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CLASSIFICATION OF HETEROCLINIC ORBITS OF SEMILINEAR PARABOLIC EQUATIONS WITH A POLYNOMIAL NONLINEARITY

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Abstract. For a given semilinear parabolic equation with polynomial nonlinearity, many solutions blow up in finite time. For a certain class of these equations, we show that some of the solutions which do not blow up actually tend to equilibria. The characterizing property of such solutions is a finite energy constraint, which comes about from the fact that this class of equations can be written as the flow of the L^2 gradient of a certain functional.

1. Introduction

Dynamics of semilinear parabolic equations play an important role in certain applications, and have a long history of study. Perhaps the earliest occurrence is in a pair of articles [4, 8], where the long-time behavior of solutions is addressed in the context of population biology. The behavior of global solutions that approach equilibria for positive and *negative* time is also of special interest. These *heteroclines* capture the admissible transitions between equilibria, which is important for assembling a dynamical picture of spaces of solutions.

In this article, the global behavior of smooth solutions to the semilinear parabolic equation

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) - u^N + \sum_{i=0}^{N-1} a_i(x) u^i(t,x) = \Delta u + P(u), \tag{1.1}$$

for $(t,x) \in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$ is considered, where $N \geq 2$ and $a_i \in L^\infty(\mathbb{R}^n)$ are smooth with all derivatives of all orders bounded. It suffices to note that the existence of multiple equilibria (see [11]) quickly foils any hope for a common limit for all global solutions, and paves the way for more complicated dynamics.

1.1. Article highlights.

Our main result (Theorem 3.1) is that heteroclinic orbits of (1.1) connecting two finite action equilibrium solutions (Definition 2.1) are characterized by finite energy (Definition 2.2). (Time limits of solutions will be understood in the sense of uniform convergence on compact subsets.) That this characterization is necessary at all comes from the fact that the spatial domain of (1.1) is unbounded. For bounded spatial domains, all bounded global solutions

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converge to equilibria. [7] The strength of our result comes from the fact that the finite energy constraint makes solutions behave rather well. Therefore, our result is much sharper than what has typically been obtained in the past, and it applies to more complicated nonlinear terms.

Heteroclinic orbits of (1.1) are rare: that there are any such solutions at all is shown in [9]. Under certain conditions on the a_i, the space of heteroclinic orbits is a finite-dimensional cell complex [10], contained in an time-weighted (but not spatially-weighted) Sobolev space. As an example of the rarity of heteroclines, we note in passing that travelling wave solutions do not have finite energy. Even though a travelling wave will often converge locally to equilibria, at least one of those equilibria will not have finite action. On the other hand, we can exclude travelling waves from the solution set of (1.1) if we require that all the coefficients a_i decay fast enough and only consider one spatial dimension. Then our result establishes an equivalence between all heteroclinic orbits and the finite energy solutions, as all equilibria have finite action. (See [11] for a demonstration that decay of the a_i implies finite action.)

A crucial requirement in this article is that the equilibria be *isolated*. While it is unlikely that the equilibria of (1.1) are always isolated, they are in many interesting cases. We therefore examine some sufficient conditions for isolatedness of equilibria (Lemma 3.4).

1.2. **Historical comments.** The study of (1.1) on unbounded domains is not new. Blow-up behavior for equations like (1.1) was examined in a classic paper by Fujita. [6] This line of classical reasoning was studied by many authors, and is summarized in [14]. For somewhat more restricted nonlinearities, Du and Ma were able to use squeezing methods to obtain similar results to what we obtain here. In particular, they also show that certain kinds of solutions approach equilibria. [2] A major difference between the results obtained by Du and Ma and the work presented here is that in our case there is a lack of uniqueness, both in the global solutions themselves and also in their limits. Generally speaking, there will be many equilibrium solutions to which global solutions may tend, each with different dynamical properties.

In a somewhat different setting, Floer used a finite energy constraint for solutions and a regularity constraint on equilibria to characterize heteroclinic orbits of an elliptic problem. [5] The techniques of Floer were subsequently used by Salamon to provide a new characterization of solutions to gradient flows on finite-dimensional manifolds. [12] In this article, we recast some of Salamon's work into a parabolic setting, and of course work within an infinite-dimensional space.

1.3. **Outline of the article.** In Section 2, we present definitions of *energy* and *action* for solutions to (1.1) and timeslices of solutions, respectively. Our characterization theorem is proven in Section 3, and discussed in Section 3.1.

2. Finite energy constraints

It is well-known that solutions to (1.1) exist along strips of the form $(t, x) \in I \times \mathbb{R}^n$ for sufficiently small, positive t-intervals I. One might hope to extend such solutions to all of $\mathbb{R}^{n+1}$, but for certain choices of initial conditions such global solutions may fail to exist. [6] We will specifically avoid blow-up by considering only global solutions to (1.1). By global solutions, we mean those which are defined for all

$\mathbb{R}^{n+1}$, have one continous partial derivative in time, and two continous partial derivatives in space. It should be noted that global solutions to (1.1) are quite rare: the backwards-time Cauchy problem contains a heat operator, and so most solutions will not extend to all of $\mathbb{R}^{n+1}$. Existence of global solutions (as in [9]) is therefore a feature of the nonlinear term in (1.1).

Definition 2.1. Our analysis of (1.1) will make considerable use of the fact that it is a gradient differential equation. Observe that the right side of (1.1) is the $L^2(\mathbb{R}^n)$ gradient of the following *action functional*, defined for all $f \in C^1(\mathbb{R}^n)$:

$$A(f) = \int_{\mathbb{R}^n} -\frac{1}{2}\|\nabla f(x)\|^2 - \frac{f^{N+1}(x)}{N+1} + \sum_{i=0}^{N-1} \frac{a_i(x)}{i+1} f^{i+1}(x)dx. \tag{2.1}$$

It is then evident that along a solution $u(t) \in L^2(\mathbb{R}^n)$ to (1.1),

$$\begin{aligned}
\frac{dA(u(t))}{dt} &= dA|_{u(t)}\big(\frac{\partial u}{\partial t}\big) \\
&= \langle \nabla A(u(t)), \frac{\partial u}{\partial t}\rangle \\
&= \langle \Delta u + P(u), \frac{\partial u}{\partial t}\rangle \\
&= \|\frac{\partial u}{\partial t}\|_2^2 \geq 0,
\end{aligned}$$

so $A(u(t))$ is a monotone function. As an immediate consequence, nonconstant t-periodic solutions to (1.1) do not exist.

Definition 2.2. The *energy functional* is the following quantity defined on the space S of functions $\mathbb{R}^{n+1} \to \mathbb{R}$ with one continuous partial derivative in the first variable (t), and two continuous partial derviatives in the rest (x):

$$E(u) = \frac{1}{2}\int_{-\infty}^{\infty}\int |\frac{\partial u}{\partial t}|^2 + |\Delta u + P(u)|^2 dx\, dt. \tag{2.2}$$

It is evident that S is a Banach space under the appropriate norm, which is

$$\|u\|_S = \|u\|_\infty + \|\frac{\partial u}{\partial t}\|_\infty + \sum_{i=1}^{n} \|\frac{\partial u}{\partial x_i}\|_\infty + \sum_{i,j=1}^{n} \|\frac{\partial^2 u}{\partial x_i \partial x_j}\|_\infty.$$

Calculation 2.3. Suppose $u \in S$ is in the domain of definition for the energy functional, then

$$\begin{aligned}
E(u) &= \frac{1}{2}\int_{-\infty}^{\infty}\int |\frac{\partial u}{\partial t}|^2 + |\Delta u + P(u)|^2 dx\, dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\int \Big(\frac{\partial u}{\partial t} - \Delta u - P(u)\Big)^2 + 2\frac{\partial u}{\partial t}(\Delta u + P(u))dx\, dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\int \Big(\frac{\partial u}{\partial t} - \Delta u - P(u)\Big)^2 dx\, dt + \int_{-\infty}^{\infty} \langle \frac{\partial u}{\partial t}, \Delta u + P(u)\rangle dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\int \Big(\frac{\partial u}{\partial t} - \Delta u - P(u)\Big)^2 dx\, dt + \int_{-\infty}^{\infty} \langle \frac{\partial u}{\partial t}, \nabla A(u(t))\rangle dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\int \Big(\frac{\partial u}{\partial t} - \Delta u - P(u)\Big)^2 dx\, dt + \int_{-\infty}^{\infty} \frac{d}{dt}A(u(t))dt
\end{aligned}$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \int \Big(\frac{\partial u}{\partial t} - \Delta u - P(u) \Big)^2 dx\, dt + A(u(T)) \Big|_{T=-\infty}^{\infty}.$$

This calculation shows that finite energy solutions to (1.1) minimize the energy functional over functions in S with t-boundary conditions being equilibria of (1.1), and x-boundary conditions enforced by finiteness of the integrals. If a solution to (1.1) is a heteroclinic connection between two equilibria, then the energy functional measures the difference between the values of the action functional evaluated at the two equilibria. *The main result of this article is the converse, that finite energy characterizes the solutions which connect equilibria.*

Finite energy solutions to (1.1) are even more rare than global solutions. However, the set of finite energy solutions is not entirely vacuous, as equilibrium solutions automatically have finite energy. Not every equation of the form (1.1) will have equilibria, but some do. Consider

$$\frac{\partial u}{\partial t} = \Delta u - u^2,$$

which evidently has the zero function as an equilibrium. Indeed, the zero function is the *only* finite energy solution [6].

It is well-known that equations like (1.1) exhibiting translational symmetry in space may support travelling wave solutions of the form $u(t, x) = U(x - ct)$ for some $c \in \mathbb{R}$. [3] As a result, it is immediate that travelling waves will have infinite energy. On the other hand, they also evidently connect equilibria (as measured using the topology of uniform convergence on compact subsets, as opposed to the topology of S defined earlier). Calculation 2.3 shows that a necessary condition for travelling waves is that there exists at least one equilibrium whose action is infinite. In this article, we will consider only equilibria with finite action, and solutions with finite energy.

3. Convergence to equilibria

In this section, we show that finite energy solutions tend to equilibria as $|t| \to \infty$, culminating in the proof of the following theorem.

Theorem 3.1. *Suppose that either $n = 1$ or N is odd and that equilibria are isolated. A smooth global solution u to* (1.1) *has finite energy if and only if each of the following hold:*

- *each of $U_\pm(x) = \lim_{t\to\pm\infty} u(t, x)$ exists and converges with its first derivatives uniformly on compact subsets of $\mathbb{R}^n$,*
- *$U_\pm$ are bounded, continuous equilibrium solutions to* (1.1)*,*
- *and either $|A(U_+) - A(U_-)| < \infty$ or $U_+ = U_-$.*

We follow Floer in [5] which leads us through an essentially standard parabolic bootstrapping argument. We begin, however, with a result that is consequence of the fact that $W^{k,\infty}(\mathbb{R}^{n+1})$ is a Banach algebra. This permits a straightforward treatment of the polynomial nonlinearity in (1.1).

Convention. We employ the standard multi-index notation D^k to refer to the set of order k derivative operators, to be taken over the principal directions in $\mathbb{R}^{n+1}$. For instance, D^1 refers to the set $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots \frac{\partial}{\partial x_n}\}$.

Lemma 3.2. *Let $U \subseteq \mathbb{R}^n$ and $u \in W^{k,p}(U)$ satisfy $\|D^j u\|_\infty \le C < \infty$ for $0 \le j \le k$ (in particular, u is bounded). If $P(u) = \sum_{i=1}^N a_i u^i$ with $a_i \in L^\infty(U)$ then there exists a C' such that $\|P(u)\|_{k,p} \le C'\|u\|_{k,p}$. (Recall that we also assume that $\|D^j a_i\|_\infty$ exist and are all finite.)*

We note that this is a local result, and therefore does not require decay conditions on the a_i.

Proof. First, using the definition of the Sobolev norm,

$$\|P(u)\|_{k,p} = \sum_{j=0}^{k} \|D^j P(u)\|_p \le \sum_{j=0}^{k}\sum_{i=1}^{N} \|D^j a_i u^i\|_p.$$

Now $|D^j a_i u^i| \le P_{i,j}(u, Du, \dots, D^j u)$, which is a polynomial in j variables with constant coefficients, and no constant term. (The constant coefficients is a consequence of the bounded derivatives of the a_i.) Additionally,

$$\begin{aligned} \|(D^m u)^q D^j u\|_p &= \Big(\int |(D^m u)^q D^j u|^p\Big)^{1/p} \\ &\le \|D^m u\|_\infty^q \Big(\int |D^j u|^p\Big)^{1/p} \le C^q \|D^j u\|_p, \end{aligned}$$

so by collecting terms,

$$\|P(u)\|_{k,p} \le \sum_{j=0}^{k}\sum_{i=1}^{N} \|D^j a_i u^i\|_p \le \sum_{j=0}^{k} A_j \|D^j u\|_p \le C'\|u\|_{k,p}.$$

□

The following result is a parabolic bootstrapping argument that does most of the work. In it, we follow Floer in [5], replacing "elliptic" with "parabolic" as necessary.

Lemma 3.3. *Suppose that all of the equilibria of* (1.1) *are isolated. If u is a finite energy solution to* (1.1) *with $\|D^j u\|_{L^\infty((-\infty,\infty)\times V)} \le C < \infty$ for $0 \le j \le k$ with $k \ge 1$ on each compact $V \subset \mathbb{R}^n$, then each of $\lim_{t\to\pm\infty} D^j u(t,x)$ exists, and each converges uniformly on compact subsets of $\mathbb{R}^n$. Further, the limits are equilibrium solutions to* (1.1)*. (Again, we assume $\|D^j a_i\|_\infty < \infty$ as before.)*

Proof. Define $u_m(t,x) = u(t_m, x)$ where $t_m \to \infty$. Suppose $U \subset \mathbb{R}^{n+1}$ is a bounded open set and $K \subset U$ is compact. Let β be a smooth bump function whose support is $\bar{U}$, takes the value 1 on K, and is nonzero within U. We take $p > 1$ such that $kp > n+1$. Then we can consider $u_m \in W^{k,p}(U)$ (recall that u and its first k derivatives of u are bounded on the closure of U), and we have

$$\|u_m\|_{W^{k+1,p}(K)} \le \|\beta u_m\|_{W^{k+1,p}(U)}.$$

Then using the standard regularity for the parabolic operators,

$$\|\beta u_m\|_{W^{k+1,p}(U)} \le C_1 \Big\| \Big(\frac{\partial}{\partial t} - \Delta + \frac{2}{\beta}\nabla\beta\cdot\nabla\Big)(\beta u_m)\Big\|_{W^{k,p}(U)}.$$

The usual product rule yields the following:

$$\Big(\frac{\partial}{\partial t} - \Delta + \frac{2}{\beta}\nabla\beta\cdot\nabla\Big)(\beta u_m) = u_m\Big(\frac{\partial}{\partial t} - \Delta + \frac{2}{\beta}\nabla\beta\cdot\nabla\Big)\beta + \beta\Big(\frac{\partial}{\partial t} - \Delta\Big)u_m$$

which implies that

$$\begin{aligned}
&\|\Big(\frac{\partial}{\partial t} - \Delta + \frac{2}{\beta}\nabla\beta\cdot\nabla\Big)(\beta u_m)\|_{W^{k,p}(U)} \\
&\le \big\|u_m\Big(\frac{\partial}{\partial t} - \Delta + \frac{2}{\beta}\nabla\beta\cdot\nabla\Big)\beta\|_{W^{k,p}(U)} + \big\|\beta\Big(\frac{\partial}{\partial t} - \Delta\Big)u_m\big\|_{W^{k,p}(U)}.
\end{aligned}$$

Let $P'(u) = -u^N + \sum_{i=1}^{N-1} a_i u^i$, noting carefully that we have left out the a_0 term. Hence, as suggested in [12] we obtain

$$\begin{aligned}
&\|u_m\|_{W^{k+1,p}(K)} \\
&\le C_1\|\beta\Big(\frac{\partial}{\partial t} - \Delta\Big)u_m\|_{W^{k,p}(U)} + C_2\|u_m\|_{W^{k,p}(U)} \\
&\le C_1\|\beta\Big(\frac{\partial}{\partial t} - \Delta\Big)u_m + \beta P'(u_m) - \beta P'(u_m)\|_{W^{k,p}(U)} + C_2\|u_m\|_{W^{k,p}(U)} \\
&\le C_1\|\beta a_0\|_{W^{k,p}(U)} + C_1\|\beta P'(u_m)\|_{W^{k,p}(U)} + C_2\|u_m\|_{W^{k,p}(U)} \\
&\le C_1\|\beta a_0\|_{W^{k,p}(U)} + C_3\|u_m\|_{W^{k,p}(U)},
\end{aligned}$$

where the last inequality is a consequence of Lemma 3.2. By the hypotheses on u and a_0, this implies that there is a finite bound on $\|u_m\|_{W^{k+1,p}(K)}$, which is independent of m. Now by our choice of p, the general Sobolev inequality implies that $\|u_m\|_{C^{k+1-(n+1)/p}(K)}$ is uniformly bounded. By choosing p large enough, there is a subsequence $\{v_{m'}\} \subset \{u_m\}$ such that $v_{m'}$ (and its first k derivatives) converge uniformly on K, say to v. For any $T > 0$, we observe

$$\begin{aligned}
\int_{-T}^{T}\int |\frac{\partial v}{\partial t}|^2 dx\,dt &= \lim_{m'\to\infty}\int_{-T}^{T}\int |\frac{\partial v_{m'}}{\partial t}|^2 dx\,dt \\
&= \lim_{m'\to\infty}\int_{t'_m-T}^{t'_m+T}\int |\frac{\partial u}{\partial t}|^2 dx\,dt = 0,
\end{aligned}$$

where the last equality is by the finite energy condition. Hence $|\frac{\partial v}{\partial t}| = 0$ almost everywhere, which implies that v is an equilibrium.

To prove that $\lim_{t\to\infty} u(t,x) = v(x)$, we follow a relatively standard line of reasoning, as outlined in [1, Proposition 3.19]. Suppose on the contrary, that $\lim_{t\to\infty} u(t,x) \ne v(x)$. (Evidently, v is still an accumulation point of u.) Since we assume that the equilibria are isolated, let $U \subset C^k(K)$ be a closed set with nonempty interior containing v and no other equilibria. Since v is not a limit of u, we can find an open neighborhood $V \subset U \subset C^k(K)$ of v and a sequence $\{t''_m\}$ with $t''_m \to \infty$ such that $u(t''_m) \in U - V$ for all m. However, the same argument as given in the previous section of the proof implies that there is an accumulation point v' of $\{u(t''_m)\}$, and that $v' \in U - V$. We must conclude that v' is an equilibrium in U, which is a contradiction.

Similar reasoning works for $t \to -\infty$ as we simply then take $t_m \to -\infty$ in the definition of u_m. □

It is not terribly restrictive to assume that the equilibria be isolated. In one spatial dimension, an equilibrium f of (1.1) is isolated in $C^{2,\alpha}(\mathbb{R})$ (for $\alpha > 0$) when the Schrödinger operator $(\frac{d^2}{dx^2} - 2f)$ is injective. This follows from the finiteness of the point spectrum, which is a consequence of Sturm-Liuoville theory. Observe that in particular, the injectivity of the Schrödinger operator is therefore generic.

Indeed, we have the following concrete result which ensures that there are plentiful choices of PDE like (1.1) in which equilibria are isolated.

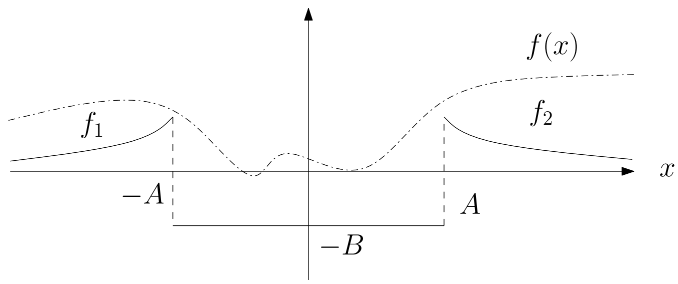

FIGURE 1. Lower bound for equilibrium f in Lemma 3.4

Lemma 3.4. *Suppose that f is an equilibrium of (1.1) which has the form*

$$f(x) \geq \begin{cases} f_1(x) & \text{for } x < -A \\ f_2(x) & \text{for } x > A \\ -B & \text{for } -A \leq x \leq A \end{cases}$$

where $f_1, f_2 > 0$, $f_1' > 0$, $f_2' < 0$ are all continuous, and $A, B \geq 0$. (See Figure 1.) Then for sufficiently small, but nonzero A, B, the Schrödinger operator $H = (\frac{d^2}{dx^2} - 2f)$ is injective on the subspace of $C^2(\mathbb{R})$ which consists of functions that decay to zero.

Proof. Suppose that $Hu = 0$ for some nonzero $u \in C^2(\mathbb{R})$ and that $u(x)$ tends to zero as $|x| \to \infty$. The maximum principle applied to $\frac{d^2u}{dx^2} = 2fu$ implies that u is of one sign on $x < -A$. Without loss of generality, we assume u is positive. Indeed, u will be monotonic increasing.

Now suppose that $u'(-A) = u_0' > 0$. We solve for a v, lower bound on u defined by

$$\begin{gathered} \frac{d^2v}{dx^2} = -2Bv \quad \text{on } -A \leq x \leq A, \\ v(-A) = u_0 > 0 \\ v'(-A) = u_0' > 0 \end{gathered}$$

noting that we should require $v''(-A) > 0$ by the fact that $u \in C^2(\mathbb{R})$. Of course, this has the general solution $v = c_1 \cos x\sqrt{2B} + c_2 \sin x\sqrt{2B}$. So if $2A\sqrt{2B} < \pi$, there can be at most one inflection point of u in $-A \leq x \leq A$. By the continuity of u'', this means that $u''(+A) > 0$. As a result, the maximum principle implies that u is positive on all of $\mathbb{R}$. On the other hand, since $f_2 > 0$, $u(x)$ cannot tend to zero as $x \to +\infty$, a contradiction. $\square$

We would like to relax the bounds on u and its derivatives, by showing that they are consequences of the finite energy condition. The following proposition implies Theorem 3.1 immediately.

Proposition 3.5. *Suppose that the equilibria of* (1.1) *are all isolated, and that either* $n = 1$ *(one spatial dimension) or* N *is odd. If* u *is a finite energy solution to* (1.1)*, then the the limits* $\lim_{t\to\pm\infty} u(t,x)$ *exist uniformly on compact subsets, and additionally,*

- *u is bounded,*
- *the derivatives Du are bounded,*
- *and therefore the limits are continuous equilibrium solutions.*

Proof. Since

$$E(u) = \frac{1}{2}\int_{-\infty}^{\infty}\int |\frac{\partial u}{\partial t}|^2 + |\Delta u + P(u)|^2 dx\, dt < \infty,$$

we have that for any $\epsilon > 0$,

$$\lim_{T\to\infty} \frac{1}{2}\int_{T-\epsilon}^{T+\epsilon}\int |\frac{\partial u}{\partial t}|^2 + |\Delta u + P(u)|^2 dx\, dt = 0, \tag{3.1}$$

whence $\lim_{t\to\infty} |\frac{\partial u}{\partial t}| = 0$ for almost all x. This gives that the limit is an equilibrium almost everywhere. Of course, this argument works for $t \to -\infty$.

When N is odd, a comparison principle shows that solutions to (1.1) are always bounded. Observe that for large $|u|$, the $-u^N$ term dominates the other terms in $P(u)$, which imposes a kind of asymptotic convexity on the problem.

We need to consider the case with N even. In that case, a comparison principle on (1.1) shows that u is bounded from *above*: assume that for a fixed t_0, $u(t_0,\cdot)$ attains a maximum at x_0, then

$$\begin{aligned}\frac{\partial u(t_0,x_0)}{\partial t} &= \Delta u(t_0,x_0) - u^N(t_0,x_0) + \sum_{i=0}^{N-1} a_i(x_0)u^i(t_0,x_0)\\ &\le -u^N(t_0,x_0) + \sum_{i=0}^{N-1} a_i(x_0)u^i(t_0,x_0).\end{aligned}$$

If we assume that u is not bounded from above, then the u^N term will eventually dominate (since all of the a_i are bounded) resulting in a contradiction.

On the other hand, if N is even we have assumed that $n = 1$ in this case, and it follows from an asymptotic ODE argument that unbounded equilibria are bounded from *below*. In one spatial dimension, equilibrium solutions must satisfy

$$u'' = u^N - \sum_{i=0}^{N-1} a_i u^i. \tag{3.2}$$

Observe that for $|u|$ sufficiently large, the u^N term will dominate, since all the a_i are bounded. Therefore, for $|u|$ large, (3.2) we have

$$au^N \le u'' \le Au^N,$$

for some $a, A > 0$ whose solutions (when $|u|$ is large) are easily found (explicitly) to each have a lower bound.

As a result, we must conclude that if a solution to (1.1) tends to any equilibrium, that equilibrium (and hence u also) must be bounded.

Now observe that $|\frac{\partial u}{\partial t}| \to 0$ as $t \to \pm\infty$ on almost all of any compact $K \subset \mathbb{R}^n$ (by (3.1)), and that $|\frac{\partial u}{\partial t}| \le a < \infty$ for some finite a on $\{(t,x)|t = 0, x \in K\}$ by the smoothness of u. By the compactness of K, this means that if $\|\frac{\partial u}{\partial t}\|_{L^\infty((-\infty,\infty)\times K)} =$

∞, there must be a (t^*, x^*) such that $\lim_{(t,x)\to(t^*,x^*)} |\frac{\partial u}{\partial t}| = \infty$. This contradicts smoothness of u, so we conclude $|\frac{\partial u}{\partial t}|$ is bounded on the strip $(-\infty,\infty)\times K$. On the other hand, the finite energy condition also implies that for each $v \in \mathbb{R}^n$,

$$\lim_{s\to\infty} \int_{-\infty}^{\infty} \int_{K+sv} |\frac{\partial u}{\partial t}|^2 dx\, dt = 0,$$

whence we must conclude that $\lim_{s\to\infty} |\frac{\partial u(t,x+sv)}{\partial t}| = 0$ for almost every $t \in \mathbb{R}$ and $x \in K$. Thus the smoothness of u implies that $|\frac{\partial u}{\partial t}|$ is bounded on all of $\mathbb{R}^{n+1}$.

Next, note that since $|\frac{\partial u}{\partial t}|$ and u are both bounded, then so is Δu: by (1.1)

$$\|\Delta u\|_\infty \le \|\frac{\partial u}{\partial t}\|_\infty + \|u\|_\infty^N + \sum_{i=0}^{N-1} \|a_i\|_\infty \|u\|_\infty^i,$$

which uses the boundedness of the a_i. Taken together, this implies that all the spatial first derivatives of u are also bounded.

As a result, we have on K a bounded equicontinuous family of functions, so Ascoli's theorem implies that they (after extracting a suitable subsequence) converge uniformly on compact subsets of K to a continuous limit. As in the end of the proof of Lemma 3.3, the existence of $\lim_{t\to\infty} u(t,x)$ relies on the equilibria being isolated. □

3.1. **Discussion.** The point of employing the bootstrapping argument of Lemma 3.3 is only to extract uniform convergence of the first derivatives of the solution. As can be seen from the proof of Proposition 3.5, such regularity arguments are unneeded to obtain good convergence of the solution only.

While Theorem 3.1 is probably true for all spatial dimensions, the proof given here cannot be generalized to higher dimensions. In particular, Véron in [13] shows that in the case of $P(u) = -u^N$, there are solutions to the equilibrium equation $\Delta u - u^N = 0$ which are *unbounded below* and *bounded above* when the spatial dimension is greater than one. This breaks the proof of Proposition 3.5, that the limiting equilibria of finite energy solutions are bounded for N even, since the proof requires exactly the opposite.

On the other hand, the case of $P(u) = -u|u|^{N-1} + \sum_{i=0}^{N-1} a_i u^i$ is considerably easier than what we have considered here. In particular, all solutions to (1.1) are then bounded. In that case, the proof of Theorem 3.1 works for all spatial dimensions.

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