}^+(g)$ we obtain

$$\frac{d}{dt} R(u+t\phi)|_{t=0} = 0.$$

Therefore,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \phi = \lambda_1 \int_\Omega g\,|u|^{p-2} u\,\phi, \quad \forall\,\phi \in \mathcal{C}_c^\infty(\Omega).$$

Now we use the density of $\mathcal{C}_c^\infty(\Omega)$ in $\mathcal{D}_0^{1,p}(\Omega)$ to conclude that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v = \lambda_1 \int_\Omega g\,|u|^{p-2} u\,v, \quad \forall\,v \in \mathcal{D}_0^{1,p}(\Omega).$$

□

4.2. **Qualitative properties of λ_1.** First we prove that the eigenfunctions corresponding to λ_1 are of constant sign. Since the eigenfunctions are not regular enough, the classical strong maximum principle is not applicable here. In [6], for $p = 2$, we use a strong maximum principle due to Brezis and Ponce [11] to show that first eigenfunctions are of constant sign. A similar strong maximum principle is obtained in [18], for quasilinear operators. From [18, Proposition 3.2] we have the following lemma.

Lemma 4.4 (Strong Maximum principle for Δ_p)**.** *Let $u \in \mathcal{D}_0^{1,p}(\Omega)$, $V \in L^1_{\rm loc}(\Omega)$ be such that $u, V \geq 0$ a.e in Ω. If $V|u|^{p-1} \in L^1_{\rm loc}(\Omega)$ and u satisfies the following differential inequality(in the sense of the distributions)*

$$-\Delta_p(u) + V(x)u^{p-1} \geq 0 \quad \textit{in } \Omega,$$

then either $u \equiv 0$ or $u > 0$ a.e.

Now using the above lemma we prove the following result.

Lemma 4.5. *The eigenfunctions of* (1.1) *corresponding to λ_1 are of constant sign.*

Proof. It is clear that the eigenfunctions corresponding to λ_1 are the minimizers of R_p on $\mathcal{D}_p^+(g)$. Let u be a minimizer of R_p on $\mathcal{D}_p^+(g)$. Since $u \neq 0$ either u^+ or u^- is non zero. Without loss of generality we may assume that $u^+ \neq 0$. Now by taking u^+ as a test function in (1.2), we see that u^+ also minimizes R_p on $\mathcal{D}_p^+(g)$. Thus by Proposition 4.3, u^+ also solves (1.1) in the weak sense,

$$-\Delta_p u^+ - \lambda_1 g(u^+)^{p-1} = 0, \quad \text{in } \Omega.$$

In particular, we have the following differential inequality in the sense of distributions:

$$-\Delta_p u^+ + \lambda_1 g^-(u^+)^{p-1} = \lambda_1 g^+(u^+)^{p-1} \geq 0, \quad \text{in } \Omega.$$

It is clear that g^- and u^+ satisfy all the assumptions of Lemma 4.4, provided $g^-(u^+)^p \in L^1_{\rm loc}(\Omega)$. Since $g|u|^p \in L^1(\Omega)$, we have $(g^-)^{1/q}(u^+)^{p-1} \in L^q(\Omega)$, where q is the conjugate exponent of p. Further, $(g^-)^{1/p} \in L^p_{\rm loc}(\Omega)$, since $g \in L^1_{\rm loc}(\Omega)$. Let us write

$$g^-(u^+)^{p-1} = (g^-)^{1/p}(g^-)^{1/q}(u^+)^{p-1}.$$

Now we use Hölder inequality to conclude that $g^-(u^+)^{p-1} \in L^1_{\rm loc}(\Omega)$. Now in view of Lemma 4.4 we obtain $u^+ > 0$ a.e. and hence $u = u^+$. Moreover, the zero set of u is of measure zero. □

Indeed, the above lemma shows that λ_1 is a principal eigenvalue of (1.1). Next we prove the uniqueness of the positive principal eigenvalue, using the Picone's identity for the p-Laplacian. In [4], Picone's identity is proved for $\mathcal{C}^1$ functions. However it is not hard to obtain a similar identity for less regular functions.

Lemma 4.6 (Picone's identity). *Let $u \geq 0, v > 0$ a.e. and let $|\nabla v|, |\nabla u|$ exist as measurable functions. Then the following identity holds a.e.*

$$\begin{aligned} &|\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla v \\ &= |\nabla u|^p - \nabla(\frac{u^p}{v^{p-1}}) \cdot |\nabla v|^{p-2}\nabla v. \end{aligned}$$

Further, the left hand side of the above identity is nonnegative.

Now we prove the uniqueness of the positive principal eigenvalue.

Lemma 4.7. *Let $g \in L(N/p, \infty)$ and let $\lambda > 0$ be a positive principal eigenvalue of (1.1). Then*

$$\lambda = \lambda_1 = \inf\{\int_\Omega |\nabla u|^p : u \in M\}.$$

Proof. Let $v \in \mathcal{D}_0^{1,p}(\Omega)$ be a positive eigenfunction of (1.1) corresponding to λ. Let $u \in M$ and let $\{\phi_n\}$ in $\mathcal{C}_c^\infty(\Omega)$ be such that $\|u - \phi_n\|_{\mathcal{D}_0^{1,p}(\Omega)} \to 0$ and $\int_\Omega g|u|^p = 1$. Note that $\frac{|\phi_n|^p}{v+\varepsilon} \in \mathcal{D}_0^{1,p}(\Omega)$. Thus by the Picone's identity (see Lemma 4.6), we have

$$0 \leq \int_\Omega |\nabla \phi_n|^p - \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\big(\frac{|\phi_n|^p}{(v+\varepsilon)^{p-1}}\big). \tag{4.3}$$

Since v is an eigenfunction of (1.1) corresponding to λ, we have

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\Big(\frac{\phi_n^p}{(v+\varepsilon)^{p-1}}\Big) = \lambda \int_\Omega g v^{p-1} \frac{|\phi_n|^p}{(v+\varepsilon)^{p-1}}. \tag{4.4}$$

Now from (4.3) and (4.4) we

$$0 \leq \int_\Omega |\nabla \phi_n|^p - \lambda \int_\Omega g v^{p-1} \frac{|\phi_n|^p}{(v+\varepsilon)^{p-1}}. \tag{4.5}$$

By letting $\varepsilon \to 0$, the dominated convergence theorem yields

$$0 \leq \int_\Omega |\nabla \phi_n|^p - \lambda \int_\Omega g|\phi_n|^p.$$

Now we let $n \to \infty$ to obtain the inequality

$$0 \leq \int_\Omega |\nabla u|^p - \lambda \int_\Omega g u^p.$$

Therefore,

$$\lambda \leq \int_\Omega |\nabla u|^p, \quad \forall\, u \in M. \tag{4.6}$$

This completes the proof. □

Remark 4.8. Using Lemma 4.5, we see that λ_1 is a positive principal eigenvalue and Lemma 4.7 shows that λ_1 is the unique positive principal eigenvalue of (1.1). In particular, the eigenfunctions corresponding to other eigenvalues of (1.1) must change sign.

When Ω is connected, for the simplicity of λ_1, we refer to [18, Theorem 1.3]. There, the authors obtained the simplicity of the first eigenvalue of (1.1), if it exists, even for g in $L^1_{\text{loc}}(\Omega)$.

4.3. **Radial symmetry of the eigenfunctions.** Now we give sufficient conditions for the radial symmetry of the eigenfunctions corresponding to the eigenvalue λ_1 of (1.1). Here we assume that the domain Ω is a ball centered at origin or $\mathbb{R}^N$. Bhattacharya [8] proved the radial symmetry of the first eigenfunctions of (1.1), when $g \equiv 1$ and Ω is ball.

Here we prove that all the positive eigenfunctions corresponding to λ_1 are radial and radially decreasing, provided g is nonnegative, radial and radially decreasing. Thus our result is a two fold generalization of results of Bhattacharya, as we allow more general weight functions and the domain can be $\mathbb{R}^N$. Our result uses certain rearrangement inequalities. We emphasize that here we are not assuming any conditions on g that ensures λ_1 is an eigenvalue.

Theorem 4.9. *Let Ω be a ball centered at origin or $\mathbb{R}^N$. Let g be nonnegative, radial and radially decreasing measurable function. If λ_1 is an eigenvalue of* (1.1)*, then any positive eigenfunction corresponding to λ_1 is radial and radially decreasing.*

Proof. Let u be a positive eigenfunction of (1.1) corresponding to λ_1. Let u_* and g_* be the symmetric decreasing rearrangement of u and g respectively. Since g is nonnegative, radial and radially decreasing, we use property (a) of Proposition 2.3 to conclude that $g = g_*$ a.e. Further, as u is positive by property (c) of Proposition 2.3 we obtain $(u^p)_* = (u_*)^p$ a.e. Now by the Hardy-Littlewood inequality,

$$\int_\Omega g\, u^p \le \int_\Omega g_*(u^p)_* = \int_\Omega g(u_*)^p.$$

Also due to Polya-Szego, we have the following inequality:

$$\int_\Omega |\nabla u_*|^p \le \int_\Omega |\nabla u|^p.$$

Thus

$$\frac{1}{\int_\Omega g(u_*)^p} \int_\Omega |\nabla u_*|^p \le \frac{1}{\int_\Omega g(u)^p} \int_\Omega |\nabla u|^p. \tag{4.7}$$

Since u is a minimizer of R_p on $\mathcal{D}_p^+(g)$, equality holds in (4.7) and hence u_* also minimizes R_p on $\mathcal{D}_p^+(g)$. Now as λ_1 is simple, we obtain $u_* = \alpha u$ a.e. for some $\alpha > 0$. This shows that u is radial, radially decreasing. $\square$

Using the above lemma we see that for $g(x)) = \frac{1}{|x|^p}$, $x \in \mathbb{R}^N$ (1.1) does not admit a positive principal eigenvalue. A proof for the case $p = 2$ is given in [17].

Porposition 4.10. *Let $g(x) = 1/|x|^p$, $x \in \mathbb{R}^N$. Then* (1.1) *does not admit a positive principal eigenvalue.*

Proof. From Lemma 4.7, we know that, if $\lambda > \lambda_1$ then λ is not a principal eigenvalue of (1.1). Thus, it is enough to show that λ_1 is not an eigenvalue of (1.1), when $g(x) = \frac{1}{|x|^p}$. By [18, Theorem 1.3], if λ_1 is an eigenvalue of (1.1), then λ_1 is simple. Further, if u is an eigenfunction of (1.1) corresponding λ_1, then using the scale invariance of (1.1), for each $\alpha \in \mathbb{R}$, one can verify that

$$v_\alpha(x) = u(\alpha x)$$

is also an eigenfunction of (1.1) corresponding to λ_1. Now using the simplicity of λ_1, we obtain

$$u(x) = |x|^{1-\frac{N}{p}} u(1).$$

A contradiction as $|x|^{1-\frac{N}{p}} \notin \mathcal{D}_0^{1,p}(\mathbb{R}^N)$. $\square$

Remark 4.11. In particular, the above Lemma shows that the best constant in the Hardy's inequality

$$\int_{\mathbb{R}^N} |\nabla u|^p \le C \int_{\mathbb{R}^N} \frac{1}{|x|^p} |u|^p$$

is not attained for any $u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)$.

5. An infinite set of eigenvalues

In this section we discuss the existence of infinitely many eigenvalues of (1.1), using the Ljusternik-Schnirelmann theory on $\mathcal{C}^1$ manifold due to Szulkin [27]. Before stating his result we briefly describe the notion of P.S. condition and genus.

Let $\mathcal{M}$ be a $\mathcal{C}^1$ manifold and $f \in \mathcal{C}^1(\mathcal{M};\mathbb{R})$. Denote the differential of f at u by $df(u)$. Then $df(u)$ is an element of $(T_u\mathcal{M})^*$, the cotangent space of $\mathcal{M}$ at u (see [12, section 27.4] for definition and properties).

We say that a map $f \in \mathcal{C}^1(\mathcal{M};\mathbb{R})$ satisfies Palais-Smale (P.S. for short) condition on $\mathcal{M}$, if a sequence $\{u_n\} \subset \mathcal{M}$ is such that $f(u_n) \to \lambda$ and $df(u_n) \to 0$ then $\{u_n\}$ possesses a convergent subsequence.

Let A be a closed symmetric (i.e, $-A = A$) subset of $\mathcal{M}$, the *krasnoselski* genus $\gamma(A)$ is defined to be the smallest integer k for which there exists a non-vanishing odd continuous mapping from A to $\mathbb{R}^k$. If there exists no such map for any k, then we define $\gamma(A) = \infty$ and we set $\gamma(\emptyset) = 0$. For more details and properties of genus we refer to [25].

From [27, Corollary 4.1] one can deduce the following theorem.

Theorem 5.1. *Let $\mathcal{M}$ be a closed symmetric $\mathcal{C}^1$ submanifold of a real Banach space X and $0 \notin \mathcal{M}$. Let $f \in \mathcal{C}^1(\mathcal{M};R)$ be an even function which satisfies P.S. condition on $\mathcal{M}$ and bounded below. Define*

$$c_j := \inf_{A \in \varGamma_j} \sup_{x \in A} f(x),$$

where $\varGamma_j = \{A \subset \mathcal{M} : A \text{ is compact and symmetric about origin}, \gamma(A) \ge j\}$. If for a given j, $c_j = c_{j+1} \cdots = c_{j+p} \equiv c$, then $\gamma(K_c) \ge p+1$, where $K_c = \{x \in M : f(x) = c\,, df(x) = 0\}$.

Note that the set $M = \{u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega g|u|^p = 1\}$ may not even possess a manifold structure from the topology of $\mathcal{D}_0^{1,p}(\Omega)$, due to the weak assumptions on g^-. However, we show that M admits a $\mathcal{C}^1$ Banach manifold structure from a subspace contained in $\mathcal{D}_0^{1,p}(\Omega)$.

For $g^- \in L^1_{\rm loc}(\Omega)$, we define

$$\|u\|_X^p := \int_\Omega |\nabla u|^p + \int_\Omega g^- |u|^p.$$

$$X := \{u \in \mathcal{D}_0^{1,p}(\Omega) : \|u\|_X < \infty\}.$$

Then one can easily verify the following:

- X is a Banach space with the norm $\|\cdot\|_X$ and X is reflexive.

- Since g^- is locally integrable, $\mathcal{C}_c^\infty(\Omega)$ is contained in X.
- Let $g \in L^1_{\rm loc}(\Omega)$ and $g^+ \in \mathcal{F}_{N/p}$. Then $\mathcal{D}_p^+(g)$ is contained in X. This can be seen as
$$\int_\Omega g^-|u|^p < \int_\Omega g^+|u|^p \le C\|g^+\|_{(\frac{N}{p},\infty)}\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega)} < \infty, \tag{5.1}$$
where C is the constant involving the constants that are appearing in the Lorentz-Sobolev embedding and the Hölder inequality. Note that the first inequality follows as $\int_\Omega g|u|^p > 0$, for $u \in \mathcal{D}^+(g)$.
- X is continuously embedded into $\mathcal{D}_0^{1,p}(\Omega)$. Thus X embedded continuously into the Lorentz space $L(p^*, p)$ and embedded compactly into $L^p_{\rm loc}(\Omega)$.

We denote the dual space of X by X' and the duality action by $\langle\cdot,\cdot\rangle$.

Using the definition of the norm one can easily see that, the map G_p^-, defined by
$$G_p^-(u) := \frac{1}{p}\int_\Omega g^-|u|^p,$$
is continuous on X. Further, using the dominated convergence theorem one can verify that G_p^- is continuously differentiable on X and its derivative is given by
$$\langle {G_p^-}'(u), v\rangle = \int_\Omega g^-|u|^{p-2}u\,v.$$

Similarly using the Sobolev embedding and the Hölder inequality one can easily verify that G_p^+ is $\mathcal{C}^1$ in $\mathcal{D}_0^{1,p}(\Omega)$ and in particular on X. The derivative of G_p^+ is given by
$$\langle {G_p^+}'(u), v\rangle = \int_\Omega g^+|u|^{p-2}u\,v.$$
Note that for $u \in M$, $\langle G_p'(u), u\rangle = p$ and hence the map $G_p'(u) \ne 0$. Recall that, $c \in \mathbb{R}$ is called a regular value of G_p, if $G_p'(u) \ne 0$ for all u such that $G_p(u) = c$. Thus we have the following lemma.

Lemma 5.2. *Let Ω be a domain in $\mathbb{R}^N$ with $N > p$. Let $g \in L^1_{\rm loc}(\Omega)$ be such that $g^+ \in \mathcal{F}_{N/p} \setminus \{0\}$. Then the map G_p is in $\mathcal{C}^1(X;\mathbb{R})$ and $G_p' : X \to X'$ is given by*
$$\langle G_p'(u), v\rangle = \int_\Omega g|u|^{p-2}u\,v.$$
Further, 1 is a regular value of G_p.

Remark 5.3. In view of [12, Example 27.2], the above lemma shows that M is a $\mathcal{C}^1$ Banach submanifold of X. Note that M is symmetric about the origin as the map G_p is even.

Next we show that J_p satisfies all the conditions to apply Theorem 5.1.

Lemma 5.4. *J_p is a $\mathcal{C}^1$ functional on M and the derivative of J_p is given by*
$$\langle J_p'(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v$$

The proof is straight forward and is omitted.

Remark 5.5. Using [13, Proposition 6.4.35], one can deduce that
$$\|dJ_p(u)\| = \min_{\lambda\in\mathbb{R}}\|J_p'(u) - \lambda G_p'(u)\|. \tag{5.2}$$

Thus $dJ_p(u_n) \to 0$ if and only if there exists a sequence $\{\lambda_n\}$ of real numbers such that $J_p'(u_n) - \lambda_n G_p'(u_n) \to 0$.

In the next lemma we prove the compactness of the map G_p^+, that we use for showing that the map J_p satisfies P.S. condition on M.

Lemma 5.6. *The map ${G_p^+}' : X \to X'$ is compact.*

Proof. Let $u_n \rightharpoonup u$ in X and $v \in X$. Let q be the conjugate exponent of p. Now using the Lorentz-Sobolev embedding and the Hölder inequality available for the Lorentz spaces, one can verify the following:

$$
\begin{gathered}
(|u_n|^{p-2}u_n - |u|^{p-2}u) \in L\Big(\frac{p^*}{p-1}, \frac{p}{p-1}\Big), \\
(g^+)^{1/q}(|u_n|^{p-2}u_n - |u|^{p-2}u) \in L(\frac{p}{p-1}, \frac{p}{p-1}) \\
(g^+)^{1/p}|v| \in L(p\,, p) \\
\big\|(g^+)^{1/p}v\big\|_p \leq C\|g^+\|_{(N/p,\infty)}^{1/p}\|v\|_{(p^*,p)}
\end{gathered}
$$

where C is a constant that depends only on p, N. Now by using the usual Hölder inequality we obtain

$$
\begin{aligned}
&|\langle G_p'(u_n) - G_p'(u), v\rangle| \\
&\leq \int_\Omega g^+|(|u_n|^{p-2}u_n - |u|^{p-2}u|\,|v| \\
&\leq \Big(\int_\Omega g^+|(|u_n|^{p-2}u_n - |u|^{p-2}u)|^{p/(p-1)}\Big)^{(p-1)/p}\Big(\int_\Omega g^+|v|^p\Big)^{1/p} \\
&\leq \|g^+\|_{(N/p,\infty)}^{1/p}\|v\|_{(p^*,p)}\Big(\int_\Omega g^+|(|u_n|^{p-2}u_n - |u|^{p-2}u)|^{p/(p-1)}\Big)^{(p-1)/p}
\end{aligned}
$$

Thus

$$
\|G_p'(u_n) - G_p'(u)\| \leq \|g^+\|_{(N/p,\infty)}^{1/p}\Big(\int_\Omega g^+|(|u_n|^{p-2}u_n - |u|^{p-2}u)|^{p/(p-1)}\Big)^{(p-1)/p}
$$

Now it is sufficient to show that

$$
\Big(\int_\Omega g^+|(|u_n|^{p-2}u_n - |u|^{p-2}u)|^{p/(p-1)}\Big)^{(p-1)/p} \to 0, \quad \text{as } n \to \infty.
$$

Let $\varepsilon > 0$ and $g_\varepsilon \in C_c^\infty(\Omega)$ be arbitrary.

$$
\begin{aligned}
&\int_\Omega g^+|(|u_n|^{p-2}u_n - |u|^{p-2}u)|^{p/(p-1)} \\
&= \int_\Omega g_\varepsilon|(|u_n|^{p-2}u_n - |u|^{p-2}u)|^{p/(p-1)} + \int_\Omega (g^+ - g_\varepsilon)|(|u_n|^{p-2}u_n - |u|^{p-2}u)|^{p/(p-1)}
\end{aligned}
\tag{5.3}
$$

First we estimate the second integral. Observe that $\big|(|u_n|^{p-2}u_n - |u|^{p-2}u)\big|^{p/(p-1)}$ is bounded in $L(\frac{p^*}{p}, 1)$. Let

$$
\begin{gathered}
m = \sup_n \big\|\big|(|u_n|^{p-2}u_n - |u|^{p-2}u)\big|^{p/(p-1)}\big\|_{(\frac{p^*}{p},1)}, \\
\int_\Omega |(g^+ - g_\varepsilon)||(|u_n|^{p-2}u_n - |u|^{p-2}u)|^{p/(p-1)} \leq Cm\big\|(g^+ - g_\varepsilon)\big\|_{(N/p,\infty)}
\end{gathered}
$$

where the constant C includes all the constants that appear in the Hölder inequality and the Lorentz-Sobolev embedding. Now since $g^+ \in F_{N/p}$, from the definition of $F_{N/p}$, we can choose $g_\varepsilon \in C_c^\infty(\Omega)$ such that

$$m\|(g^+ - g_\varepsilon)\|_{(N/p,\infty)} < \frac{\varepsilon}{2C}$$

Thus we can make the second integral in (5.3) smaller than $\frac{\varepsilon}{2}$ for a suitable choice of g_ε. Since X is embedded compactly into $L^p_{\rm loc}(\Omega)$, the first integral converges to zero up to a subsequence $\{u_{n_k}\}$ of $\{u_n\}$. Hence we obtain $k_0 \in \mathbb{N}$ so that,

$$\int_\Omega g^+|(|u_{n_k}|^{p-2}u_n - |u|^{p-2}u)|^{p/(p-1)} < \varepsilon, \quad \forall\, k > k_0.$$

Now the uniqueness of limit of subsequence helps us to conclude, as in Lemma 4.1, that $\left(\int_\Omega g^+|(|u_n|^{p-2}u_n - |u|^{p-2}u)|^{p/(p-1)}\right)^{(p-1)/p} \to 0$ as $n \to \infty$. Hence the proof. □

Definition 5.7. For $\lambda \in \mathbb{R}^+$, we define $A_\lambda : X \to X'$ as

$$A_\lambda = {J_p}' + \lambda\, {G_p^-}'.$$

In the next proposition we show that the map J_p indeed satisfies P.S. condition on the M.

Porposition 5.8. *J_p satisfies P.S. condition on M.*

Proof. Let $\{u_n\}$ be a sequence in M, such that $J_p(u_n) \to \lambda$ and $dJ_p(u_n) \to 0$. Thus there exists a sequence $\{\lambda_n\}$ such that

$$J_p'(u_n) - \lambda_n G_p'(u_n) \to 0 \quad \text{as } n \to \infty, \tag{5.4}$$

Since $J_p(u_n)$ is bounded, using the estimate (5.1), we see that $\{G_p^-(u_n)\}$ is bounded. Thus the sequence $\{u_n\}$ is bounded in X and hence by the reflexivity we may assume passing to a subsequence that $u_n \rightharpoonup u$. Since G_p^+ is weakly continuous, we obtain $G_p^+(u_n) \to G_p^+(u)$. Now by Fatou's lemma,

$$\int_\Omega g^-|u|^p \leq \liminf \int_\Omega g^+|u_n|^p - 1 = \int_\Omega g^+|u|^p - 1. \tag{5.5}$$

Thus $\int_\Omega g|u|^p \geq 1$ and hence $u \neq 0$. Further, $\lambda_n \to \lambda$ as $n \to \infty$, since

$$p(J_p(u_n) - \lambda_n) = \langle J_p'(u_n) - \lambda_n G_p'(u_n), u_n\rangle \to 0.$$

Now we write (5.4) as

$$A_{\lambda_n}(u_n) - \lambda_n {G_p^+}'(u_n) \to 0.$$

Since $\lambda_n \to \lambda$, we obtains $A_{\lambda_n}(u_n) - A_\lambda(u_n) \to 0$. Now the compactness of ${G_p^+}'$ yields the strong convergence of $A_\lambda(u_n)$ and hence $\langle A_\lambda(u_n), u_n - u\rangle \to 0$. Since $u_n \rightharpoonup u$, using [24, Lemma 4.3] one obtain $u_n \to u$. □

We borrow an idea from [18, Proposition 4.2], for the proof of the following lemma.

Lemma 5.9. *For each $n \in \mathbb{N}$, the set $\Gamma_n \neq \emptyset$.*

Proof. The idea is to construct odd continuous maps from $S^{n-1} \to M$, for each $n \in \mathbb{N}$. Let $\Omega^+ = \{x : g^+(x) > 0\}$. Since $|\Omega^+| > 0$, using the Lebesgue-Besicovitch differentiation theorem, one can choose n points $x_1, x_2, \dots x_n$ in Ω^+ such that

$$\lim_{r\to 0} \frac{1}{|B_r(x_i)|} \int_{B_r(x_i)} g(y)dy = g(x_i) > 0.$$

Thus there exists $R > 0$, such that $B_R(x_i) \cap B_R(x_j) = \emptyset$ and

$$\int_{B_r(x_i)} g(y)dy > 0, \quad \text{for } 0 < r < R.$$

Now one can choose r such that $0 < r < R$ and

$$\int_{B_R(x_i)\setminus B_r(x_i)} |g(y)|dy < \int_{B_r(x_i)} g(y)dy \tag{5.6}$$

Let $u_i \in \mathcal{C}_c^\infty(B_R(x_i))$ such that $0 \le u_i(x) \le 1$ and $u_i \equiv 1$ on $B_r(x_i)$. Now using (5.6) we have the following

$$\int_{B_R(x_i)} g|u_i|^p = \int_{B_r(x_i)} g + \int_{B_R(x_i)\setminus B_r(x_i)} g|u_i|^p \ge \int_{B_r(x_i)} g - \int_{B_R(x_i)\setminus B_r(x_i)} |g| > 0$$

Thus we obtain $v_i = u_i/(\int_\Omega g|u_i|^p)^{1/p} \in M$. Note that the support of v_is are disjoint. Now for $\alpha = (\alpha_1, \alpha_2, \dots \alpha_n) \in \mathbb{R}^n$ with $\sum |\alpha_i|^p = 1$, we have $\sum \alpha_i v_i \in C_c^\infty(\Omega)$ and $\int_\Omega g|\sum \alpha_i v_i|^p = 1$. It is easy to see that the map $\phi(\alpha) = \sum \alpha_i u_i$ is an odd continuous map from S^{n-1} into M. Thus $\phi(S^{n-1})$ is compact and symmetric about origin. Now from the definition of genus it follows that $\gamma(\phi(S^{n-1})) \ge \gamma(S^{n-1}) = n$. $\square$

Now we are in a position to adapt the Ljusternik-Schnirelmann theorem available for $\mathcal{C}^1$ manifold in our situation and prove the existence of infinitely many eigenvalues for (1.1).

Proof of Theorem 1.3. Since J and M satisfy all the requirements of Theorem 5.1, for each $j \in \mathbb{N}$, we have $\gamma(K_{c_j}) \ge 1$. Thus $K_{c_j} \ne \emptyset$ and hence there exist $u_j \in M$ such that $dJ(u_j) = 0$ and $J(u_j) = c_j$. Therefore c_j is an eigenvalue of (1.1) and u_j is an eigenfunction corresponding to c_j.

A proof for the unboundedness of the sequence $\{c_n\}$ is given in [9](see Theorem 2). For the sake of completeness we adapt their idea in our situation. Recall that the space X is separable (see [1, (3.5)]) and hence X admits a biorthogonal system $\{e_m, e_m^*\}$, (see [21, Proposition 1.f.3]) such that

$$\overline{\{e_m, m :\in \mathbb{N}\}} = X, \quad e_m^* \in X', \quad \langle e_m^*, e_n\rangle = \delta_{n,m},$$
$$\langle e_m^*, x\rangle = 0, \quad \forall m \Rightarrow x = 0.$$

Let $E_n = \operatorname{span}\{e_1, e_2, \dots, e_n\}$ and let

$$E_n^\perp = \overline{\operatorname{span}\{e_{n+1}, e_{n+2}, \dots\}}.$$

Since $E_{n-1}^\perp$ is of codimension $n-1$, for any $A \in \Gamma_n$ we have $A \cap E_{n-1}^\perp \ne \emptyset$ (see [25, Proposition 7.8]). Let

$$\mu_n = \inf_{A\in\Gamma_n} \sup_{A\cap E_{n-1}^\perp} J(u), \quad n = 1, 2, \dots$$

Now we show that $\mu_n \to \infty$. If possible let $\{\mu_n\}$ be bounded, then there exists $u_n \in E_{n-1}^\perp \cap M$ such that $\mu_n \le J(u_n) < c$ for some constant $c > 0$. Since $u_n \in M$, the estimate (5.1) shows that u_n is indeed bounded in X. Thus $u_n \rightharpoonup u$ for some $u \in X$. Now by the choice of biorthogonal system, for each m, $\langle e_m^*, u_n\rangle \to 0$ as $n \to \infty$. Thus $u_n \rightharpoonup 0$, in X and hence $u = 0$, a contradiction to $\int_\Omega g|u|^p \geq 1$ (see the conclusion followed estimate (5.5)). Therefore, $\mu_n \to \infty$ and hence $c_n \to \infty$ as $\mu_n \leq c_n$. $\square$

Remark 5.10. If $g^- \in \mathcal{F}_{N/p} \setminus \{0\}$, then there exists a sequence μ_n of negative eigenvalues of (1.1) tending to $-\infty$. Further, μ_1 is simple and it is the unique negative principal eigenvalue of (1.1).

6. Remarks

In this section we remark about possible extensions and applications of weighted eigenvalue problems for the p-Laplacian.

One can study the existence of ground states for the Δ_p operator with a more general subcritical nonlinearities in the right hand side. More precisely, for given locally integrable functions V, g on a domain $\Omega \subset \mathbb{R}^N$ with $V \geq 0$ but g allowed to change sign, we look for the positive solutions in $\mathcal{D}_0^{1,p}(\Omega)$ for the problem

$$\Delta_p u + V|u|^{p-2}u = \lambda g|u|^{q-2}u, \quad u \in \mathcal{D}_0^{1,p}(\Omega), \tag{6.1}$$

where $q \in [p, p^*)$ and $1 < p < N$.

Remark 6.1. Indeed, one can show that if $g^+ \in \mathcal{F}_{\widetilde{p}} \setminus \{0\}$ with $\frac{1}{\widetilde{p}} + \frac{q}{p^*} = 1$, then (6.1) has a positive solution. If one verify that $G(u) = \int_\Omega g^+|u|^q$ is compact, then by arguing as in Proposition 4.2, it is immediate that $\int_\Omega\{|\nabla u|^p + V|u|^p\}$ has a positive minimizer on $M_q = \{u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega g|u|^q = 1\}$. Also using the homogeneity of the Rayleigh quotient $R = \frac{\int_\Omega\{|\nabla u|^p + V|u|^p\}}{(\int_\Omega g|u|^q)^{\frac{p}{q}}}$ corresponding to (6.1) we obtain a minimizer of R on $\{u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega g|u|^q > 0\}$ and hence a positive solution of (6.1). For the positivity of this minimizer one can use [18, Proposition 5.3].

Remark 6.2. Let g be as in the above remark. Then the following generalized Hardy-Sobolev inequality holds

$$\Big(\int_\Omega g|u|^q\Big)^{q/p} \leq \frac{1}{\lambda_1}\int_\Omega\{|\nabla u|^p + V|u|^p\}, \quad \forall\, u \in \mathcal{D}_0^{1,p}(\Omega),\ \int_\Omega g|u|^q > 0 \tag{6.2}$$

where λ_1 is the minimum of $\int_\Omega\{|\nabla u|^p + V|u|^p\}$ on M_q. Further the best constant is attained. This extends the results of Visciglia [31] for $p \neq 2$.

Remark 6.3. The existence of a simple eigenvalue for (1.1) can be applied to study the bifurcation phenomena of the solutions for the semilinear problem of the type

$$-\Delta_p u = \lambda\big(a(x)u + b(x)r(u)\big), \quad u \in \mathcal{D}_0^{1,p}(\Omega) \tag{6.3}$$

for a real parameter λ when a, b are in certain sub class of weak Lebesgue space with a suitable growth condition on r. Such a result is available for $p = 2$ see in [6]. We deal with this question in a subsequent work.

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