Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 50, pp. 1–6.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

GENERALIZED FRAMEWORKS FOR FIRST-ORDER EVOLUTION INCLUSIONS BASED ON YOSIDA APPROXIMATIONS

RAM U. VERMA

Abstract. First, general frameworks for the first-order evolution inclusions are developed based on the A-maximal relaxed monotonicity, and then using the Yosida approximation the solvability of a general class of first-order nonlinear evolution inclusions is investigated. The role the A-maximal relaxed monotonicity is significant in the sense that it not only empowers the first-order nonlinear evolution inclusions but also generalizes the existing Yosida approximations and its characterizations in the current literature.

1. Preliminaries

The notion of the A-maximal relaxed monotonicity [6] is not only limited to the first-order evolution equations/inclusions in conjunction with Yosida approximations, but goes way beyond, including the fields of optimization and control theory, variational inequality and variational inclusion problems, and unify a greater degree of investigations relating to other fields as well. The obtained results seem to be general in nature, and have a greater potential for applications. For more details, we refer the reader to the references in this article. Consider a real separable Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$.

We study a general class of first-order nonlinear evolution inhomogeneous inclusions of the form

$$\begin{gathered} u'(t) + Mu(t) \ni f(t) \quad \text{for almost all } t \in (0,T), \\ u(0) = u_0, \end{gathered} \tag{1.1}$$

where $M : X \to 2^X$ is a multivalued mapping on X, $f \in W_2^1(0,T;X)$, T is fixed, $0 < T < \infty$, and $u : [0,\infty) \to X$ is a continuous function such that the above inclusion problem holds.

Definition 1.1. Let $M : X \to 2^X$ be a set-valued mapping on a real Hilbert space X, and let $A : X \to X$ be (r)-strongly monotone. Then M is said to be accretive if $R^M_{\rho,A}$ is single-valued and $(\frac{1}{r-\rho m})$-Lipschitz continuous for $r-\rho m > 0$. Furthermore,

2000 *Mathematics Subject Classification.* 49J40, 65B05.
Key words and phrases. Generalized first-order evolution inclusions; variational problems; maximal monotone mapping; A-maximal relaxed monotone mapping; generalized resolvent operator; general Yosida approximations.

Submitted December 25, 2010. Published April 11, 2011.

M is m-accretive (or A-maximal accretive) if M is accretive and $R^M_{\rho,A}$ exists for every $\rho > 0$ on X, where $R^M_{\rho,A}$ is the resolvent of M.

Lemma 1.2. *Let $M : X \to 2^X$ be a set-valued mapping on a real Hilbert space X. Then following properties are equivalent:*

- (i) *M is monotone.*
- (ii) *M is accretive.*

Lemma 1.3. *Let $M : X \to 2^X$ be a set-valued mapping on a real Hilbert space X. Then we have the following implications equivalent:*

- (i) *M is A-maximal relaxed monotone.*
- (ii) *M is monotone and $R(A + \rho M) = X$.*
- (iii) *M is m-accretive.*

We plan to explore the solvability of the inclusion problem (1.1) based on the notion of the A-maximal relaxed monotonicity [6] and the generalized Yosida approximations. The generalized Yosida approximation turns out to be Lipschitz continuous, while we explore the solvability of the inclusion problem (1.1). The obtained results seem to be application-enhanced to problems arising from other fields, including optimization theory, decision and management sciences, engineering science, variational inequality and variational inclusion problems. There are also some detailed results that are investigated on the generalized Yosida approximations to the context of the A-maximal relaxed monotonicity frameworks. For more details, we refer the reader to the references in this article.

2. Auxiliary results

Definition 2.1. Let $A : X \to X$ be an (r)-strongly monotone single-valued mapping and $M : X \to 2^X$ be a set-valued mappings. The map $M : X \to 2^X$ is said to be A-maximal relaxed monotone if

- (i) M is (m)-relaxed monotone; i.e.,
$$\langle u^* - v^*, u - v\rangle \geq -m\|u - v\|^2 \quad \forall (u, u^*), (v, v^*) \in M,$$
- (ii) $R(A + \rho M) = X$ for $\rho > 0$.

Definition 2.2. Let $A : X \to X$ be a single-valued mapping and $M : X \to 2^X$ be a set-valued mapping. Let A be (r)-strongly monotone. The map M is said to be accretive iff $(A + \rho M)^{-1}$ is single-valued and $(A + \rho M)^{-1}$ is $(\frac{1}{r-\rho m})$-Lipschitz continuous for all $\rho > 0$ and $r - \rho m > 0$.

Proposition 2.3 ([6])**.** *Let $A : X \to X$ be a single-valued mapping, and $M : X \to 2^X$ be a set-valued mapping such that $D(A) \cap D(M) \neq \emptyset$. Let A be (r)-strongly monotone, and let M be an A-maximal relaxed monotone mapping. Then the generalized resolvent operator associated with M and defined by*
$$R^M_{\rho,A}(u) = (A + \rho M)^{-1}(u) \quad \forall\, u \in X,$$
is $(\frac{1}{r-\rho m})$-Lipschitz continuous.

Next, we generalize the Yosida approximation M_ρ by $M_\rho = \rho^{-1}(I - AR^M_{\rho,A})$, where $A : X \to X$ is an (r)-strongly monotone mapping on X for $\rho > 0$, and for $R^M_{\rho,A} = (A + \rho M)^{-1}$, which reduces to the Yosida approximation of M for $A = I$:
$$M_\rho = \rho^{-1}(I - R^M_\rho),$$

where I is the identity and $R_\rho^M = (I+\rho M)^{-1}$.

Lemma 2.4. *Let $A: X \to X$ and $M: X \to 2^X$ be mappings such that $D(A) \cap D(M) \neq \emptyset$. Let A be (r)-strongly monotone and $AoR_{\rho,A}^M$ be cocoercive, and let M be an A-maximal relaxed monotone mapping. Then the generalized Yosida approximation M_ρ of M defined by*

$$M_\rho = \rho^{-1}(I - AR_{\rho,A}^M),$$

where

$$R_{\rho,A}^M(u) = (A+\rho M)^{-1}(u) \quad \forall\, u \in D(A)\cap D(M),$$

is $(\frac{1}{\rho})$-Lipschitz continuous.

Proof. For any $u, v \in X$, we have

$$\begin{aligned}
&\langle M_\rho(u) - M_\rho(v), u - v\rangle\\
&= \langle M_\rho(u) - M_\rho(v), \rho[M_\rho(u) - M_\rho(v) - (M_\rho(u) - M_\rho(v))] + u - v\rangle\\
&= \rho\|M_\rho(u) - M_\rho(v)\|^2\\
&\quad - \langle \rho^{-1}[u - v - (AR_{\rho,A}^M(u) - AR_{\rho,A}^M(v))], -(AR_{\rho,A}^M(u) - AR_{\rho,A}^M(v))\rangle\\
&= \rho\|M_\rho(u) - M_\rho(v)\|^2 + \rho^{-1}\langle u - v, AR_{\rho,A}^M(u) - AR_{\rho,A}^M(v)\rangle\\
&\quad - \rho^{-1}\langle AR_{\rho,A}^M(u) - AR_{\rho,A}^M(v), AR_{\rho,A}^M(u) - AR_{\rho,A}^M(v)\rangle\\
&\geq \rho\|M_\rho(u) - M_\rho(v)\|^2 + \rho^{-1}\|AR_{\rho,A}^M(u) - AR_{\rho,A}^M(v)\|^2\\
&\quad - \rho^{-1}\|AR_{\rho,A}^M(u) - AR_{\rho,A}^M(v)\|^2\\
&\geq \rho\|M_\rho(u) - M_\rho(v)\|^2.
\end{aligned}$$

□

For $A = I$, Lemma 2.4 reduces to

Lemma 2.5. Let $M: X \to 2^X$ be a set-valued mapping. Let M be a maximal monotone mapping. Then the Yosida approximation M_ρ of M defined by

$$M_\rho = \rho^{-1}(I - R_\rho^M),$$

where

$$R_\rho^M(u) = (I+\rho M)^{-1}(u) \quad \forall\, u \in D(M),$$

is $(\frac{1}{\rho})$-Lipschitz continuous.

Proposition 2.6. *Let $A: X \to X$ and $M: X \to 2^X$ be mappings such that $D(A)\cap D(M) \neq \emptyset$. Let A be (r)-strongly monotone and $AoR_{\rho,A}^M$ be cocoercive, and let M be an A-maximal relaxed monotone mapping. Then the generalized Yosida approximation M_ρ of M defined by $M_\rho = \rho^{-1}(I - AR_{\rho,A}^M)$, satisfies:*

(i) $M_\rho(u) \in MR_{\rho,A}^M(u)$.
(ii) *M_ρ is A-maximal relaxed monotone.*
(iii) $(M_\lambda)_\mu = M_{\lambda+\mu}$.

Proof. To prove (i), consider

$$w = R_{\rho,A}^M(u) \Rightarrow u \in (A+\rho M)(w) \Rightarrow \rho M_\rho(u) = u - A(w) \in \rho M(w).$$

The proofs of (ii) and (iii) follow, respectively, from the Lipschitz continuity of the generalized resolvent $R_{\rho,A}^M(u)$ and the definition of M_ρ. □

Proposition 2.7. *Let $A : X \to X$ and $M : X \to 2^X$ be mappings such that $D(A) \cap D(M) \neq \emptyset$. Let A be (r)-strongly monotone and $AoR^M_{\rho,A}$ be cocoercive, and let M be an A-maximal relaxed monotone mapping. Then the generalized Yosida approximation M_ρ of M defined by $M_\rho = \rho^{-1}(I - AR^M_{\rho,A})$, satisfies: for all $u \in D(M)$,*

$$M_\rho(u) \to M_0(u), \quad \|M_\rho(u)\| \uparrow \|M_0(u)\| \quad \text{as } \rho \downarrow 0,$$
$$\|M_\rho(u) - M_0(u)\|^2 \leq \|M_0(u)\|^2 - \|M_\rho(u)\|^2 \quad \text{for all } \rho > 0.$$

3. Generalized First-Order Evolution Inclusions

Let $A : X \to X$ be a single-valued mapping, and $M : X \to 2^X$ be a multivalued mapping. In this section, we consider the solvability of first-order nonlinear evolution inclusions of the form

$$\begin{gathered} u'(t) + Mu(t) \ni f(t) \quad \text{for almost all } t \in (0, T) \\ u(0) = u_0, \end{gathered} \tag{3.1}$$

where $M : X \to 2^X$ is a multivalued mapping on X, $f \in W^1_2(0, T; X)$, T is fixed, $0 < T < \infty$, and $u : [0, \infty) \to X$ is a continuous function such that (3.1) holds. Here M is A-maximal relaxed monotone and the Yosida approximation of M is defined by

$$M_\rho = \rho^{-1}(I - A(R^M_{\rho,A})).$$

We consider the main result on the first-order evolution inclusions based on A-maximal relaxed monotonicity framework in conjunction with generalized Yosida approximations.

Theorem 3.1. *Let $A : X \to X$ be (r)-strongly monotone, and let $M : X \to 2^X$ be A-maximal relaxed monotone on a separable Hilbert space X. Let $AoR^M_{\rho,A}$ be cocoercive, where $R^M_{\rho,A}) = (A + \rho M)^{-1}$ for $\rho > 0$. Suppose that the given*

$$u_0 \in D(M), \quad f \in W^1_2(0, T; X)$$

are fixed. Then (3.1) has exactly one solution $u \in W^1_2(0, T; X)$ such that $M : X \to 2^X$ is A-maximal relaxed monotone.

Proof. The proof is based on the results from Section 2, especially Lemma 2.4 and Proposition 2.3. First, we consider the regularized problems

$$u'_\rho(t) + M_\rho u_\rho(t) = f(t), \quad u_\rho(0) = u_0, \rho > 0. \tag{3.2}$$

As the function f is continuous on [0,T] by the hypotheses, and M_ρ is $(\frac{1}{\rho})$-Lipschitz continuous by Lemma 2.4, problems (3.2) can be solved as for first-order evolution equations. To achieve that goal, we need to arrive at *a priori* estimate

$$\|u'_\rho(t)\| \leq C \quad \forall\, \rho > 0,\ t \in [0, T]. \tag{3.3}$$

Now we differentiate (3.2) by setting $g_\rho(t) = M_\rho o u_\rho(t)$ as follows:

$$u''_\rho(t) + g'_\rho(t) = f'(t) \quad \text{for almost all } t. \tag{3.4}$$

Under the hypotheses all the derivatives exist. Since $A \circ R^M_{\rho,A}$ is cocoercive (and hence $A \circ R^M_{\rho,A}$ is nonexpansive), it implies

$$\langle A(R^M_{\rho,A}(u)) - A(R^M_{\rho,A}(u)), u - v\rangle \leq \|u - v\|^2.$$

It follows that M_ρ is monotone, and thus we have

$$\langle M_\rho u_\rho(t+h) - M_\rho u_\rho(t), u_\rho(t+h) - u_\rho(t)\rangle \geq 0.$$

It follows that

$$\langle g'_\rho(t), u'_\rho(t)\rangle \geq 0.$$

Therefore,

$$\begin{aligned}
\langle u''_\rho(t), u'_\rho(t)\rangle &\leq -\langle g'_\rho(t), u'_\rho(t)\rangle + \langle f'(t), u'_\rho(t)\rangle \\
&\leq \langle f'(t), u'_\rho(t)\rangle \\
&\leq \|f'(t)\| \, \|u'_\rho(t)\| \\
&\leq \frac{1}{2}\|f'(t)\|^2 + \frac{1}{2}\|u'_\rho(t)\|^2.
\end{aligned}$$

Applying integration by parts to $\langle u''_\rho(t), u'_\rho(t)\rangle$, we have

$$\begin{aligned}
2\int_0^t \langle u''_\rho(s), u'_\rho(s)\rangle ds &= \|u'_\rho(t)\|^2 - \|u'_\rho(0)\|^2 \\
&\leq \|f\|_Y^2 + \int_0^t \|u'_\rho(s)\|^2 ds,
\end{aligned}$$

where $Y = W_2^1(0,T;X)$. This is equivalent to

$$\|u'_\rho(t)\|^2 - \|u'_\rho(0)\|^2 \leq \|f\|_Y^2 + \int_0^t \|u'_\rho(s)\|^2 ds,$$

where $Y = W_2^1(0,T;X)$. Now by Gronwall lemma,

$$\|u'_\rho(t)\|^2 \leq c(\|u'_\rho(0)\|^2 + \|f\|_Y^2).$$

Finally, using (3.2), we have $u'_\rho(0) = -M_\rho(u_0) + f(0)$ and $\|M_\rho u_0\| \leq \|M_0 u_0\|$, and thus, it follows that (3.3) holds. $\square$

Corollary 3.2. *Let $M : X \to 2^X$ be maximal monotone on a separable Hilbert space X. Suppose that the given*

$$u_0 \in D(M), \quad f \in W_2^1(0,T;X)$$

are fixed. Then (3.1) *has exactly one solution $u \in W_2^1(0,T;X)$ such that $M : X \to 2^X$ is maximal monotone.*

Concluding Remarks. The obtained results on the first-order evolution inclusions can further be generalized to the case of a real Banach space setting in terms of accretivity and m-accretivity. More importantly, the solution concept is also changed as an integral solution based on the difference method belonging to (3.1) as backward differences. The uniqueness proof assures that each classical solution of (3.1) is also an integral solution.

References

[1] Ahmed, N. U.; *Optimization and Identification of Systems Governed by Evolution Equations on Banach Spaces*, Pitman Research Notes in Mathematics, Volume 184, Longman Scientific and Technical, 1988.

[2] Fang, Y. P.; Huang, N. J.; *H-monotone operators and system of variational inclusions*, Communications on Applied Nonlinear Analysis 11 (1) (2004), 93–101.

[3] Komura, Y.; *Nonlinear semigroups in Hilbert space*, Journal of Mathematical Society of Japan 19 (1967), 493–507.

[4] Nakajo,k; Takahashi, W.; *Strong convergence theorems for nonxpansive mappings and nonexpansive semigroups*, Journal of Mathematical Analysis and Applications 279 (2003), 372–379.
[5] Verma, R. U.; *A-monotonicity and applications to nonlinear inclusion problems*, Journal of Applied Mathematics and Stochastic Analysis 17 (2004), 193–195.
[6] Verma, R. U.; *A-monotonicity and its role in nonlinear variational inclusions*, Journal of Optimization Theory and Applications 129 (3)(2006), 457–467.
[7] Verma, R. U.; *First-order evolution equations and Yosida approximations based on the relative maximal monotonicity*, International Journal of Evolution Equations 5 (2) (2010), 1–19.
[8] Verma, R. U.; *General system of A-monotone nonlinear variational inclusions with applications*, Journal of Optimization Theory and Applications 131 (1) (2006), 151–157.
[9] Verma, R. U.; *Approximation solvability of a class of nonlinear set-valued inclusions involving (A, η)-monotone mappings*, Journal of Mathematical Analysis and Applications 337 (2008), 969–975.
[10] Yosida, K.; *On differentiability and the representation of one-parameter semigroups of linear operators*, Journal of Mathematical Society of Japan 1 (1948), 15–21.
[11] Yosida, K.; *Functional Analysis*, Springer-Verlag, Berlin, 1965.
[12] Zeidler, E.; *Nonlinear Functional Analysis and its Applications I*, Springer-Verlag, New York, New York, 1984.
[13] Zeidler, E.; *Nonlinear Functional Analysis and its Applications II/A*, Springer-Verlag, New York, New York, 1990.
[14] Zeidler, E.; *Nonlinear Functional Analysis and its Applications II/B*, Springer-Verlag, New York, New York, 1990.
[15] Zeidler, E.; *Nonlinear Functional Analysis and its Applications III*, Springer-Verlag, New York, New York, 1985.

Ram U. Verma
Texas A&M University, Department of Mathematics, Kingsville, TX 78363, USA
E-mail address: verma99@msn.com