

FIXED SET THEOREMS FOR DISCRETE DYNAMICS AND NONLINEAR BOUNDARY-VALUE PROBLEMS

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ABSTRACT. We consider self-mappings of Hausdorff topological spaces which map compact sets to compact sets and establish the existence of invariant (fixed) sets. The fixed set results are used to provide fixed set analogues of well-known fixed point theorems. An algorithm is employed to compute the existence of fixed sets which are self-similar in a generalized sense. Some numerical examples are given. The utility of the abstract result is further illustrated via the study of a boundary value problem for a system of differential equations

1. INTRODUCTION

Suppose (\mathbb{M}, d) is a complete metric space, and

$$f_i : \mathbb{M} \rightarrow \mathbb{M}, \quad i = 1, 2, \dots, n,$$

are contraction mappings. It follows from the contraction mapping principle (Banach fixed point theorem) (see e.g. [2, 8, 4]) that each f_i has a unique fixed point. If we consider the *function system*

$$F(X) := \cup_{i=1}^n f_i(X), \quad X \subseteq \mathbb{M}, \quad (1.1)$$

then a theorem of Hutchinson [10] (see also [1, 2, 16]) says that if we restrict F to \mathcal{H} , the collection of nonempty, compact subsets of \mathbb{M} endowed with the Hausdorff metric (more details and definitions are given in Section 6 below), then F is a contraction mapping on a complete metric space and hence has a unique fixed point (set) in \mathcal{H} . That is, there exists a nonempty, closed, and bounded set $B \subseteq \mathbb{M}$ such that

$$F(B) = \cup_{i=1}^n f_i(B) = B.$$

Notice that B is the union of images of itself; should the set-up be such that each f_i is a similarity transformation, one concludes that B is the union of sets similar to itself. Since, for contraction mappings, the unique fixed point may be computed using an iteration scheme, Hutchinson's theorem has given rise to the computation of self-similar sets using iterated function systems. This has been used effectively for the representation and computation of many fractal sets (again, see [1, 16]).

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In the following sections, we will develop an extension of Hutchinson's theorem for mappings which take compact subsets of Hausdorff topological spaces (see [11]) to compact subsets of these spaces. More precisely we shall discuss the following result and variants thereof.

Theorem 1.1. *Let \mathbb{M} be a Hausdorff topological space and let F be a mapping $F : \mathcal{C} \rightarrow \mathcal{C}$, where*

$$\mathcal{C} := \{A \subseteq \mathbb{M} : A \text{ is compact, } A \neq \emptyset\},$$

and F is a monotone mapping; i.e.,

$$F(A) \subseteq F(B), \quad \text{whenever } A \subseteq B, \quad A, B \in \mathcal{C}.$$

If there exists $A \in \mathcal{C}$ such that $F(A) \subseteq A$, then there exists $B(\subseteq A) \in \mathcal{C}$ such that

$$F(B) = B;$$

i.e., there exists a nonempty compact set B which is a fixed set for F .

Remark 1.2. Note that in the above theorem, it is not required that F satisfy any continuity properties, except that it map compact sets to compact sets.

As a special case of this result we have the following result.

Theorem 1.3. *Suppose \mathbb{M} is a Hausdorff topological space and*

$$f_i : \mathbb{M} \rightarrow \mathbb{M}, \quad i = 1, 2, \dots, n,$$

is a collection of continuous functions. If there exists a nonempty compact set $C \subseteq \mathbb{M}$ such that

$$f_i(C) \subseteq C, \quad i = 1, 2, \dots, n,$$

then there exists a nonempty compact set $B \subseteq C \subseteq \mathbb{M}$ such that

$$F(B) := \cup_{i=1}^n f_i(B) = B. \tag{1.2}$$

Remark 1.4. In the above result, the requirement that each f_i be a continuous function, may be replaced by the requirement that each f_i be a mapping of the type given in Theorem 1.1. The fixed set given by these theorems may, however, not be unique.

We further develop an iteration scheme that 'converges' to the fixed sets given by the theorem in an interesting way. The sets computed show a fractal-like structure.

There exists a scattered set of results of this type guaranteeing the existence of sets which are fixed sets for such mappings. We shall cite several of those as illustrative examples of Theorem 1.1 below. Our purpose also is to give a partial survey of fixed set results and show how iteration schemes may be devised to generate fixed sets which have self-similarity properties as discussed above. For $n = 1$, the result, Theorem 1.1 is stated as Exercise 13, page 84 of [4] and was previously used in [3] as communicated to us by Professor R. Cain in [5]. Fixed set theorems of the above type have found applications in various disciplines, see, e.g., [19], where applications to interval arithmetic are given, and [13], where applications to economics and game theory are discussed. The requirement that the underlying topological space be a Hausdorff space may be relaxed and more general theorems may be obtained. For such results we refer the interested reader to the notes by Ok [15], where many interesting and related fixed point theorems (e.g. the Theorem of Abian-Brown) are developed.

The paper is organized as follows. We first give a couple of examples to show that there are T_1 topological spaces and mappings satisfying the assumptions of Theorem 1.1, which, however, may have a fixed compact set or not. We then proceed to give a proof of Theorem 1.1 using the Axiom of Choice. Next we provide an alternate and constructive proof of Theorem 1.3. Many of the standard fixed point theorems (such as Brouwer's and Schauder's) have obvious fixed set analogues; several of such are given. Using the constructive approach to the proof of Theorem 1.3 algorithms may be devised to compute fixed sets (in fact the iteration schemes given yield fixed sets for any initial choice for the scheme). We provide three examples of computations. In the final section of the paper we illustrate the use of the abstract result, Theorem 1.1, by using it to study a boundary value problem given by a system of second order ordinary differential equations subject to Dirichlet boundary conditions. The example studied also suggests avenues for the study of more general boundary value problems for nonlinear elliptic partial differential equations.

2. WHY HAUSDORFF SPACES?

The question arises whether it is necessary to assume that the topological space \mathbb{M} be a Hausdorff space, in order to have Theorem 1.1 hold for a function $f : \mathbb{M} \rightarrow \mathbb{M}$ or whether weaker separation assumptions suffice. Below we give two examples which illustrate that there are T_1 spaces in which functions exist which satisfy our assumptions, yet which do not leave any compact set fixed, and also that there are T_1 spaces in which the theorem holds.

2.1. Example. In the following we let $\mathbb{N} := \{1, 2, 3, \dots\}$,

$$\Upsilon := \{\emptyset\} \cup \{A \subseteq \mathbb{N} : A^c \text{ is finite}\}$$

(here for a given set A the set A^c is the complement of A). We easily see that (\mathbb{N}, Υ) is a topological space; it has the following properties:

Proposition 2.1. (1) (\mathbb{N}, Υ) is a T_1 space (points are closed), but it is not a T_2 (Hausdorff) space.

- (2) The closed subsets are \mathbb{N} and all finite subsets.
- (3) If $A \subseteq \mathbb{N}$ is not finite, then $\overline{A} = \mathbb{N}$.
- (4) No two non-empty open subsets of \mathbb{N} are disjoint.
- (5) Every subset of \mathbb{N} is compact.
- (6) $f : \mathbb{N} \ni n \mapsto n + 1 \in \mathbb{N}$ is a continuous function.
- (7) $f(A) \subseteq A$, if and only if, A is inductive; i.e.,

$$A = \{n : n \geq k \text{ for some } k \in \mathbb{N}\}.$$

- (8) If A is a nonempty subset of \mathbb{N} , then $f(A) \neq A$.

We leave the proof to the reader, but we remark that all inductive subsets of \mathbb{N} are mapped into themselves by the function f , yet there is no fixed set for f .

2.2. Example. Let $X_0 = (0, 1]$, $P = \{p, q\}$, where $p, q \in \mathbb{R}^2$ are given by $p = (0, 1)$, $q = (0, -1)$. Finally let $X = X_0 \cup P$. For $x \in X$ we define neighborhoods as follows:

$$\begin{aligned} x \in X_0, U(x) &:= \{(x - \epsilon, x + \epsilon) \cap X_0 : \epsilon > 0\}, \\ x \in P, U(x) &:= \{\{x\} \cup (0, \epsilon) : \epsilon > 0\}, \end{aligned}$$

and let Υ be the topology generated by these neighborhoods. One verifies that (X, Υ) is a T_1 space (points are closed) and except for the points p and q all points may be separated by neighborhoods (hence, the space is “almost” a T_2 space).

- Remark 2.2.** (1) X is a compact space with respect to the topology Υ .
 (2) X_0 , with respect to the relative topology may be identified with $(0, 1] \subset \mathbb{R}$ with respect to the relative topology induced by the topology of \mathbb{R} .
 (3) For $x = p$ or $x = q$ the space $X_0 \cup \{x\}$ may be identified with $[0, 1] \subset \mathbb{R}$ with its “usual” topology.

Proposition 2.3. *A set $A \subseteq X$ is compact, if and only if, $A \cap X_0$ is closed in X_0 and either*

- $A \cap X_0$ is compact; or
- $A \cap X_0$ is not compact, but $A \cap P \neq \emptyset$.

For $x = p$ or $x = q$ the set $X_0 \cup \{x\}$ is compact but not closed (note that $q \in X_0 \cup \{p\}$).

We again leave the proof to the reader. The main fact of this section is as follows.

Proposition 2.4. *Let $f : X \rightarrow X$ be a continuous function. Then there exists a compact set $K \subseteq X$ such that*

$$f(K) = K.$$

Proof. If $f(P) \subseteq P$, then either $f(p) = p$, or $f(p) = q$. In the first case p is a fixed point, hence a fixed set and in the second, either $f(q) = q$ or $f(q) = p$. Thus, in any case, either f has a fixed point or the fixed set P . If P is not mapped into itself by f , then $f(p) = a \in (0, 1]$. It follows that $f(q) = a$, as well, and

$$\lim_{x \rightarrow 0^+} f(x) = a.$$

If $f(1) < 1$, then by continuity, there must exist $x \in (0, 1)$ such that $f(x) = x$. If $f(1) = 1$, then 1 is a fixed point for f . Hence again, in either case, f has a fixed point and consequently a fixed set. \square

3. PROOFS

3.1. Proof of Theorem 1.1.

Proof. Let \mathcal{C} be the collection of all nonempty compact subsets of \mathbb{M} and let \mathbb{C} the subcollection of \mathcal{C} such that

$$F(B) \subseteq B, \quad \forall B \in \mathbb{C}.$$

Then, by hypothesis, the collection \mathbb{C} is not empty. We partially order \mathbb{C} as follows

$$C_1 \prec C_2 \iff C_2 \subseteq C_1, \quad C_1, C_2 \in \mathbb{C}.$$

According to the Hausdorff maximal principal (equivalently Zorn’s lemma) (see [11, 17]), there exists a maximal linearly ordered subcollection $\{B_\alpha : \alpha \in I\}$, where I is an index set. We let

$$B = \bigcap_{\alpha \in I} B_\alpha.$$

Then, since B_α is compact $\forall \alpha \in I$ and the subcollection is linearly ordered, it follows that B is nonempty and compact (the space \mathbb{M} is a Hausdorff topological

space!). On the other hand, since $F(B_\alpha) \subseteq B_\alpha$ for all $\alpha \in I$, we have that $F(B) \subseteq B$ and hence,

$$F(F(B)) \subseteq F(B) \subseteq B,$$

(recall that F is a monotone mapping). Also, $F(B)$ is compact; hence, by the maximality of the subcollection, it must be the case that $F(B) = B$. \square

3.2. Proof of Theorem 1.3.

Proof. We note that, since each f_i , $i = 1, \dots, n$, is a continuous function, it maps compact sets into compact sets, and for each compact subset A of \mathbb{M}

$$F(A) = \bigcup_{i=1}^n f_i(A)$$

is a compact set. We hence may apply Theorem 1.1 to complete the proof. \square

3.3. An alternate proof of Theorem 1.3. We next provide a proof of Theorem 1.3 that is constructive, and hence provides us with insight into the nature of the fixed sets.

The following two lemmas are both consequences of the fact that a continuous function maps compact sets into compact sets.

Lemma 3.1. *Let f_i , $i = 1, \dots, n$, and F be as in Theorem 1.3. If $\{S_j\}$ is a nested sequence of nonempty compact sets, then*

$$F(\bigcap_{j=1}^{\infty} S_j) = \bigcap_{j=1}^{\infty} F(S_j).$$

Proof. If $y \in F(\bigcap_{j=1}^{\infty} S_j)$, then there exists $x \in \bigcap_{j=1}^{\infty} S_j$ such that $y \in F(x)$. Hence $y \in F(S_j)$, $j = 1, 2, 3, \dots$; i.e., $y \in \bigcap_{j=1}^{\infty} F(S_j)$, and therefore $F(\bigcap_{j=1}^{\infty} S_j) \subseteq \bigcap_{j=1}^{\infty} F(S_j)$. If $y \in \bigcap_{j=1}^{\infty} F(S_j)$, then, for each $j = 1, 2, 3, \dots$, there exists i_j , $1 \leq i_j \leq n$ and $x_{j,i_j} \in S_j$ such that $f_{i_j}(x_{j,i_j}) = y$. Consider the sequence $\{x_{j,i_j}\}$. Since the sequence $\{S_j\}$ is a nested sequence, the sequence is contained in S_1 . Since S_1 is a compact set, $\{x_{j,i_j}\}$ must have a cluster point $x \in S_1$. It is now an easy argument to conclude that, in fact, $x \in \bigcap_{j=1}^{\infty} S_j$ and there exists i , $1 \leq i \leq n$ such that $f_i(x) = y$. Therefore $y \in f_i(\bigcap_{j=1}^{\infty} S_j)$, $y \in F(\bigcap_{j=1}^{\infty} S_j)$, and $\bigcap_{j=1}^{\infty} F(S_j) \subseteq F(\bigcap_{j=1}^{\infty} S_j)$. \square

We shall also need the following lemma.

Lemma 3.2. *Let F be as above. If $S \subseteq \mathbb{M}$ is such that \bar{S} is compact, then $F(\bar{S}) = \overline{F(S)}$.*

Proof. A continuous function maps compact sets into compact sets, so $F(\bar{S}) = \bigcup_{i=1}^n f_i(\bar{S})$ is the union of compact sets and hence compact. Compact sets are closed, so $F(\bar{S}) = \overline{F(\bar{S})}$. It is clear that $\overline{F(S)} \subseteq \overline{F(\bar{S})}$, and therefore $\overline{F(S)} \subseteq F(\bar{S})$.

If $x \in S$, then $F(x) \subseteq \overline{F(S)}$. If $x \in \bar{S} \setminus S$, then x is a limit point of S . A continuous function maps limit points into limit points, so $F(x)$ is a collection of limit points of $F(S)$. Therefore $F(x) \subseteq \overline{F(S)}$ for all $x \in \bar{S}$, and $F(\bar{S}) \subseteq \overline{F(S)}$. \square

We are now ready to reprove Theorem 1.3.

Proof. Suppose A is a compact subset of C . We define

$$B_n := \overline{\bigcup_{i=n}^{\infty} F^i(A)}, \quad B := \bigcap_{n=1}^{\infty} B_n.$$

We claim that $B \neq \emptyset$. Since closed subsets of compact sets are compact, it follows that $B_n \subseteq C$ is compact for $n = 1, 2, \dots$. It is clear that

$$B_1 \supseteq B_2 \supseteq \dots \supseteq B_n \supseteq \dots$$

Further, since the intersection of a nested sequence of nonempty compact sets is nonempty (the space is a Hausdorff space!), we conclude that

$$\bigcap_{n=1}^{\infty} B_n = B \neq \emptyset.$$

The following shows that $F(B) = B$.

$$\begin{aligned} F(B) &= F(\bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} F^i(A)}) \\ &= \bigcap_{n=1}^{\infty} F(\overline{\bigcup_{i=n}^{\infty} F^i(A)}) \quad (\text{by Lemma 3.1}) \\ &= \bigcap_{n=1}^{\infty} \overline{F(\bigcup_{i=n}^{\infty} F^i(A))} \quad (\text{by Lemma 3.2}) \\ &= \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} F^{i+1}(A)} \\ &= \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n+1}^{\infty} F^i(A)} = B. \end{aligned}$$

□

Remark 3.3. In the above proof we have shown, that for any compact subset $A \subseteq C$, the set B , defined above (called the ω limit set of A with respect to F in the theory of dynamical systems, see, e.g., [9]) defines a fixed set for the mapping F . If it is the case that $B \subseteq A$ (e.g., if $A = C$), then it is the case that

$$B = \bigcap_{n=1}^{\infty} F^n(A).$$

Proof. Since $F^n(B) = B$, $n = 1, \dots$, it follows that $B \subseteq F^n(A)$, $n = 1, 2, \dots$, and hence

$$B \subseteq \bigcap_{n=1}^{\infty} F^n(A).$$

However,

$$F^n(A) \subseteq \bigcup_{i=n}^{\infty} F^i(A) \subseteq B_n,$$

for $n = 1, 2, \dots$. Hence,

$$\bigcap_{n=1}^{\infty} F^n(A) \subseteq \bigcap_{n=1}^{\infty} B_n = B.$$

□

Remark 3.4. In the particular case that $A = C$, we have that

$$\dots \subseteq F^{i+1}(C) \subseteq F^i(C) \subseteq \dots \subseteq F(C) \subseteq C,$$

and hence we need to compute the asymptotic behavior (as $n \rightarrow \infty$) of $F^n(C)$, whereas, if $B \subseteq A$, then the asymptotic behavior of $\bigcap_{i=1}^n F^i(A)$ will determine the limit set B .

4. RESULTS MOTIVATED BY FIXED POINT THEOREMS

Fixed point theorems offer important tools in the study of nonlinear equations, be they algebraic, differential, or integral equations (see [8]). We state here, for comparison, Brouwer's and Schauder's fixed point theorems and conclude with a theorem of Krasnosel'skii and related fixed point theorems and provide some fixed set analogues of these (see again [8]).

For finite dimensional spaces, we have Brouwer's fixed point theorem:

Theorem 4.1. *Let $B \subset \mathbb{R}^N$ be a nonempty compact convex set (or a set homeomorphic to such) and let $f : B \rightarrow B$ be a continuous function. Then f has a fixed point in B ; i.e., there exists $x \in B$ such that $f(x) = x$.*

The fixed set analogue is given by:

Theorem 4.2. *Let $B \subset \mathbb{R}^N$ be a nonempty compact set and let $f : B \rightarrow B$ be a continuous function. Then f has a fixed set in B ; i.e., there exists a compact set $A \subset B$ such that $f(A) = A$.*

For infinite dimensional spaces we have Schauder's fixed point theorem:

Theorem 4.3. *Let \mathbb{E} be a Banach space and let $B \subset \mathbb{E}$ be a nonempty compact convex set (or a set homeomorphic to such). Assume $f : B \rightarrow B$ is a continuous function. Then f has a fixed point in B ; i.e., there exists $x \in B$ such that $f(x) = x$.*

Its fixed set analogue is given by:

Theorem 4.4. *Let \mathbb{E} be a Banach space and let $B \subset \mathbb{E}$ be a nonempty compact set. Assume $f : B \rightarrow B$ is a continuous function. Then f has a fixed set in B ; i.e., there exists a compact set $A \subset B$ such that $f(A) = A$.*

A mapping $f : \mathbb{E} \rightarrow \mathbb{E}$ is called *compact* if it maps bounded sets into sets with compact closure; it is called *completely continuous* if it is both continuous and compact. For such mappings we have the following version of Schauder's fixed point theorem:

Theorem 4.5. *Let \mathbb{E} be a Banach space and let $B \subset \mathbb{E}$ be a nonempty closed and bounded convex set (or a set homeomorphic to such). Assume $f : B \rightarrow B$ is a completely continuous function. Then f has a fixed point in B ; i.e., there exists $x \in B$ such that $f(x) = x$.*

A fixed set analogue is the following result:

Theorem 4.6. *Let \mathbb{E} be a Banach space and let $B \subset \mathbb{E}$ be a nonempty closed and bounded set. Assume $f : B \rightarrow B$ is a completely continuous mapping. Then f has a fixed set in B ; i.e., there exists a compact subset $A \subset B$ such that $f(A) = A$.*

Proof. Since f is a compact mapping, it follows that $\overline{f(B)} \subset B$ is a compact set. We also have that

$$f : \overline{f(B)} \rightarrow \overline{f(B)}$$

and hence, by Theorem 1.1, f has a fixed compact set in $\overline{f(B)}$. \square

A fixed point theorem due to Krasnosel'skii is the following:

Theorem 4.7. *Let \mathbb{E} be a Banach space and let $B \subset \mathbb{E}$ be a nonempty closed and bounded convex set. Assume*

$$f_1(B) + f_2(B) \subseteq B,$$

where f_1 is a contraction mapping and f_2 is a completely continuous function. Then $f := f_1 + f_2$ has a fixed point in B , i.e., there exists $x \in B$ such that $f(x) = x$.

A fixed set analogue of this result has been considered by Ok in [14] (see also [13]). We state here one such possible version.

Theorem 4.8. *Let \mathbb{E} be a Banach space and let $B \subset \mathbb{E}$ be a nonempty closed and bounded set. Assume*

$$f_1(B) + f_2(B) \subseteq B, \tag{4.1}$$

where f_1 is a contraction mapping and f_2 is a completely continuous function. Then, f , defined by

$$f(X) := f_1(X) + f_2(X), \quad X \subseteq B,$$

has a fixed set in B , i.e., there exists a set $A \subset B$ such that

$$f(A) = f_1(A) + f_2(A) = A.$$

Proof. Since f_1 is a contraction mapping, for any fixed $z \in B$, $f_1 + f_2(z)$ is a contraction mapping, as well. Because of (4.1),

$$f_1 + f_2(z) : B \rightarrow B;$$

hence, if we let id denote the identity mapping, then, since the mapping

$$\text{id} - f_1 : \mathbb{E} \rightarrow \mathbb{E}$$

is continuous bijective with a continuous inverse, the mapping

$$h := (\text{id} - f_1)^{-1} f_2 : \mathbb{E} \rightarrow \mathbb{E},$$

being the composition of a continuous with a completely continuous mapping, is completely continuous and satisfies $h(B) \subseteq B$. It follows from Theorem 4.6 that there exists a nonempty compact set $K \subseteq B$ such that

$$h(K) = K. \quad (4.2)$$

From which, we conclude that

$$K \subseteq f(K) = f_1(K) + f_2(K)$$

and, since $K \subseteq B$, $f(K) \subseteq f(B)$. Hence, for $n, m = 2, 3, \dots$,

$$K \subseteq f(K) \subseteq \dots \subseteq f^n(K) \subseteq \dots \subseteq f^m(B) \subseteq \dots \subseteq f(B) \subseteq B. \quad (4.3)$$

We therefore obtain that

$$\cup_{n=1}^{\infty} f^n(K) =: C \subseteq \cap_{n=1}^{\infty} f^n(B).$$

It follows from (4.3) that $f(C) \subseteq C$, and hence, since f is a monotone mapping,

$$f^n(K) \subseteq f^n(C) \subseteq f(C), \quad n = 1, 2, \dots,$$

and consequently $C \subseteq f(C)$; i.e. C is a fixed set for f .

We note immediately that f_1 may be replaced by any continuous function with the property that $\text{id} - f_1 : \mathbb{E} \rightarrow \mathbb{E}$ be bijective having a continuous inverse and

$$(\text{id} - f_1)^{-1} f_2 : B \rightarrow B.$$

The proof above uses arguments, as originally used by Kransosel'skii (see also [14]).

□

Remark 4.9. Theorem 4.7 still holds if we replace (4.1) by the more general condition

$$(f_1 + f_2)(B) \subseteq B$$

(see e.g. [8]). Whether or not an analogue like Theorem 4.8 may be obtained under such a hypothesis, is an open question. Also it is not known what the topological properties of the fixed set C are.

Remark 4.10. We note that more general fixed set theorems may be obtained by replacing in a suitable manner the functions used above by function systems as in Theorem 1.3.

5. A THEOREM ON PERIODIC POINTS

The next example concerns a problem posed by Stefanov [18]; its solution was published in [6]. We present here a solution using an argument based on Theorem 1.1 and on [6].

Theorem 5.1. *Let \mathbb{M} be a countable compact Hausdorff topological space and let $f : \mathbb{M} \rightarrow \mathbb{M}$ be a continuous mapping. Then there exists $p \in \mathbb{M}$ and $m \in \{1, 2, 3, \dots\}$ such that*

$$f^m(p) = p;$$

i.e., f has a periodic point.

Proof. Choose $x \in \mathbb{M}$ and denote by $A(x)$ the orbit of x under f ; i.e.,

$$A(x) := \{x, f(x), f^2(x), f^3(x), \dots\}.$$

If $A(x)$ is a finite set, x is a periodic point for f . If $A(x)$ is not finite, then $B(x)$, the set of all cluster points of $A(x)$ is not empty and is a compact set. Furthermore, it is the case that

$$f(B(x)) \subseteq B(x).$$

Hence, it follows from Theorem 1.1 that there exists a compact set $K \subseteq B(x)$ such that $f(K) = K$. It also follows from the proof of Theorem 1.1 (in particular the Hausdorff maximal principal) that the set K is minimal, in the sense that if $A \subseteq K$ is a compact set such that $f(A) \subseteq A$, then $A = K$. It follows from [12, Theorem 6.5], that K must have an isolated point, say y . If y is a periodic point, then the proof is complete. If y is not a periodic point for f , then $B(y) \neq \emptyset$ and $y \notin B(y)$ and hence, $B(y)$ is a proper subset of K . On the other hand, $f(B(y)) \subseteq B(y)$, contradicting the minimality of K , as described above.

We note that, since the point set determined by a periodic orbit of f is invariant under the mapping f , it must be the case that K is, in fact, the point set of a periodic orbit of f . \square

6. NUMERICAL EXAMPLES

The iteration scheme laid out by Hutchinson in [10] has led to the computation of many beautiful self-similar sets (see [1, 16]). Most of these constructions have consisted of affine linear contraction mappings in the plane, although Hutchinson's theorem applies to any function system defined by finitely many contraction mappings. The proof of Theorem 1.3 is constructive in nature, so the question arises if there a similar algorithm to compute the fixed sets of the functions described in the theorem. Unfortunately, we have no such algorithm, but we do have a simple iteration scheme that may give a "glimpse into the nature" of the fixed sets given by the theorem. For this purpose we will assume that we are working in a complete metric space.

First, we will need to make precise what we mean by the distance between two (closed and bounded) sets. For a more detailed development and proofs of the remarks see [1, 10, 16].

Definition 6.1. Suppose A, B are in \mathcal{H} . We define, for $\epsilon > 0$,

$$A_\epsilon = \{x \in \mathbb{M} : d(x, y) < \epsilon \text{ for some } y \in A\},$$

$$D(A, B) := \inf\{\epsilon : A \subset B_\epsilon\}.$$

Proposition 6.2. *If $A, B \in \mathcal{H}$, $h : \mathcal{H} \times \mathcal{H} \rightarrow [0, \infty)$ is defined by*

$$h(A, B) := \max\{D(A, B), D(B, A)\}.$$

Then h is a metric on \mathcal{H} (the Hausdorff metric). Additionally, if (\mathbb{M}, d) is a complete metric space, then (\mathcal{H}, h) is a complete metric space.

Remark 6.3. The Hausdorff metric may be defined in an alternative but equivalent way which is often useful in simplifying certain proofs. Namely, if we define

$$d(x, A) := \inf\{d(x, y) \mid y \in A\},$$

then

$$D(A, B) := \sup\{d(x, B) \mid x \in A\}.$$

Remark 6.4. If $A, B \in \mathcal{H}$ and $A \subseteq B$, then $h(A, B) = D(B, A)$.

Lemma 6.5. *Suppose $\{B_n\}$ is a nested sequence of nonempty compact sets in a complete metric space. Then*

$$\lim_{n \rightarrow \infty} B_n = \bigcap_{i=1}^{\infty} B_i = B$$

(with respect to the Hausdorff metric h).

Proof. It suffices to show $h(B, B_n)$, which equals $D(B_n, B)$, since $B \subseteq B_n$ for each $n = 1, 2, \dots$, converges to zero. We need to show that, given $\epsilon > 0$, we can find an integer k such that

$$D(B_k, B) = \inf\{\delta > 0 : B_b \subseteq B_\delta\} < \epsilon.$$

Since for all $n \geq k$ we have $B \subseteq B_n \subseteq B_k$ this will imply that for all $n \geq k$ $h(B, B_n) < \epsilon$.

We observe that B_ϵ is open, so that $S_n := B_n \setminus B_\epsilon$ is compact for every n . Further $\{S_n\}$ is a decreasing sequence of compact subsets of the compact set B_1 and

$$\bigcap_{n=1}^{\infty} S_n = \bigcap_{n=1}^{\infty} B_n \setminus B_\epsilon = B \setminus B_\epsilon = \emptyset.$$

Therefore, there exists k such that $S_k = \emptyset$; i.e., $B_k \subseteq B_\epsilon$. This says that

$$\inf\{\delta > 0 : B_k \subseteq B_\delta\} < \epsilon$$

and so $h(B, B_k) < \epsilon$, from which the conclusion $h(B, B_n) \rightarrow 0$ follows. \square

Let F be a function system satisfying the conditions of Theorem 1.3, with $F(C) \subseteq C$ and $A \subseteq C$.

Proposition 6.6. *The set $\overline{F^n(A)}$ is related to the set*

$$B := \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} F^i(A)}$$

(which is invariant under F) for “large values of n ” in the following sense:

- (1) *If $\epsilon > 0$, then there exists a positive integer N such that for each $x \in \overline{F^n(A)}$, $d(x, B) < \epsilon$ whenever $n > N$; i.e.,*

$$\lim_{n \rightarrow \infty} D(\overline{F^n(A)}, B) = 0.$$

- (2) *If $x \in B$, then there exists a sequence $\{x_n\}$ with $x_n \in \overline{F^n(A)}$ for $n = 1, 2, \dots$, such that some subsequence of $\{x_n\}$ converges to x .*

Proof. (1) First note that $\overline{F^n(A)} \subseteq B_n = \overline{\cup_{i=n}^\infty F^i(A)}$, which implies that

$$D(\overline{F^n(A)}, B) \leq D(B_n, B),$$

whereas it follows from Lemma 6.5 that $\lim_{n \rightarrow \infty} h(B_n, B) = 0$, and so

$$\lim_{n \rightarrow \infty} D(B_n, B) = 0.$$

(2) If $x \in B$, then $x \in B_n = \overline{\cup_{i=n}^\infty F^i(A)}$ for $n = 1, 2, \dots$. We construct, inductively, a sequence as follows: Since $x \in \overline{\cup_{i=1}^\infty F^i(A)}$, there exists an integer k_1 and $x_{k_1} \in F^{k_1}(A)$ such that $d(x, x_{k_1}) < 1$. Since $x \in \overline{\cup_{i=k_1+1}^\infty F^i(A)}$, there exists an integer $k_2 > k_1$ and $x_{k_2} \in F^{k_2}(A)$ such that $d(x, x_{k_2}) < 1/2$. And, inductively, there exists an integer k_n such that $k_n > k_{n-1}$ and there exists $x_{k_n} \in F^{k_n}(A)$ such that $d(x_{k_n}, x) < 1/n$. In this manner, we build a sequence such that $x_{k_i} \rightarrow x$ as $i \rightarrow \infty$ and each $x_{k_i} \in F^{k_i}(A)$. \square

We shall consider here three different examples of function systems for which we have implemented the iteration process, given above, using MAPLE. That is, we chose an appropriate (may be chosen arbitrarily) initial set A and compute $\overline{F^n(A)}$ for a given value of n . This set is related to a fixed set of the function F in the sense of Proposition 6.6.

As a first example consider the functions $f_1, f_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined, respectively, by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto 0.5 \begin{pmatrix} \sin 2x + \cos 2y \\ \cos 2x + \sin 2y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto 0.5 \begin{pmatrix} -\sin 2x + \cos 2y \\ \cos 2x + \sin 2y \end{pmatrix}.$$

Let

$$C := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

It is easy to verify that $f_1(C) \subseteq C$ and that $f_2(C) \subseteq C$. Both f_1 and f_2 are continuous, and so by Theorem 1.3 the function system

$$F(X) := f_1(X) \cup f_2(X)$$

has a fixed set. For the computation we have conveniently chosen A to be a singleton set $A = \{(0.5, 0.5)\}$, and $n = 14$. Computing the set $\overline{F^{14}(A)}$ gives us Figure 1. It should be stressed that this “fixed set” obtained for the function system may not be unique.

As a second example consider the following two continuous functions f_1, f_2 which map the unit square in \mathbb{R}^2 into itself.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y/(1+x^2) \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x(1-y) \\ x/(1+y^2) \end{pmatrix}.$$

Figures 2, 3, 4 show computations of $\overline{F^n(A)}$ using

$$F(X) := f_1(X) \cup f_2(X)$$

and initial sets $A = \{(0.1, 0.9)\}$, $A = \{(0.5, 0.5)\}$, and $A = \{(0.25, 0.1)\}$ with $n = 14$.

Our last example uses a function system consisting of three functions $f_1, f_2, f_3 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto 0.5 \begin{pmatrix} \sin x + \cos y \\ \cos x - \sin y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto 0.5 \begin{pmatrix} -\sin x + \cos y \\ \cos x + \sin y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -x \end{pmatrix}.$$

We use the initial set $A = \{(0.25, 0.1)\}$, and compute $\overline{F^{10}(A)}$.

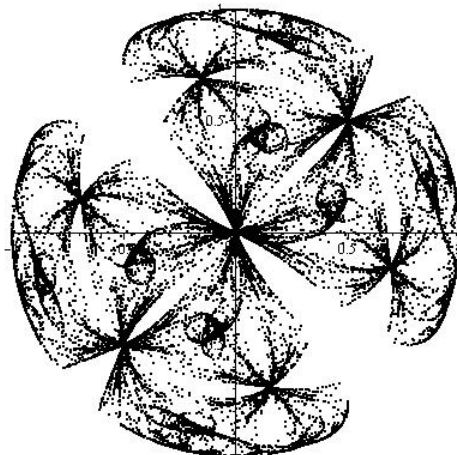


FIGURE 1. Set $\overline{F^{14}(A)}$ for initial set $A = \{(0.5, 0.5)\}$

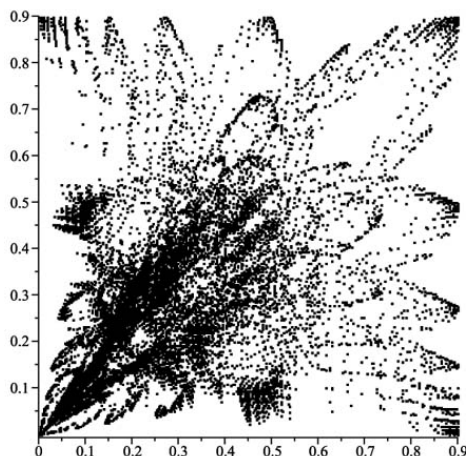


FIGURE 2. Set $\overline{F^n(A)}$ for initial set $A = \{(0.1, 0.9)\}$

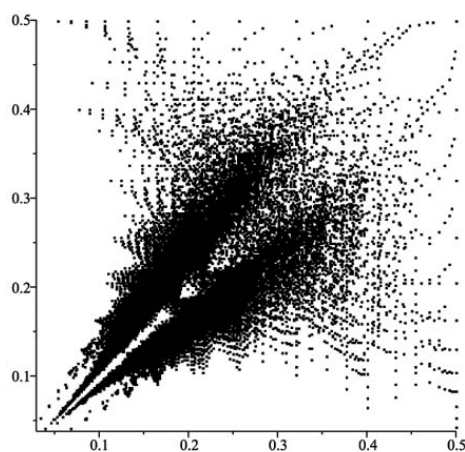
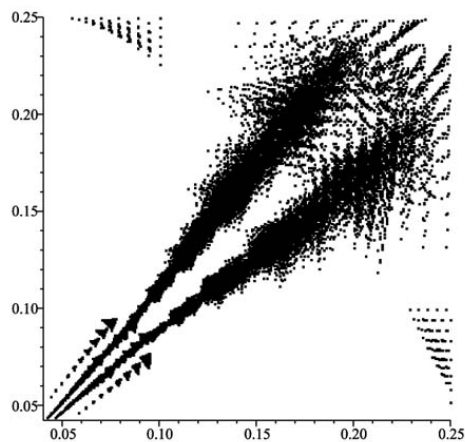
7. A BOUNDARY-VALUE PROBLEM

In this section we shall consider an application of Theorem 1.1 to the study of boundary value problems for ordinary differential equations. The example given is very specific and has been constructed to illustrate the utility of that theorem; we do not strive for generality but indicate that much more general results for local and nonlocal problems for nonlinear elliptic partial differential equations may be obtained in this fashion.

We consider the boundary-value problem

$$-u'' + u^3 = h, \quad u(0) = 0 = u(1), \quad (7.1)$$

$$-v'' + v^5 = h, \quad v(0) = 0 = v(1), \quad (7.2)$$

FIGURE 3. Set $\overline{F^n(A)}$ for initial set $A = \{(0.5, 0.5)\}$ FIGURE 4. Set $\overline{F^n(A)}$ for initial set $A = \{(0.25, 0.1)\}$

where $h \in C[0, 1]$ is a given function. We shall consider this problem in the space $C[0, 1]$ endowed with the usual norm $\|u\| = \max_{[0,1]} |u(x)|$.

It follows from basic existence theorems based on sub-supersolution methods (upper and lower solution methods) (see [7]) that both equations have solutions contained in $C^2[0, 1]$. Using the boundary conditions, and the fact that solutions are weak solutions, as well, we obtain the following a priori bounds

$$\|u\|^2 \leq \|u'\|_{L^2}^2 \leq \|h\|_{L^2} \|u\|_{L^2} \leq \|h\| \|u\|$$

from which follows that $\|u\| \leq \|h\|$, and, mutatis mutandis, $\|v\| \leq \|h\|$.

For a given h we let

$$f_1(h) = \{u : u \text{ solves (7.1)}\}, \quad f_2(h) = \{v : v \text{ solves (7.2)}\},$$

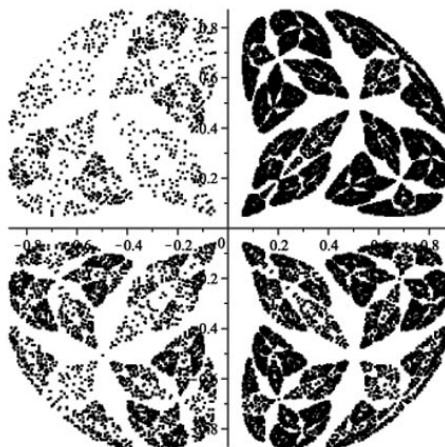


FIGURE 5. Set $\overline{F^{10}(A)}$ for initial set $A = \{(0.25, 0.1)\}$

then $f_1(h) \neq \emptyset$, $f_2(h) \neq \emptyset$. It follows (the arguments are based on the integral representation of solutions using Green's functions and the Theorem of Ascoli-Arzelà) that f_1 and f_2 map closed and bounded sets to precompact sets.

We hence may conclude that if $B := \{h : \|h\| \leq r\}$, $r > 0$, then

$$\overline{f_1(B)} \cup \overline{f_2(B)} \subseteq B$$

and there exists a compact set A , $A \subset B$, such that

$$f_1(A) \cup f_2(A) = A.$$

We note that the conclusion tells us that each element in the set A must be a solution of either (7.1) or (7.2) for some right hand side from the set A , and conversely, every solution of (7.1) or (7.2), given any right hand side from A , must be an element of A .

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REFERENCES

- [1] M. Barnsley; *Fractals Everywhere*, Academic Press, New York, 1988.
- [2] R. Brooks and K. Schmitt; *The Contraction Mapping Principle And Some Applications*, Electronic J. of Differential Equations, Monograph 09, 2009.
- [3] G. L. Cain and R. H. Kasriel; *Fixed and periodic points of local contractions on probabilistic metric spaces*, Theory of Computing Systems, 9 (1976), 289–297.
- [4] G. L. Cain; *Introduction to General Topology*, Addison Wesley, Reading, 1994.
- [5] G. L. Cain; *Personal communication*, March 22, 2010.
- [6] J. Cobb; *Unavoidable periodicity; Solution of Problem 10476*, Amer. Math. Monthly, 105(1998), 370.
- [7] C. De Coster and P. Habets, *Two-Point Boundary Value Problems: Lower and Upper Solutions*, Elsevier, 2006.
- [8] K. Deimling; *Nonlinear Functional Analysis*, Springer Verlag, Berlin, 1985.

- [9] J. Hale; *Asymptotic Behavior of Dissipative Systems*, Math Survey Monographs, Number 25, American Math. Soc., Providence, 1988.
- [10] J. Hutchinson; *Fractals and self similarity*, Indiana University Mathematics Journal, 30(1981), 713–747.
- [11] J. Kelley; *General Topology*, Van Nostrand Co., Princeton, 1955.
- [12] J. R. Munkres; *Topology: A First Course*, Prentice Hall, 1975.
- [13] E. A. Ok; *Fixed set theory for closed correspondences with applications to self-similarity and games*, Nonlinear Analysis, 56(2004), 309–330.
- [14] E. A. Ok; *Fixed set theorems of Krasnoselskiĭ type*, Proc. Amer. Math. Soc., 137(2009), 511–518.
- [15] E. A. Ok; *Order Theory*, <http://homepages.nyu.edu/~eo1/books.html>.
- [16] H. O. Peitgen, H. Jürgens, and D. Saupe; *Chaos and Fractals: New Frontier of Science*, Springer Verlag, New York, 1992.
- [17] W. Rudin; *Real and Complex Analysis*, McGraw Hill, New York, 1974.
- [18] S. T. Stefanov; *Problem 10476*, Amer. Math. Monthly, 102(1995), 746.
- [19] E. Zeidler; *Nonlinear Functional Analysis and its Applications, Fixed Point Theory*, Springer, New York, 1984.

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