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LARGE TIME BEHAVIOR OF A CAHN-HILLIARD-BOUSSINESQ SYSTEM ON A BOUNDED DOMAIN

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ABSTRACT. We study the asymptotic behavior of classical solutions to an initial-boundary value problem (IBVP) for a coupled Cahn-Hilliard-Boussinesq system on bounded domains with large initial data. A sufficient condition is established under which the solutions decay exponentially to constant states as time approaches infinity.

1. Introduction

As one of the fundamental modelling equations, the Cahn-Hilliard equation [6, 7] plays an important role in the mathematical study of multi-phase flows, and has been studied intensively in the literature both analytically and numerically (see e.g. [3, 4, 8, 12, 13, 19, 20, 25, 26, 27, 29]). The couplings of the Cahn-Hilliard equation with other basic modelling equations have been proposed in various situations to study complicated phenomena in fluid mechanics involving phase transition. For example, the coupled Cahn-Hilliard-Navier-Stokes (CHNS) system and its variations, which describe the motion of an incompressible two-phase flow under shear through an order parameter formulation, have been used in order to understand the phenomena of phase transition in incompressible fluid flows (c.f. [14, 18, 24]). Recently, a closely related model to the CHNS system has been developed in [10, 11, 15, 16] to understand the spinodal decomposition of binary fluid in a Hele-Shaw cell, tumor growth, cell sorting, and two phase flows in porous media, which is referred as the Cahn-Hilliard-Hele-Shaw (CHHS) system. In this paper, we consider the following system of equations:

$$
\begin{gathered}
\phi_t + U\cdot\nabla\phi = \Delta\mu, \quad \mathbf{x}\in\mathbb{R}^n,\ t>0,\\
\mu = -\alpha\Delta\phi + F'(\phi),\\
U_t + U\cdot\nabla U + \nabla P = \mu\nabla\phi + \theta\mathbf{e}_n,\\
\theta_t + U\cdot\nabla\theta = \kappa\Delta\theta,\\
\nabla\cdot U = 0,
\end{gathered}
\tag{1.1}
$$

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which is a system of strongly coupled partial differential equations obtained by coupling the Cahn-Hilliard equation to the inviscid heat-conductive Boussinesq equations. It describes the motion of an incompressible inviscid two-phase flow subject to convective heat transfer under the influence of gravitational force through order parameter formulation. Here, $U = (u_1, \dots, u_n)$, P and θ denotes the velocity, pressure and temperature respectively. ϕ is the order parameter and μ is a chemical potential derived from a coarse-grained study of the free energy of the fluid (c.f. [17]). The constant $\kappa > 0$ and $\alpha > 0$ models heat conduction and diffusion respectively, and $\mathbf{e}_n$ is the n-th unit vector in $\mathbb{R}^n$. The function F usually has a physical-relevant, double-well structure, each of them representing the two phases of the fluid. A typical example of F takes the form (c.f. [9, 17]): $F(z) = \frac{1}{4}(z^2-1)^2$. In this paper, we consider a general scenario by imposing appropriate growth conditions on F. We remark that, system (1.1) reduces to the CHHS model if the temperature equation and the hydrodynamic effect are dropped. On the other hand, (1.1) becomes the CHNS system if the temperature equation is removed and the viscosity of fluid is added to the velocity equation.

In the real world, flows often move in bounded domains with constraints from boundaries, where the initial-boundary value problems appear. The solutions of the initial-boundary value problems usually exhibit different behaviors and much richer phenomena comparing with the Cauchy problem. In this paper, we consider system (1.1) on a bounded domain in $\mathbb{R}^n$. The system is supplemented by the following initial and boundary conditions:

$$\begin{gathered}(\phi, \mu, U, \theta)(\mathbf{x}, 0) = (\phi_0, \mu_0, U_0, \theta_0)(\mathbf{x}), \\ \nabla\phi \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \nabla\mu \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad U \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \theta|_{\partial\Omega} = \bar{\theta},\end{gathered} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $\mathbf{n}$ is the unit outward normal to $\partial\Omega$ and $\bar{\theta}$ is a constant.

The initial-boundary value problem (1.1)–(1.2) was first studied in [30], where the global existence and uniqueness of classical solutions are established, for large initial data with finite energy in 2D. However, the large time asymptotic behavior of the solutions is not investigated due to the lack of uniform-in-time estimates of the solutions. We give definite answer to this unsolved issue in current paper for the 2D case, based on new findings of the structure of the system.

Suggested by the conservation of total mass and the boundary conditions, it is expected that the global attractors of ϕ and θ should be $\bar{\phi} = \frac{1}{|\Omega|}\int_\Omega \phi_0(\mathbf{x})d\mathbf{x}$ and $\bar{\theta}$, respectively, due to diffusion and boundary effects. In this paper, we provide a sufficient condition that guarantees the decay of the solution. We will show that when the diffusion coefficient α passes a threshold value determined by F and Ω, the functions ϕ and θ will converge exponentially in time to $\bar{\phi}$ and $\bar{\theta}$, respectively, regardless of the magnitude of the initial perturbation. To be precise, we shall assume that $\alpha - F_3 c_0 > 0$, where $F_3 > 0$ is a constant such that $F'' \geq -F_3$, and c_0 is the constant in Poincaré inequality on Ω. This condition is crucial in our analysis due to the fact that it produces a positive constant multiple of $\|\phi - \bar{\phi}\|_{H^2}^2$ which is one of the major dissipative terms controlling the exponential decay of ϕ. The condition will trigger a chain reaction leading the energy estimate performed in [30] to a whole new scenario. As consequences of the convergence of ϕ and θ, we will show that the velocity and vorticity are uniformly bounded in time.

Throughout this paper, $\|\cdot\|_{L^p}$, $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_{W^{s,p}}$ denote the norms of the usual Lebesgue measurable function spaces L^p $(1 \le p < \infty)$, L^∞ and the usual Sobolev space $W^{s,p}$, respectively. For $p = 2$, we denote the norm $\|\cdot\|_{L^2}$ by $\|\cdot\|$ and $\|\cdot\|_{W^{s,2}}$ by $\|\cdot\|_{H^s}$, respectively. The function spaces under consideration are $C([0,T]; H^r(\Omega))$ and $L^2([0,T]; H^s(\Omega))$, equipped with norms $\sup_{0\le t\le T}\|\Psi(\cdot,t)\|_{H^r}$ and $\big(\int_0^T \|\Psi(\cdot,\tau)\|_{H^s}^2 d\tau\big)^{1/2}$, respectively, where r, s are positive integers. Unless specified, c_i will denote generic constants which are independent of ϕ, μ, U, θ and t, but may depend on α, κ, Ω and initial data.

For the sake of completeness, we first state the results obtained in [30].

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Suppose that $F(\cdot)$ satisfies the following conditions:*

- *$F(\cdot)$ is of C^6 class and $F(\cdot) \ge 0$;*
- *There exist constants $F_1, F_2 > 0$ such that $|F^{(n)}(\phi)| \le F_1|\phi|^{p-n} + F_2$, $n = 1, \dots, 6$, $\forall\, 6 \le p < \infty$ and $\phi \in \mathbb{R}$;*
- *There exists a constant $F_3 \ge 0$ such that $F'' \ge -F_3$.*

If the initial data $\phi_0(\mathbf{x}) \in H^5(\Omega)$, $\mu_0(\mathbf{x}) \in H^3(\Omega)$ and $(\theta_0(\mathbf{x}), U_0(\mathbf{x})) \in H^3(\Omega)$ are compatible with the boundary conditions, then there exists a unique solution (ϕ, μ, θ, U) of (1.1)–(1.2) globally in time such that

$$\phi \in C([0,T]; H^5(\Omega)) \cap L^2([0,T]; H^7(\Omega)),$$
$$\mu \in C([0,T]; H^3(\Omega)) \cap L^2([0,T]; H^5(\Omega)),$$

$U \in C([0,T]; H^3(\Omega))$ and $\theta \in C([0,T]; H^3(\Omega)) \cap L^2([0,T]; H^4(\Omega))$ for $0 < T < \infty$.

The next theorem is the main result of this paper regarding the large-time asymptotic behavior of the solution obtained in Theorem 1.1.

Theorem 1.2. *Suppose that the assumptions in Theorem 1.1 are in force and assume that the constant $\alpha - F_3 c_0 > 0$, where c_0 is the constant in Poincaré inequality on Ω. Then the solution to (1.1)–(1.2) satisfies*

$$\|\phi(\cdot,t) - \bar\phi\|_{H^5} + \|\mu(\cdot,t) - F'(\bar\phi)\|_{H^3} + \|\theta(\cdot,t) - \bar\theta\|_{H^3} \le \gamma e^{-\beta t};$$
$$\|U(\cdot,t)\|_{W^{1,p}} \le \gamma(p), \quad \forall\, 1 \le p < \infty; \quad \|\omega(\cdot,t)\|_{L^\infty} \le \bar\gamma, \quad \forall\, t \ge 0,$$

for some constants $\gamma, \beta, \gamma(p), \bar\gamma > 0$ independent of t, where $\bar\phi = \frac{1}{|\Omega|}\int_\Omega \phi_0(\mathbf{x})d\mathbf{x}$ and $\omega = v_x - u_y$ is the 2D vorticity.

Remark 1.3. It should be pointed out that, in the theorems obtained above, no smallness restriction is put upon the initial data.

Remark 1.4. We observe that, by assuming small initial perturbation around the equilibrium state and by exploring the structure of the function F, one can show the exponential decay of the solution. However, the asymptotic result obtained in Theorem 1.2 has an obvious advantage over the case for small perturbation. Indeed, Theorem 1.2 provides a convenient criterion for determining whether the solution collapses to a constant state as time evolves. Based on the result, one only needs to measure the volume of the domain, instead of measuring the "smallness" of the initial perturbation which is usually laborious to perform, to determine whether the solution decays or not when other system parameters are fixed.

Remark 1.5. It is well-known that the Cahn-Hilliard equation is an effective model in the study of sharp interfaces in two-phase fluid flows. However, based on our results, the order parameter ϕ tends to a uniform constant $\bar{\phi}$ instead of ± 1. This suggests that under the conditions of Theorem 1.1 and Theorem 1.2, the modelling equations (1.1) indeed fail to model the sharp interfacial phenomenon. Therefore, our results exhibit some bifurcation phenomena on the effectiveness of the modelling equations.

The proof of Theorem 1.2 involves a series of accurate combinations of energy estimates. The estimates are delicate mainly due to the coupling between the equations by convection, gravitational force and boundary effects. Great efforts have been made to simplify the proof. Current proof involves intensive applications of Sobolev embeddings and Ladyzhenskaya type inequalities, see Lemma 2.1. Roughly speaking, because of the lack of the spatial derivatives of the solution at the boundary, our energy framework proceeds as follows: We first apply the standard energy estimate on the solution and the temporal derivatives of the solution. We then apply standard results on elliptic equations to recover estimates of the spatial derivatives. Such a process will be repeated up to third order, and then with the aid of the assumption on α, the carefully coupled estimates will be composed into a desired one leading to the exponential decay of the solution.

The rest of the paper is organized as follows. In Section 2, we give some basic facts that will be used in the proof of Theorem 1.2. In Section 3, we prove some uniform-in-time energy estimates of the solution based on which the combinations of energy estimates will be performed. We then complete the proof of Theorem 1.2 in Section 4.

2. Preliminaries

In this section, we shall collect several facts which will be used in the proof of Theorem 1.2. First, we recall some inequalities of Sobolev and Ladyzhenskaya type (c.f. [1, 21]).

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^2$ be any bounded domain with smooth boundary $\partial\Omega$. Then*

(i) $\|f\|_{L^\infty} \le c_1 \|f\|_{H^2}$;
(ii) $\|f\|_{L^\infty} \le c_2 \|f\|_{W^{1,p}}$ *for all* $p > 2$;
(iii) $\|f\|_{L^p} \le c_3 \|f\|_{H^1}$ *for all* $1 \le p < \infty$;
(iv) $\|f\|_{L^4}^2 \le c_4 \big(\|f\| \|\nabla f\| + \|f\|^2\big)$;
(v) $\|f\|_{L^8}^2 \le c_5 \big(\|f\| \|\nabla f\|_{L^4} + \|f\|^2\big)$,

for some constants $c_i = c_i(p, \Omega)$, $i = 1, \dots, 5$.

Next, we recall some classical results on elliptic equations (c.f. [2, 22, 23]), which are useful in the estimation of θ.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^2$ be any bounded domain with smooth boundary $\partial\Omega$. Consider the Dirichlet problem*

$$\kappa \Delta \Theta = f \quad \text{in } \Omega,$$
$$\Theta = 0 \quad \text{on } \partial\Omega.$$

If $f \in W^{m,p}$, then $\Theta \in W^{m+2,p}$ and there exists a constant $c_6 = c_6(p, \kappa, m, \Omega)$ such that

$$\|\Theta\|_{W^{m+2,p}} \le c_6 \|f\|_{W^{m,p}}$$

for any $p \in (1, \infty)$ *and the integer* $m \geq -1$.

The next lemma is useful for the estimation of the velocity field (see [5]).

Lemma 2.3. *Let* $\Omega \subset \mathbb{R}^2$ *be any bounded domain with smooth boundary* $\partial\Omega$, *and let* $F \in W^{s,p}(\Omega)$ *be a vector-valued function satisfying* $F \cdot \mathbf{n}|_{\partial\Omega} = 0$, *where* $\mathbf{n}$ *is the unit outward normal to* $\partial\Omega$. *Then there exists a constant* $c_7 = c_7(s, p, \Omega)$ *such that*

$$\|F\|_{W^{s,p}} \leq c_7(\|\nabla \times F\|_{W^{s-1,p}} + \|\nabla \cdot F\|_{W^{s-1,p}} + \|F\|_{L^p})$$

for any $s \geq 1$ *and* $p \in (1, \infty)$.

Finally, we recall some Poincaré type inequalities, which will be used in the estimation of ϕ, whose proof is straightforward.

Lemma 2.4. *Let* $\Omega \subset \mathbb{R}^n$ *be any bounded domain with smooth boundary* $\partial\Omega$. *Then, for any function* $H^s(\Omega) \ni f : \Omega \to \mathbb{R}$, *there exists a constant* $c_8 = c_8(s, \Omega) > 0$ *such that*

(1) $\|f - \bar{f}\|_{H^{2s}} \leq c_8\|\Delta^s f\|$ *and* $\|f - \bar{f}\|_{H^{2s+1}} \leq c_8\|\nabla\Delta^s f\|$, $s \geq 1$, *if* $\nabla f \cdot \mathbf{n}|_{\partial\Omega} = 0$;
(2) $\|f\|_{H^{2s}} \leq c_8\|\Delta^s f\|$ *and* $\|f\|_{H^{2s+1}} \leq c_8\|\nabla\Delta^s f\|$, $s \geq 1$, *if* $f|_{\partial\Omega} = 0$,

where $\bar{f} = \frac{1}{|\Omega|}\int_\Omega f d\mathbf{x}$.

3. Uniform energy estimates

In this section, we establish some uniform-in-time energy estimates of the solution under the condition $\alpha - F_3 c_0 > 0$, based on which the exponential decay rate of the solution will be proved. The results are stated as a sequence of lemmas and the proofs are carried out by carefully exploring the condition $\alpha - F_3 c_0 > 0$ and delicate applications of Cauchy-Schwarz and Gronwall inequalities.

To study the asymptotic behavior, we first reformulate the original problem to get the one for the perturbations. For this purpose, let $\Phi = \phi - \bar{\phi}$ and $\Theta = \theta - \bar{\theta}$. After plugging Φ and Θ into (1.1) we obtain

$$\begin{gathered}
\Phi_t + U \cdot \nabla\Phi = \Delta\mu, \\
\mu = -\alpha\Delta\Phi + F'(\phi), \\
U_t + U \cdot \nabla U + \nabla\tilde{P} = \mu\nabla\Phi + \Theta\mathbf{e}_2, \\
\Theta_t + U \cdot \nabla\Theta = \kappa\Delta\Theta, \\
\nabla \cdot U = 0,
\end{gathered} \tag{3.1}$$

which is equivalent to (1.1) for sufficiently smooth solutions, where $\tilde{P} = P - \bar{\theta}y$, and the initial and boundary conditions become

$$\begin{gathered}
(\Phi, \mu, U, \Theta)(\mathbf{x}, 0) = (\Phi_0, \mu_0, U_0, \Theta_0)(\mathbf{x}) \equiv (\phi_0 - \bar{\phi}, \mu_0, U_0, \theta_0 - \bar{\theta})(\mathbf{x}), \\
\nabla\Phi \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \nabla\mu \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad U \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \Theta|_{\partial\Omega} = 0.
\end{gathered} \tag{3.2}$$

We begin with the uniform estimate of $\|\Phi\|_{L^2}$.

Lemma 3.1. *Under the assumptions of Theorem 1.2, it holds that*

$$\|\Phi(\cdot, t)\|^2 + \int_0^t \|\Phi(\cdot, \tau)\|_{H^2}^2 d\tau \leq c_9, \quad \forall\, t \geq 0. \tag{3.3}$$

Proof. Taking L^2 inner product of $(3.1)_1$ with Φ, after integrating by parts we have

$$\frac{1}{2}\frac{d}{dt}\|\Phi\|^2 = \int_\Omega \Phi\Delta\mu\mathbf{x} = -\int_\Omega \nabla\mu\cdot\nabla\Phi d\mathbf{x} = -\alpha\|\Delta\Phi\|^2 - \int_\Omega F''(\phi)|\nabla\Phi|^2 d\mathbf{x}, \quad (3.4)$$

which gives

$$\frac{1}{2}\frac{d}{dt}\|\Phi\|^2 + \alpha\|\Delta\Phi\|^2 \le F_3\|\nabla\Phi\|^2, \quad (3.5)$$

where we have used the condition on F. Since $\Phi = \phi - \bar\phi$ and $\bar\phi$ is a constant, using the boundary conditions, Cauchy-Schwarz and Poincaré inequalities we have

$$\|\nabla\Phi\|^2 = -\int_\Omega \Phi\Delta\Phi d\mathbf{x} \le \frac{1}{2c_0}\|\Phi\|^2 + \frac{c_0}{2}\|\Delta\Phi\|^2 \le \frac{1}{2}\|\nabla\Phi\|^2 + \frac{c_0}{2}\|\Delta\Phi\|^2, \quad (3.6)$$

which implies

$$\|\nabla\Phi\|^2 \le c_0\|\Delta\Phi\|^2, \quad (3.7)$$

where c_0 is the constant in Poincaré inequality on Ω. Let $\alpha_1 \equiv \alpha - F_3c_0 > 0$. Substituting (3.7) in (3.5) we have

$$\frac{1}{2}\frac{d}{dt}\|\Phi\|^2 + \alpha_1\|\Delta\Phi\|^2 \le 0. \quad (3.8)$$

Upon integrating (3.8) in time over $[0,t]$ and using Lemma 2.4 we obtain (3.3). This completes the proof. □

Remark 3.2. The estimate (3.8) already implies the decay of $\|\Phi\|^2$. However, our ultimate goal is to show the decay rate of the higher order derivatives of the solution. Hence, for the sake of completeness, we leave the proof of the decay rate in the next section.

Next, we prove uniform estimates of Θ, which will be used to settle down the uniform bound of $\|U\|_{H^1}$.

Lemma 3.3. *Under the assumptions of Theorem 1.2, there exists a constant β_0 independent of t such that for any $t \ge 0$, it holds that*

$$\|\Theta(\cdot,t)\|^2 \le \|\Theta_0\|^2 e^{-2\beta_0 t}, \quad \int_0^t \|\nabla\Theta(\cdot,\tau)\|^2 e^{\beta_0\tau}d\tau \le \frac{1}{\kappa}\|\Theta_0\|^2. \quad (3.9)$$

Proof. Taking the L^2 inner product of $(3.1)_4$ with Θ we have

$$\frac{1}{2}\frac{d}{dt}\|\Theta\|^2 + \kappa\|\nabla\Theta\|^2 = 0. \quad (3.10)$$

Since $\Theta|_{\partial\Omega} = 0$, Poincaré's inequality implies

$$\frac{d}{dt}\|\Theta\|^2 + \frac{2\kappa}{c_0}\|\Theta\|^2 \le 0, \quad (3.11)$$

which yields immediately

$$\|\Theta(\cdot,t)\|^2 \le \|\Theta_0\|^2 e^{-2\beta_0 t}, \quad (3.12)$$

where $\beta_0 = \kappa/c_0$. This proves the first part of (3.9).

Next, we multiply (3.10) by $e^{\beta_0 t}$ and use (3.12) to obtain

$$\frac{d}{dt}\big(e^{\beta_0 t}\|\Theta\|^2\big) + 2\kappa e^{\beta_0 t}\|\nabla\Theta\|^2 \le \beta_0 e^{-\beta_0 t}\|\Theta_0\|^2. \quad (3.13)$$

For any $t > 0$, upon integrating (3.13) in time we obtain

$$e^{\beta_0 t}\|\Theta(\cdot,t)\|^2 - \|\Theta_0\|^2 + 2\kappa\int_0^t e^{\beta_0\tau}\|\nabla\Theta(\cdot,\tau)\|^2 d\tau \le \big(1-e^{-\beta_0 t}\big)\|\Theta_0\|^2, \tag{3.14}$$

which implies the second part of (3.9) immediately. This completes the proof. □

With the help of Lemma 3.3 we now prove the uniform estimates of $\|U\|_{H^1}$ and $\|\Phi\|_{H^1}$.

Lemma 3.4. *Under the assumptions of Theorem 1.2, for all $t \ge 0$,*

$$\|U(\cdot,t)\|_{H^1}^2 + \|\Phi(\cdot,t)\|_{H^1}^2 + \int_0^t \big(\|\nabla\mu(\cdot,\tau)\|^2 + \|\Phi(\cdot,\tau)\|_{H^3}^2\big)d\tau \le c_{10}. \tag{3.15}$$

Proof. **Step 1.** Note that due to Lemma 2.3 and the boundary condition on U, it suffices to estimate $\|U\|^2$ and $\|\omega\|^2$, in order estimate $\|U(\cdot,t)\|_{H^1}^2$. Taking the L^2 inner product of $(3.1)_3$ with U we have

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 = \int_\Omega \mu(\nabla\Phi\cdot U)d\mathbf{x} + \int_\Omega \Theta\mathbf{e}_2\cdot U d\mathbf{x}. \tag{3.16}$$

Taking L^2 inner product of $(3.1)_1$ with μ we have

$$\frac{d}{dt}\Big(\frac{\alpha}{2}\|\nabla\Phi\|^2 + \int_\Omega F(\phi)d\mathbf{x}\Big) + \|\nabla\mu\|^2 = -\int_\Omega \mu(\nabla\Phi\cdot U)d\mathbf{x}. \tag{3.17}$$

Adding (3.16) and (3.17), we obtain

$$\frac{d}{dt}\Big(\frac{1}{2}\|U\|^2 + \frac{\alpha}{2}\|\nabla\Phi\|^2 + \int_\Omega F(\phi)d\mathbf{x}\Big) + \|\nabla\mu\|^2 = \int_\Omega \Theta\mathbf{e}_2\cdot U d\mathbf{x}. \tag{3.18}$$

Applying Cauchy-Schwarz inequality to the right-hand side of (3.18) and using (3.12), we obtain

$$\frac{d}{dt}\Big(\frac{1}{2}\|U\|^2 + \frac{\alpha}{2}\|\nabla\Phi\|^2 + \int_\Omega F(\phi)d\mathbf{x}\Big) + \|\nabla\mu\|^2 \le e^{-\beta_0 t}\|U\|^2 + e^{-\beta_0 t}\|\Theta_0\|^2. \tag{3.19}$$

After dropping $\|\nabla\mu\|^2$ from the left hand side (LHS) of (3.19), we have

$$\frac{d}{dt}\Big(\frac{1}{2}\|U\|^2 + \frac{\alpha}{2}\|\nabla\Phi\|^2 + \int_\Omega F(\phi)d\mathbf{x}\Big) \le e^{-\beta_0 t}\|U\|^2 + e^{-\beta_0 t}\|\Theta_0\|^2. \tag{3.20}$$

Since $F \ge 0$, Gronwall's inequality then gives

$$\frac{1}{2}\|U\|^2 + \frac{\alpha}{2}\|\nabla\Phi\|^2 + \int_\Omega F(\phi)d\mathbf{x} \le c_{11}, \quad \forall\, t \ge 0. \tag{3.21}$$

Applying (3.21) to (3.19) and integrating with respect to t we obtain

$$\|U\|^2 + \|\Phi\|_{H^1}^2 + \int_\Omega F(\phi)d\mathbf{x} + \int_0^t \|\nabla\mu(\cdot,\tau)\|^2 d\tau \le c_{12}, \quad \forall\, t \ge 0. \tag{3.22}$$

Step 2. By the definition of Φ, Lemma 2.4 and $(3.1)_2$, we observe that

$$\|\Phi\|_{H^3}^2 \le c_8\|\nabla(\Delta\Phi)\|^2 \le c_{13}\big(\|\nabla\mu\|^2 + \|F''(\phi)\nabla\Phi\|^2\big). \tag{3.23}$$

Using the condition on F, Hölder inequality, Lemma 2.1 (iii) and (3.22) we have

$$\begin{aligned}\|F''(\phi)\nabla\Phi\|^2 &\le c_{14}\big(\|\phi\|_{L^{4(p-2)}}^{2(p-2)}\|\nabla\Phi\|_{L^4}^2 + \|\nabla\Phi\|^2\big)\\ &\le c_{15}\big((\|\Phi\|_{H^1}^{2(p-2)} + \|\bar\phi\|_{H^1}^{2(p-2)})\|\nabla\Phi\|_{H^1}^2 + \|\nabla\Phi\|^2\big)\\ &\le c_{16}\|\Phi\|_{H^2}^2.\end{aligned} \tag{3.24}$$

substituting (3.24) in (3.23) we have

$$\|\Phi\|_{H^3}^2 \le c_{17}\big(\|\nabla\mu\|^2 + \|\Phi\|_{H^2}^2\big), \tag{3.25}$$

which, together with (3.3) and (3.22), implies

$$\int_0^t \|\Phi(\cdot,\tau)\|_{H^3}^2 d\tau \le c_{18}, \quad \forall\, t \ge 0. \tag{3.26}$$

Step 3. By taking the curl of the velocity equation, we obtain

$$\omega_t + U\cdot\nabla\omega = \mu_x\Phi_y - \mu_y\Phi_x + \Theta_x, \tag{3.27}$$

where $\omega = v_x - u_y$ is the 2D vorticity. Taking L^2 inner product of (3.27) with ω and applying Hölder inequality we have

$$\frac{d}{dt}\|\omega\| \le 2\|\nabla\mu\|\|\nabla\Phi\|_{L^\infty} + \|\nabla\Theta\|. \tag{3.28}$$

Upon integrating (3.28) in time using Hölder and Sobolev inequalities we have

$$\begin{aligned}\|\omega(\cdot,t)\| &\le \int_0^t \big(2\|\nabla\mu\|\|\nabla\Phi\|_{L^\infty} + \|\nabla\Theta\|\big)d\tau + \|\omega_0\| \\ &\le c_{19}\Big(\int_0^t \|\nabla\mu\|^2 d\tau\Big)^{1/2}\Big(\int_0^t \|\Phi\|_{H^3}^2 d\tau\Big)^{1/2} \\ &\quad + \Big(\int_0^t e^{\beta_0\tau/2}\|\nabla\Theta\|^2 d\tau\Big)^{1/2}\Big(\int_0^t e^{-\beta_0\tau/2} d\tau\Big)^{1/2} + \|\omega_0\|.\end{aligned} \tag{3.29}$$

Since the right hand side of (3.29) is uniformly bounded in time by virtue of previous estimates, we have

$$\|\omega(\cdot,t)\| \le c_{20}, \quad \forall\, t \ge 0. \tag{3.30}$$

Thus, (3.15) follows from (3.22), (3.26) and (3.30). This completes the proof. □

With the aid of Lemma 3.4 we are now able to improve the estimates of Φ and μ. Due to the lack of spatial derivatives of the solution on $\partial\Omega$, we shall alternatively work on the temporal derivatives and use an iteration program to recover the spatial derivatives.

Lemma 3.5. *Under the assumptions of Theorem 1.2, it holds that*

$$\|\Phi(\cdot,t)\|_{H^2}^2 + \|\mu(\cdot,t)\|^2 + \int_0^t \big(\|\Phi_t\|^2 + \|\nabla\mu\|_{H^1}^2\big)d\tau \le c_{21}, \quad \forall\, t \ge 0. \tag{3.31}$$

Proof. **Step 1.** By taking L^2 inner product of $(3.1)_1$ with Φ_t we have

$$\|\Phi_t\|^2 + \int_\Omega \Phi_t(U\cdot\nabla\Phi)d\mathbf{x} = \int_\Omega \Phi_t\Delta\mu d\mathbf{x}. \tag{3.32}$$

Using the boundary conditions we calculate the RHS of (3.32) as:

$$\int_\Omega \Phi_t\Delta\mu d\mathbf{x} = -\frac{d}{dt}\Big(\frac{\alpha}{2}\|\Delta\Phi\|^2 + \frac12\int_\Omega F''(\phi)|\nabla\Phi|^2 d\mathbf{x}\Big) + \frac12\int_\Omega F'''(\phi)\Phi_t|\nabla\Phi|^2 d\mathbf{x}. \tag{3.33}$$

Substituting (3.33) in (3.32) we obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{\alpha}{2}\|\Delta\Phi\|^2 + \frac12\int_\Omega F''(\phi)|\nabla\Phi|^2 d\mathbf{x}\Big) + \|\Phi_t\|^2 \\ &= \frac12\int_\Omega F'''(\phi)\Phi_t|\nabla\Phi|^2 d\mathbf{x} - \int_\Omega \Phi_t(U\cdot\nabla\Phi)d\mathbf{x}.\end{aligned} \tag{3.34}$$

Using Cauchy-Schwarz inequality and Lemma 3.4 we estimate the first term on the RHS of (3.34) as

$$\begin{aligned}\Big|\frac{1}{2}\int_\Omega F'''(\phi)\Phi_t|\nabla\Phi|^2d\mathbf{x}\Big| &\le \frac{1}{4}\|\Phi_t\|^2+\frac{1}{4}\int_\Omega|F'''(\phi)|^2|\nabla\Phi|^4d\mathbf{x}\\ &\le \frac{1}{4}\|\Phi_t\|^2+c_{22}\|\phi\|^{2(p-3)}_{L^{4(p-3)}}\|\nabla\Phi\|^4_{L^8}+c_{22}\|\nabla\Phi\|^4_{L^4}\\ &\le \frac{1}{4}\|\Phi_t\|^2+c_{23}\big(\|\nabla\Phi\|^4_{L^8}+\|\nabla\Phi\|^4_{L^4}\big).\end{aligned} \tag{3.35}$$

Lemma 2.1 (iii)–(v) and Lemma 3.4 then give

$$\begin{aligned}\|\nabla\Phi\|^4_{L^4}+\|\nabla\Phi\|^4_{L^8} &\le c_{24}\big(\|\nabla\Phi\|^2\|\nabla^2\Phi\|^2+\|\nabla\Phi\|^4+\|\nabla\Phi\|^2\|\nabla^2\Phi\|^2_{L^4}+\|\nabla\Phi\|^4\big)\\ &\le c_{25}\big(\|\nabla^2\Phi\|^2+\|\nabla\Phi\|^2+\|\nabla^2\Phi\|^2_{H^1}\big)\\ &\le c_{26}\|\Phi\|^2_{H^3}.\end{aligned} \tag{3.36}$$

So we update (3.35) as

$$\Big|\frac{1}{2}\int_\Omega F'''(\phi)\Phi_t|\nabla\Phi|^2d\mathbf{x}\Big| \le \frac{1}{4}\|\Phi_t\|^2+c_{27}\|\Phi\|^2_{H^3}. \tag{3.37}$$

The second term on the RHS of (3.34) is estimated as

$$\begin{aligned}\Big|-\int_\Omega\Phi_t(U\cdot\nabla\Phi)d\mathbf{x}\Big| &\le \frac{1}{4}\|\Phi_t\|^2+c_{28}\|U\|^2_{H^1}\|\nabla\Phi\|^2_{H^1}\\ &\le \frac{1}{4}\|\Phi_t\|^2+c_{29}\|\Phi\|^2_{H^2},\end{aligned} \tag{3.38}$$

where we have used Lemma 3.4. Combining (3.34), (3.37) and (3.38) we have

$$\frac{d}{dt}\Big(\frac{\alpha}{2}\|\Delta\Phi\|^2+\frac{1}{2}\int_\Omega F''(\phi)|\nabla\Phi|^2d\mathbf{x}\Big)+\frac{1}{2}\|\Phi_t\|^2 \le c_{30}\|\Phi\|^2_{H^3}. \tag{3.39}$$

Upon integrating (3.39) in time and using (3.26) we have

$$\frac{\alpha}{2}\|\Delta\Phi\|^2+\frac{1}{2}\int_\Omega F''(\phi)|\nabla\Phi|^2d\mathbf{x}+\frac{1}{2}\int_0^t\|\Phi_t\|^2d\tau \le c_{31}. \tag{3.40}$$

Since $F''\ge -F_3$, we have

$$\int_\Omega F''(\phi)|\nabla\Phi|^2d\mathbf{x} \ge -F_3\|\nabla\Phi\|^2 \ge -F_3c_0\|\Delta\Phi\|^2, \tag{3.41}$$

where we have used (3.6). Substituting (3.41) in (3.40) we have

$$\frac{\alpha_1}{2}\|\Delta\Phi\|^2+\frac{1}{2}\int_0^t\|\Phi_t\|^2d\tau \le c_{31},$$

which, together with Lemma 2.4, implies

$$\|\Phi\|^2_{H^2}+\int_0^t\|\Phi_t\|^2d\tau \le c_{32}. \tag{3.42}$$

Step 2. We derive some consequences of (3.42). From $(3.1)_2$ and Lemma 2.1 (i) we see that

$$\begin{aligned}\|\mu(\cdot,t)\|^2 &\le c_{33}\big(\|\Delta\Phi\|^2+\|F'(\phi)\|^2\big)\\ &\le c_{34}\big(\|\Delta\Phi\|^2+\|\phi\|^{2(p-1)}_{L^\infty}+1\big)\\ &\le c_{35}\big(\|\Delta\Phi\|^2+\|\Phi\|^{2(p-1)}_{H^2}+\|\bar\phi\|^{2(p-1)}_{L^\infty}+1\big).\end{aligned} \tag{3.43}$$

Therefore, using (3.42) we have

$$\|\mu(\cdot,t)\|^2 \le c_{36}, \quad \forall\, t \ge 0. \tag{3.44}$$

Since $\nabla\mu\cdot\mathbf{n}|_{\partial\Omega}=0$, by Lemma 2.4 and Lemma 3.4 we have

$$\begin{aligned}\|\nabla\mu\|_{H^1}^2 \le c_{37}\|\Delta\mu\|^2 &\le c_{38}\big(\|\Phi_t\|^2+\|U\cdot\nabla\Phi\|^2\big)\\ &\le c_{39}\big(\|\Phi_t\|^2+\|U\|_{H^1}^2\|\nabla\Phi\|_{H^1}^2\big)\\ &\le c_{40}\big(\|\Phi_t\|^2+\|\Phi\|_{H^2}^2\big),\end{aligned} \tag{3.45}$$

which, together with (3.15) and (3.42), implies that

$$\int_0^t \|\nabla\mu\|_{H^1}^2 d\tau \le c_{41}. \tag{3.46}$$

Therefore, (3.31) follows from (3.42), (3.44) and (3.46). This completes the proof. □

The next lemma gives the uniform estimate of $\|U_t(\cdot,t)\|^2$ whose proof requires more careful examination of the energy estimate for the temperature.

Lemma 3.6. *Under the assumptions of Theorem 1.2, there exists a constant $\beta_1>0$ independent of t such that*

$$e^{\beta_1 t}\|\Theta(\cdot,t)\|_{H^1}^2 + \int_0^t e^{\beta_1\tau/2}\|\Theta(\cdot,\tau)\|_{H^2}^2 d\tau \le c_{42}, \quad \forall\, t\ge 0. \tag{3.47}$$

Proof. **Step 1.** Taking L^2 inner product of $(3.1)_4$ with Θ_t we have

$$\frac{\kappa}{2}\frac{d}{dt}\|\nabla\Theta\|^2 + \|\Theta_t\|^2 \le \|U\cdot\nabla\Theta\|^2 + \frac14\|\Theta_t\|^2. \tag{3.48}$$

Using Lemma 3.4 we have

$$\|U\cdot\nabla\Theta\|^2 \le c_{43}\|U\|_{H^1}^2\|\nabla\Theta\|_{L^4}^2 \le c_{44}\|\nabla\Theta\|_{L^4}^2. \tag{3.49}$$

So we update (3.48) as

$$\frac{\kappa}{2}\frac{d}{dt}\|\nabla\Theta\|^2 + \frac34\|\Theta_t\|^2 \le c_{44}\|\nabla\Theta\|_{L^4}^2. \tag{3.50}$$

The estimate of the RHS of (3.50) is tricky. First, applying Lemma 2.1 (iv) to $\nabla\Theta$ to obtain

$$c_{44}\|\nabla\Theta\|_{L^4}^2 \le c_{45}\big(\|\nabla\Theta\|\|D^2\Theta\|+\|\nabla\Theta\|^2\big) \le c_{46}(\delta)\|\nabla\Theta\|^2+\delta\|D^2\Theta\|^2, \tag{3.51}$$

where δ is a number to be determined. Since $\Theta|_{\partial\Omega}=0$, by the elliptic estimate (c.f. Lemma 2.2), we have

$$\|\Theta\|_{H^2}^2 \le c_{47}\big(\|\Theta_t\|^2+\|U\cdot\nabla\Theta\|^2\big). \tag{3.52}$$

For the second term on the RHS of (3.52), we use (3.49) and (3.51) to get

$$\|U\cdot\nabla\Theta\|^2 \le c_{48}\big(\|\nabla\Theta\|\|D^2\Theta\|+\|\nabla\Theta\|^2\big). \tag{3.53}$$

Then, using Cauchy-Schwarz inequality we update (3.52) as

$$\begin{aligned}\|\Theta\|_{H^2}^2 &\le c_{49}\big(\|\Theta_t\|^2+\|\nabla\Theta\|\|D^2\Theta\|+\|\nabla\Theta\|^2\big)\\ &\le c_{50}\big(\|\Theta_t\|^2+\|\nabla\Theta\|^2\big)+\frac12\|\Theta\|_{H^2}^2,\end{aligned} \tag{3.54}$$

which implies

$$\|\Theta\|_{H^2}^2 \le c_{51}\big(\|\Theta_t\|^2+\|\nabla\Theta\|^2\big). \tag{3.55}$$

By choosing $\delta = 1/(4c_{51})$ in (3.51), and coupling the result with (3.55) we have

$$c_{44}\|\nabla\Theta\|_{L^4}^2 \le c_{52}\|\nabla\Theta\|^2 + \frac{1}{4}\|\Theta_t\|^2. \tag{3.56}$$

Combining (3.50) and (3.56) we obtain

$$\frac{\kappa}{2}\frac{d}{dt}\|\nabla\Theta\|^2 + \frac{1}{2}\|\Theta_t\|^2 \le c_{52}\|\nabla\Theta\|^2. \tag{3.57}$$

Step 2. We multiply (3.10) by $2c_{52}/\kappa$ and add the result to (3.57) to obtain

$$\frac{d}{dt}\Big(\frac{c_{52}}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\Big) + c_{52}\|\nabla\Theta\|^2 + \frac{1}{2}\|\Theta_t\|^2 \le 0. \tag{3.58}$$

It is clear that, by Poincaré inequality, there exists a constant $c_{53} > 0$ such that

$$c_{53}\Big(\frac{c_{52}}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\Big) \le c_{52}\|\nabla\Theta\|^2. \tag{3.59}$$

Substituting (3.59) in (3.58) we have

$$\frac{d}{dt}\Big(\frac{c_{52}}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\Big) + c_{53}\Big(\frac{c_{52}}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\Big) + \frac{1}{2}\|\Theta_t\|^2 \le 0, \tag{3.60}$$

which implies (by dropping $\frac{1}{2}\|\Theta_t\|^2$ from the LHS) that

$$\Big(\frac{c_{52}}{\kappa}\|\Theta(\cdot,t)\|^2 + \frac{\kappa}{2}\|\nabla\Theta(\cdot,t)\|^2\Big) \le \Big(\frac{c_{52}}{\kappa}\|\Theta_0\|^2 + \frac{\kappa}{2}\|\nabla\Theta_0\|^2\Big)e^{-c_{53}t}. \tag{3.61}$$

Therefore, for $t \ge 0$,

$$\|\Theta(\cdot,t)\|_{H^1}^2 \le \Big(\min\{c_{52}/\kappa, \kappa/2\}\Big)^{-1}\Big(\frac{c_{52}}{\kappa}\|\Theta_0\|^2 + \frac{\kappa}{2}\|\nabla\Theta_0\|^2\Big)e^{-c_{53}t}. \tag{3.62}$$

Using (3.58) and (3.61), by repeating the same procedure as in Lemma 3.3, we have

$$\begin{aligned} &e^{c_{53}t/2}\Big(\frac{c_{52}}{\kappa}\|\Theta(\cdot,t)\|^2 + \frac{\kappa}{2}\|\nabla\Theta(\cdot,t)\|^2\Big) \\ &+ \int_0^t e^{c_{53}\tau/2}\Big(c_{52}\|\nabla\Theta(\cdot,\tau)\|^2 + \frac{1}{2}\|\Theta_t(\cdot,\tau)\|^2\Big)d\tau \\ &\le 2\Big(\frac{c_{52}}{\kappa}\|\Theta_0\|^2 + \frac{\kappa}{2}\|\nabla\Theta_0\|^2\Big), \end{aligned} \tag{3.63}$$

which yields

$$\int_0^t e^{c_{53}\tau/2}\Big(c_{52}\|\nabla\Theta(\cdot,\tau)\|^2 + \frac{1}{2}\|\Theta_t(\cdot,\tau)\|^2\Big)d\tau \le 2\Big(\frac{c_{52}}{\kappa}\|\Theta_0\|^2 + \frac{\kappa}{2}\|\nabla\Theta_0\|^2\Big). \tag{3.64}$$

In view of (3.55) we see that

$$\int_0^t e^{c_{53}\tau/2}\|\Theta(\cdot,\tau)\|_{H^2}^2 d\tau \le c_{54}. \tag{3.65}$$

Therefore, (3.47) follows from (3.62) and (3.65). This completes the proof. □

With the help of Lemma 3.6, we have the following result.

Lemma 3.7. *Under the assumptions of Theorem 1.2, for all $t \ge 0$,*

$$\|\mu(\cdot,t)\|_{H^1}^2 + \|\Phi_t(\cdot,t)\|^2 + \|U(\cdot,t)\|_{W^{1,4}}^2 + \|U(\cdot,t)\|_{L^\infty}^2 + \|U_t(\cdot,t)\|^2 \le c_{55}. \tag{3.66}$$

Proof. **Step 1.** Taking L^2 inner product of (3.27) with $|\omega|^2\omega$ and using Hölder inequality we have

$$\frac{d}{dt}\|\omega\|_{L^4} \le 2\|\nabla\mu\|_{L^8}\|\nabla\Phi\|_{L^8} + \|\nabla\Theta\|_{L^4} \le c_{56}(\|\nabla\mu\|_{H^1}\|\Phi\|_{H^2} + \|\Theta\|_{H^2}). \quad (3.67)$$

Integrating (3.67) in time and using the previous lemmas, we have

$$\begin{aligned}\|\omega(\cdot,t)\|_{L^4} &\le c_{56}\Big(\int_0^t \|\nabla\mu\|_{H^1}^2 d\tau\Big)^{1/2}\Big(\int_0^t \|\Phi\|_{H^2}^2 d\tau\Big)^{1/2}\\ &\quad + c_{56}\Big(\int_0^t e^{-\beta_1\tau/2} d\tau\Big)^{1/2}\Big(\int_0^t e^{\beta_1\tau/2}\|\Theta\|_{H^2}^2 d\tau\Big)^{1/2} + \|\omega_0\|_{L^4}\\ &\le c_{57},\end{aligned} \quad (3.68)$$

which, together with Lemma 2.3 and Sobolev embedding, implies

$$\|U(\cdot,t)\|_{W^{1,4}} + \|U(\cdot,t)\|_{L^\infty} \le c_{58}, \quad \forall\, t \ge 0. \quad (3.69)$$

For the estimate of $\|U_t\|^2$, taking L^2 inner product of $(3.1)_3$ with U_t we have

$$\|U_t\|^2 \le c_{59}\big(\|U\|_{L^\infty}^2\|\nabla U\|^2 + \|\mu\|_{H^1}^2\|\Phi\|_{H^2}^2 + \|\Theta_0\|^2\big) \le c_{60}\big(\|\nabla\mu\|^2 + 1\big), \quad (3.70)$$

where we have used (3.42), (3.44) and (3.69).

Step 2. We now deal with μ and Φ_t. Taking L^2 inner product of $(3.1)_1$ with μ_t we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mu\|^2 + \alpha\|\nabla\Phi_t\|^2 = -\int_\Omega \big[F''(\phi)\Phi_t^2 + \mu_t(U\cdot\nabla\Phi)\big]d\mathbf{x}. \quad (3.71)$$

From (3.42) and Sobolev embedding we know that

$$\|\Phi\|_{L^\infty} \le c_{61}, \quad (3.72)$$

which according to the condition on F implies

$$\|F^{(p-n)}(\phi)\|_{L^\infty} \le F_1 c(p,n)\big(\|\Phi\|_{L^\infty}^{p-n} + \|\bar{\phi}\|_{L^\infty}^{p-n}\big) + F_2 \le c_{62}, \quad n = 1,\dots,6. \quad (3.73)$$

Using (3.73) we estimate the RHS of (3.71) as follows:

$$\begin{aligned}&\Big| -\int_\Omega \big[F''(\phi)\Phi_t^2 + \mu_t(U\cdot\nabla\Phi)\big]d\mathbf{x}\Big|\\ &\le \|F''(\phi)\|_{L^\infty}\|\Phi_t\|^2 + 2\alpha\|U\|_{L^\infty}^2\|\nabla\Phi\|^2 + \frac{1}{8\alpha}\|\mu_t\|^2\\ &\le c_{62}\|\Phi_t\|^2 + c_{63}\|\nabla\Phi\|^2 + \frac{1}{8\alpha}\big(2\alpha^2\|\Delta\Phi_t\|^2 + 2\|F''(\phi)\|_{L^\infty}^2\|\Phi_t\|^2\big)\\ &\le c_{64}\|\Phi_t\|^2 + c_{65}\|\Delta\Phi\|^2 + \frac{\alpha}{4}\|\Delta\Phi_t\|^2,\end{aligned}$$

where we have used (3.69) and Lemma 2.4. We update (3.71) as

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mu\|^2 + \alpha\|\nabla\Phi_t\|^2 \le c_{64}\|\Phi_t\|^2 + c_{65}\|\Delta\Phi\|^2 + \frac{\alpha}{4}\|\Delta\Phi_t\|^2. \quad (3.74)$$

Differentiating $(3.1)_1$ with respect to t we have

$$\Phi_{tt} + U_t\cdot\nabla\Phi + U\cdot\nabla\Phi_t = \Delta\mu_t. \quad (3.75)$$

Taking L^2 inner product of (3.75) with Φ_t we have

$$\frac{1}{2}\frac{d}{dt}\|\Phi_t\|^2 + \alpha\|\Delta\Phi_t\|^2 = \int_\Omega F''(\phi)\Phi_t\Delta\Phi_t d\mathbf{x} + \int_\Omega \Phi(U_t\cdot\nabla\Phi_t)d\mathbf{x}. \quad (3.76)$$

Using (3.73) and (3.70) we estimate the RHS of (3.76) as follows:

$$
\begin{aligned}
&\Big|\int_\Omega F''(\phi)\Phi_t\Delta\Phi_t d\mathbf{x} + \int_\Omega \Phi(U_t\cdot\nabla\Phi_t)d\mathbf{x}\Big| \\
&\le \frac{\alpha}{4}\|\Delta\Phi_t\|^2 + \frac{1}{\alpha}\|F''(\phi)\|_{L^\infty}^2\|\Phi_t\|^2 + \frac{1}{2\alpha}\|\Phi\|_{L^\infty}^2\|U_t\|^2 + \frac{\alpha}{2}\|\nabla\Phi_t\|^2 \\
&\le \frac{\alpha}{4}\|\Delta\Phi_t\|^2 + c_{66}\|\Phi_t\|^2 + c_{67}\|\Phi\|_{H^2}^2(\|\nabla\mu\|^2+1) + \frac{\alpha}{2}\|\nabla\Phi_t\|^2 \\
&\le \frac{\alpha}{4}\|\Delta\Phi_t\|^2 + c_{66}\|\Phi_t\|^2 + c_{68}\big(\|\nabla\mu\|^2 + \|\Delta\Phi\|^2\big) + \frac{\alpha}{2}\|\nabla\Phi_t\|^2.
\end{aligned}
\tag{3.77}
$$

So we update (3.76) as

$$
\frac{1}{2}\frac{d}{dt}\|\Phi_t\|^2 + \frac{3\alpha}{4}\|\Delta\Phi_t\|^2 \le c_{66}\|\Phi_t\|^2 + c_{68}\big(\|\nabla\mu\|^2 + \|\Delta\Phi\|^2\big) + \frac{\alpha}{2}\|\nabla\Phi_t\|^2. \tag{3.78}
$$

Combining (3.74) and (3.78) we obtain

$$
\frac{d}{dt}\Big(\|\nabla\mu\|^2+\|\Phi_t\|^2\Big)+\alpha\Big(\|\nabla\Phi_t\|^2+\|\Delta\Phi_t\|^2\Big) \le c_{69}\big(\|\Phi_t\|^2+\|\nabla\mu\|^2+\|\Delta\Phi\|^2\big). \tag{3.79}
$$

After integrating (3.79) in time and using (3.3) and (3.31) we have

$$
\|\nabla\mu(\cdot,t)\|^2 + \|\Phi_t(\cdot,t)\|^2 + \int_0^t \big(\|\nabla\Phi_t\|^2 + \|\Delta\Phi_t\|^2\big)d\tau \le c_{70}, \quad \forall\, t\ge 0. \tag{3.80}
$$

Substituting (3.80) in (3.70) we have $\|U_t(\cdot,t)\|^2 \le c_{71}$. This completes the proof. □

As consequences of previous lemmas, we have the following result.

Lemma 3.8. *Under the assumptions of Theorem 1.2, it holds*

$$
\|\Phi(\cdot,t)\|_{H^4}^2 + \|\mu(\cdot,t)\|_{H^2}^2 \le c_{72}, \quad \forall\, t\ge 0. \tag{3.81}
$$

Proof. First, by (3.45) we have

$$
\|\mu(\cdot,t)\|_{H^2}^2 \le c_{40}\big(\|\Phi_t(\cdot,t)\|^2 + \|\Phi(\cdot,t)\|_{H^2}^2\big) + \|\mu(\cdot,t)\|^2. \tag{3.82}
$$

Then the uniform estimate of $\|\mu(\cdot,t)\|_{H^2}^2$ follows from Lemma 3.5 and Lemma 3.7.

Second, by Lemma 2.4 and $(3.1)_2$ we have

$$
\|\Phi(\cdot,t)\|_{H^4}^2 \le c_{73}\big(\|\mu(\cdot,t)\|_{H^2}^2 + \|F'(\phi)(\cdot,t)\|_{H^2}^2\big). \tag{3.83}
$$

Using the second condition (H_2) on F, (3.31) and (3.73), it is straightforward to show that $\|F'(\phi)(\cdot,t)\|_{H^2}^2 \le c_{74}$, which together with the uniform bound of $\|\mu(\cdot,t)\|_{H^2}^2$ imply that $\|\Phi(\cdot,t)\|_{H^4}^2 \le c_{75}$. This completes the proof. □

4. Large time asymptotic behavior

In this section we prove Theorem 1.2, based on a sequence of accurate combinations of energy estimates. For the convenience of the reader, we first collect some uniform-in-time estimates. From (3.31), (3.66) and (3.81) we have, for any $t\ge 0$:

$$
\big(\|\Phi\|_{H^4}^2 + \|\mu\|_{H^2}^2 + \|\Phi_t\|^2 + \|F^{(p-n)}(\phi)\|_{L^\infty}^2 + \|U\|_{W^{1,4}}^2 + \|U\|_{L^\infty}^2 + \|U_t\|^2\big)(t) \le c_{76}. \tag{4.1}
$$

4.1. **Decay of (Φ,μ). Step 1.** First, by $(3.1)_2$ we have

$$\begin{aligned}\|\mu - F'(\bar\phi)\|_{H^3}^2 &= \|-\alpha\Delta\Phi + F'(\phi) - F'(\bar\phi)\|^2 + \|\nabla\mu\|_{H^2}^2\\ &\le 2\alpha^2\|\Delta\Phi\|^2 + 2\|F'(\phi)-F'(\bar\phi)\|^2 + \|\nabla\mu\|_{H^2}^2.\end{aligned}\tag{4.2}$$

Using (4.1) and Lemma 2.3 we estimate the last two terms on the RHS of (4.2) as follows:

$$\begin{aligned}&2\|F'(\phi)-F'(\bar\phi)\|^2 + \|\nabla\mu\|_{H^2}^2\\ &\le 2\|F''(\zeta)\|_{L^\infty}^2\|\Phi\|^2 + c_{77}\|\nabla\Delta\mu\|^2\\ &\le c_{78}\|\Phi\|^2 + c_{79}\big(\|\nabla\Phi_t\|^2 + \|\nabla(U\cdot\nabla\Phi)\|^2\big)\\ &\le c_{78}\|\Phi\|^2 + c_{80}\big(\|\nabla\Phi_t\|^2 + \|\nabla U\|^2\|\nabla\Phi\|_\infty^2 + \|U\|_\infty^2\|\nabla^2\Phi\|^2\big)\\ &\le c_{81}\big(\|\nabla\Phi_t\|^2 + \|\Phi\|_{H^3}^2\big).\end{aligned}\tag{4.3}$$

Combining (4.2) and (4.3) we have

$$\|\mu - F'(\bar\phi)\|_{H^3}^2 \le c_{82}\big(\|\nabla\Phi_t\|^2 + \|\Phi\|_{H^3}^2\big).\tag{4.4}$$

Combining (3.25) and (4.4) we then have

$$\|\mu - F'(\bar\phi)\|_{H^3}^2 \le c_{83}\big(\|\Phi_t\|_{H^1}^2 + \|\Phi\|_{H^2}^2 + \|\nabla\mu\|^2\big).\tag{4.5}$$

Second, by Lemma 2.4 and $(3.1)_2$ we have

$$\|\Phi\|_{H^5}^2 \le c_8\|\nabla\Delta^2\Phi\|^2 \le c_{84}\|\nabla\Delta\mu\|^2 + c_{85}\|\nabla\Delta F'(\phi)\|^2.\tag{4.6}$$

By direct calculations and Sobolev embeddings we can show that

$$\begin{aligned}\|\nabla F'(\phi)\|_{H^2}^2 &\le c_{86}\|F'''(\phi)\|_{L^\infty}^2\big(\|\nabla\Phi\|_{H^1}^4 + 2\|\nabla\Phi\|_{H^1}^2\|\nabla\Phi\|_{H^2}^2\big)\\ &\quad + c_{87}\|F''''(\phi)\|_{L^\infty}^2\|\nabla\Phi\|_{H^2}^4\|\nabla\Phi\|^2.\end{aligned}\tag{4.7}$$

Using (4.1) we obtain from (4.7) that $\|\nabla F'(\phi)\|_{H^2}^2 \le c_{88}\|\nabla\Phi\|_{H^2}^2$, which, together with (4.6), implies that

$$\|\Phi\|_{H^5}^2 \le c_{89}\big(\|\nabla\Phi_t\|^2 + \|\Phi\|_{H^3}^2\big).\tag{4.8}$$

Combining (4.5), (4.8) and (3.25) we have

$$\|\mu - F'(\bar\phi)\|_{H^3}^2 + \|\Phi\|_{H^5}^2 \le c_{90}\big(\|\Phi_t\|_{H^1}^2 + \|\Phi\|_{H^2}^2 + \|\nabla\mu\|^2\big).\tag{4.9}$$

Therefore, it suffices to show the decay of RHS of (4.9) in order to prove the decay of Φ and μ.

Step 2. We recall (3.17),

$$\frac{d}{dt}\Big(\frac{\alpha}{2}\|\nabla\Phi\|^2 + \int_\Omega F(\phi)d\mathbf{x}\Big) + \|\nabla\mu\|^2 = -\int_\Omega \mu(\nabla\Phi\cdot U)d\mathbf{x}.\tag{4.10}$$

Due to the structure of the function $F(\cdot)$, there may be a constant term in the integral $\int_\Omega F(\phi)d\mathbf{x}$ in general, which is impossible to decay. In order to resolve this issue, we observe, since $\int_\Omega(\phi-\overline{\phi})d\mathbf{x} = 0$, it holds that

$$\begin{aligned}\int_\Omega F(\phi) - F(\overline{\phi})d\mathbf{x} &= \int_\Omega F'(\overline{\phi})(\phi-\overline{\phi})d\mathbf{x} + \frac12\int_\Omega F''(\xi)(\phi-\overline{\phi})^2d\mathbf{x}\\ &= \frac12\int_\Omega F''(\xi)\Phi^2d\mathbf{x}, \quad \text{for some } \xi \text{ between } \phi \text{ and } \bar\phi.\end{aligned}$$

Then we update (4.10) as

$$\frac{d}{dt}\Big(\frac{\alpha}{2}\|\nabla\Phi\|^2+\frac{1}{2}\int_\Omega F''(\xi)\Phi^2 d\mathbf{x}\Big)+\|\nabla\mu\|^2=-\int_\Omega \mu(\nabla\Phi\cdot U)d\mathbf{x}. \tag{4.11}$$

Using (4.1) and Lemma 2.4 we estimate the RHS of (4.11) as

$$\begin{aligned}\Big|-\int_\Omega \mu(\nabla\Phi\cdot U)d\mathbf{x}\Big| &= \Big|\int_\Omega \Phi U\cdot\nabla\mu d\mathbf{x}\Big|\\ &\le \frac{1}{2}\|\nabla\mu\|^2+\frac{1}{2}\|U\|_{L^4}^2\|\Phi\|_{L^4}^2\\ &\le \frac{1}{2}\|\nabla\mu\|^2+c_{91}\|\Phi\|_{H^1}^2\\ &\le \frac{1}{2}\|\nabla\mu\|^2+c_{92}\|\Delta\Phi\|^2.\end{aligned} \tag{4.12}$$

Substituting (4.12) in (4.11), we have

$$\frac{d}{dt}\Big(\alpha\|\nabla\Phi\|^2+\int_\Omega F''(\xi)\Phi^2 d\mathbf{x}\Big)+\|\nabla\mu\|^2\le c_{93}\|\Delta\Phi\|^2. \tag{4.13}$$

Step 3. Recalling (3.34) and using (4.1) and Lemma 2.4 we have

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{\alpha}{2}\|\Delta\Phi\|^2+\frac{1}{2}\int_\Omega F''(\phi)|\nabla\Phi|^2 d\mathbf{x}\Big)+\|\Phi_t\|^2\\ &=\frac{1}{2}\int_\Omega F'''(\phi)\Phi_t|\nabla\Phi|^2 d\mathbf{x}-\int_\Omega \Phi_t(U\cdot\nabla\Phi)d\mathbf{x}\\ &\le \frac{1}{2}\|\Phi_t\|^2+c_{94}(\|\nabla\Phi\|_{H^1}^4+\|\nabla\Phi\|_{H^1}^2)\\ &\le \frac{1}{2}\|\Phi_t\|^2+c_{95}\|\Delta\Phi\|^2,\end{aligned}$$

which yields

$$\frac{d}{dt}\Big(\alpha\|\Delta\Phi\|^2+\int_\Omega F''(\phi)|\nabla\Phi|^2 d\mathbf{x}\Big)+\|\Phi_t\|^2\le c_{96}\|\Delta\Phi\|^2. \tag{4.14}$$

Step 4. From (3.77) and (4.1) we have

$$\frac{1}{2}\frac{d}{dt}\|\Phi_t\|^2+\alpha\|\Delta\Phi_t\|^2\le\frac{\alpha}{4}\|\Delta\Phi_t\|^2+\frac{\alpha}{2}\|\nabla\Phi_t\|^2+c_{97}(\|\Phi_t\|^2+\|\Delta\Phi\|^2). \tag{4.15}$$

Combining (3.74) and (4.15), we have

$$\frac{d}{dt}\big(\|\nabla\mu\|^2+\|\Phi_t\|^2\big)+\alpha\big(\|\nabla\Phi_t\|^2+\|\Delta\Phi_t\|^2\big)\le c_{98}\big(\|\Phi_t\|^2+\|\Delta\Phi\|^2\big). \tag{4.16}$$

Step 5. Taking L^2 inner product of (3.75) with μ_t we have

$$\frac{\alpha}{2}\frac{d}{dt}\|\nabla\Phi_t\|^2+\|\nabla\mu_t\|^2=\int_\Omega (U_t\Phi+U\Phi_t)\cdot\nabla\mu_t d\mathbf{x}-\int_\Omega F''(\phi)\Phi_t\Phi_{tt}d\mathbf{x}. \tag{4.17}$$

For the last term on the RHS of (4.17), we have

$$-\int_\Omega F''(\phi)\Phi_t\Phi_{tt}d\mathbf{x}=-\frac{1}{2}\frac{d}{dt}\int_\Omega F''(\phi)\Phi_t^2 d\mathbf{x}+\frac{1}{2}\int_\Omega F'''(\phi)\Phi_t^3 d\mathbf{x}.$$

So we update (4.17) as

$$\begin{aligned}&\frac{d}{dt}\Big(\alpha\|\nabla\Phi_t\|^2+\int_\Omega F''(\phi)\Phi_t^2d\mathbf{x}\Big)+2\|\nabla\mu_t\|^2\\&=2\int_\Omega(U_t\Phi+U\Phi_t)\cdot\nabla\mu_t d\mathbf{x}+\int_\Omega F'''(\phi)\Phi_t^3d\mathbf{x}.\end{aligned}\tag{4.18}$$

Using (4.1) and Lemma 3.6 we estimate the first two terms on the RHS of (4.18) as

$$\begin{aligned}\Big|2\int_\Omega(U_t\Phi+U\Phi_t)\cdot\nabla\mu_t d\mathbf{x}\Big|&\le\|\nabla\mu_t\|^2+\|U_t\|^2\|\Phi\|_{L^\infty}^2+\|U\|_{L^\infty}^2\|\Phi_t\|^2\\&\le\|\nabla\mu_t\|^2+c_{99}\|\Phi\|_{H^2}^2+c_{76}\|\Phi_t\|^2\\&\le\|\nabla\mu_t\|^2+c_{100}\|\Delta\Phi\|^2+c_{101}\|\Delta\Phi_t\|^2.\end{aligned}\tag{4.19}$$

Similarly, for the term involving Φ_t^3, we have

$$\begin{aligned}\Big|\int_\Omega F'''(\phi)\Phi_t^3d\mathbf{x}\Big|&\le\|F'''(\phi)\|_\infty\|\Phi_t\|_{L^3}^3\\&\le c_{76}\|\Phi_t\|_\infty^2\|\Phi_t\|_{L^1}\\&\le c_{102}\|\Delta\Phi_t\|^2\|\Phi_t\|\le c_{103}\|\Delta\Phi_t\|^2.\end{aligned}\tag{4.20}$$

Substituting (4.19) and (4.20) in (4.18), we have

$$\frac{d}{dt}\Big(\alpha\|\nabla\Phi_t\|^2+\int_\Omega F''(\phi)\Phi_t^2d\mathbf{x}\Big)+\|\nabla\mu_t\|^2\le c_{104}\big(\|\Delta\Phi_t\|^2+\|\Delta\Phi\|^2\big).\tag{4.21}$$

Step 6. In this step, we make combinations of energy estimates, which will be used to prove the exponential decay of Φ and μ. First, we collect energy inequalities from **Steps 2–5**:

(4.13)	$\frac{d}{dt}\Big(\alpha\|\nabla\Phi\|^2+\int_\Omega F''(\xi)\Phi^2d\mathbf{x}\Big)+\|\nabla\mu\|^2\le c_{93}\|\Delta\Phi\|^2$
(4.14)	$\frac{d}{dt}\Big(\alpha\|\Delta\Phi\|^2+\int_\Omega F''(\phi)\lvert\nabla\Phi\rvert^2d\mathbf{x}\Big)+\|\Phi_t\|^2\le c_{96}\|\Delta\Phi\|^2$
(4.16)	$\frac{d}{dt}\big(\|\nabla\mu\|^2+\|\Phi_t\|^2\big)+\alpha\big(\|\nabla\Phi_t\|^2+\|\Delta\Phi_t\|^2\big)\le c_{98}\big(\|\Phi_t\|^2+\|\Delta\Phi\|^2\big)$
(4.21)	$\frac{d}{dt}\Big(\alpha\|\nabla\Phi_t\|^2+\int_\Omega F''(\phi)\Phi_t^2d\mathbf{x}\Big)+\|\nabla\mu_t\|^2\le c_{104}\big(\|\Delta\Phi_t\|^2+\|\Delta\Phi\|^2\big)$

First, multiply (4.16) by $\frac{2c_{104}}{\alpha}$ and then add (4.21) to obtain

$$\frac{d}{dt}\big[J_0(t)\big]+K_0(t)\le c_{105}\big(\|\Phi_t\|^2+\|\Delta\Phi\|^2\big),\tag{4.22}$$

where

$$\begin{gathered}J_0(t)\equiv\frac{2c_{104}}{\alpha}\big(\|\nabla\mu\|^2+\|\Phi_t\|^2\big)+\alpha\|\nabla\Phi_t\|^2+\int_\Omega F''(\phi)\Phi_t^2d\mathbf{x},\\K_0(t)\equiv c_{104}\big(2\|\nabla\Phi_t\|^2+\|\Delta\Phi_t\|^2\big)+\|\nabla\mu_t\|^2.\end{gathered}$$

Second, multiply (4.14) by $2c_{105}$ then add (4.22) to obtain

$$\frac{d}{dt}\big[J_1(t)\big]+K_1(t)\le c_{106}\|\Delta\Phi\|^2,\tag{4.23}$$

where

$$\begin{gathered}J_1(t)\equiv J_0(t)+2c_{105}\Big(\alpha\|\Delta\Phi\|^2+\int_\Omega F''(\phi)|\nabla\Phi|^2d\mathbf{x}\Big),\\K_1(t)\equiv c_{105}\|\Phi_t\|^2+K_0(t).\end{gathered}$$

Third, by coupling (4.23) and (4.13), we have

$$\frac{d}{dt}\big[J_2(t)\big] + K_2(t) \le c_{107}\|\Delta\Phi\|^2, \tag{4.24}$$

where

$$J_2(t) \equiv J_1(t) + \Big(\alpha\|\nabla\Phi\|^2 + \int_\Omega F''(\xi)\Phi^2 d\mathbf{x}\Big), \quad K_2(t) \equiv K_1(t) + \|\nabla\mu\|^2.$$

Recalling (3.7), we have

$$\frac{d}{dt}\|\Phi\|^2 + 2\alpha_1\|\Delta\Phi\|^2 \le 0. \tag{4.25}$$

Then, multiply (4.25) by $\frac{c_{107}}{\alpha_1}$ then add (4.24) to obtain

$$\frac{d}{dt}\big[J_3(t)\big] + K_3(t) \le 0, \tag{4.26}$$

where

$$J_3(t) \equiv J_2(t) + \frac{c_{107}}{\alpha_1}\|\Phi\|^2, \quad K_3(t) \equiv K_2(t) + c_{107}\|\Delta\Phi\|^2.$$

Step 7. In this step we apply the condition $\alpha_1 = \alpha - F_3 c_0 > 0$ to uncover the secret hidden in (4.26), which will give the desired decay estimates of Φ and μ. First, by (4.1) and Poincaré inequality we have

$$\begin{aligned}
&\frac{2c_{104}}{\alpha}\big(\|\nabla\mu\|^2 + \|\Phi_t\|^2\big) + \alpha_1\|\nabla\Phi_t\|^2\\
&\le J_0(t)\\
&\le \frac{2c_{104}}{\alpha}\big(\|\nabla\mu\|^2 + \|\Phi_t\|^2\big) + \alpha\|\nabla\Phi_t\|^2 + c_{76}\|\Phi_t\|^2,
\end{aligned}$$

which implies

$$J_0(t) \cong \|\nabla\mu\|^2 + \|\Phi_t\|_{H^1}^2.$$

where the symbol $\cong$ denotes the equivalence of quantities. Also, by Poincaré inequality, we have

$$\begin{aligned}
&c_{104}\big(c_0\|\Phi_t\|^2 + \|\nabla\Phi_t\|^2 + \|\Delta\Phi_t\|^2\big) + \|\nabla\mu_t\|^2\\
&\le K_0(t)\\
&\le c_{104}\big(\|\Phi_t\|^2 + 2\|\nabla\Phi_t\|^2 + \|\Delta\Phi_t\|^2\big) + \|\nabla\mu_t\|^2.
\end{aligned}$$

Therefore, $K_0(t) \cong \|\Phi_t\|_{H^2}^2 + \|\nabla\mu_t\|^2$.

Similarly, we have

$J_1(t) \cong \|\nabla\mu\|^2 + \|\Phi_t\|_{H^1}^2 + \|\Phi\|_{H^2}^2$	$K_1(t) \cong \|\Phi_t\|_{H^2}^2 + \|\nabla\mu_t\|^2$
$J_2(t) \cong \|\nabla\mu\|^2 + \|\Phi_t\|_{H^1}^2 + \|\Phi\|_{H^2}^2$	$K_2(t) \cong \|\Phi_t\|_{H^2}^2 + \|\nabla\mu_t\|^2 + \|\nabla\mu\|^2$
$J_3(t) \cong \|\nabla\mu\|^2 + \|\Phi_t\|_{H^1}^2 + \|\Phi\|_{H^2}^2$	$K_3(t) \cong \|\Phi_t\|_{H^2}^2 + \|\nabla\mu_t\|^2 + \|\nabla\mu\|^2 + \|\Phi\|_{H^2}^2$

Then it is clear that there exists a constant $c_{108} > 0$ independent of t such that $c_{108}J_3(t) \le K_3(t)$, which, together with (4.26), implies

$$\frac{d}{dt}\big(J_3(t)\big) + c_{108}J_3(t) \le 0,$$

which gives

$$J_3(t) \le J_3(0)e^{-c_{108}t}, \quad \forall\, t \ge 0.$$

We then have

$$\|\nabla\mu(\cdot,t)\|^2 + \|\Phi_t(\cdot,t)\|_{H^1}^2 + \|\Phi(\cdot,t)\|_{H^2}^2 \le c_{109}e^{-c_{108}t}, \quad \forall\, t \ge 0,$$

which, together with (4.9), implies

$$\|\Phi(\cdot,t)\|_{H^5}^2 + \|(\mu - F'(\bar{\phi}))(\cdot,t)\|_{H^3}^2 \le c_{110}e^{-c_{108}t}, \quad \forall\, t \ge 0, \tag{4.27}$$

4.2. **Decay of Θ.** In this section, we show the exponential decay of $\|\Theta\|_{H^3}$. Again, this will be done by combining uniform estimates of the solution.
Step 1. Since $\Theta|_{\partial\Omega} = 0$, by Lemma 2.2 and (4.1) we have

$$\begin{aligned}
\|\Theta\|_{H^3}^2 &\le c_{111}\big(\|\Theta_t\|_{H^1}^2 + \|U\cdot\nabla\Theta\|_{H^1}^2\big)\\
&\le c_{112}\big(\|\Theta_t\|_{H^1}^2 + \|U\|_{L^\infty}^2\|\nabla\Theta\|^2 + \|\nabla U\|_{L^4}^2\|\nabla\Theta\|_{L^4}^2 + \|U\|_{L^\infty}^2\|\nabla^2\Theta\|^2\big)\\
&\le c_{113}\big(\|\Theta_t\|_{H^1}^2 + \|\Theta\|_{H^2}^2\big).
\end{aligned} \tag{4.28}$$

Step 2. By taking temporal derivative of $(3.1)_4$ we obtain

$$\Theta_{tt} + U_t\cdot\nabla\Theta + U\cdot\nabla\Theta_t = \kappa\Delta\Theta_t. \tag{4.29}$$

Taking L^2 inner product of (4.29) with Θ_t and using (4.1) we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Theta_t\|^2 + \kappa\|\nabla\Theta_t\|^2 &= \int_\Omega \Theta(U_t\cdot\nabla\Theta_t)d\mathbf{x}\\
&\le \frac{\kappa}{2}\|\nabla\Theta_t\|^2 + \frac{1}{2\kappa}\|U_t\|^2\|\Theta\|_{L^\infty}^2\\
&\le \frac{\kappa}{2}\|\nabla\Theta_t\|^2 + c_{114}\|\Theta\|_{H^2}^2,
\end{aligned}$$

which gives

$$\frac{d}{dt}\|\Theta_t\|^2 + \kappa\|\nabla\Theta_t\|^2 \le c_{115}\|\Theta\|_{H^2}^2. \tag{4.30}$$

Substituting (3.55) in (4.30), we have

$$\frac{d}{dt}\|\Theta_t\|^2 + \kappa\|\nabla\Theta_t\|^2 \le c_{116}\big(\|\nabla\Theta\|^2 + \|\Theta_t\|^2\big). \tag{4.31}$$

Now, we recall (3.58),

$$\frac{d}{dt}\Big(\frac{c_{52}}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\Big) + c_{52}\|\nabla\Theta\|^2 + \frac{1}{2}\|\Theta_t\|^2 \le 0. \tag{4.32}$$

By absorbing the RHS of (4.31) into the LHS of (4.32) we have

$$\frac{d}{dt}\big(J_4(t)\big) + K_4(t) \le 0, \tag{4.33}$$

where

$$\begin{gathered}
J_4(t) \cong \|\Theta\|^2 + \|\nabla\Theta\|^2 + \|\Theta_t\|^2 \cong \|\Theta\|_{H^2}^2 + \|\Theta_t\|^2,\\
K_4(t) \cong \|\nabla\Theta\|^2 + \|\Theta_t\|^2 + \|\nabla\Theta_t\|^2 \cong \|\Theta\|_{H^2}^2 + \|\Theta_t\|_{H^1}^2.
\end{gathered} \tag{4.34}$$

Step 3. By taking L^2 inner product of (4.29) with Θ_{tt} and using Cauchy-Schwarz inequality, we have

$$\frac{\kappa}{2}\frac{d}{dt}\|\nabla\Theta_t\|^2 + \frac{1}{2}\|\Theta_{tt}\|^2 \le \|U_t\|^2\|\nabla\Theta\|_{L^\infty}^2 + \|U\|_{L^\infty}^2\|\nabla\Theta_t\|^2. \tag{4.35}$$

For the RHS of (4.35), using (4.1), Lemma 2.1 and Lemma 2.2 with $m = 0, p = 3$, we have

$$\begin{aligned}\|U_t\|^2\|\nabla\Theta\|_{L^\infty}^2 + \|U\|_{L^\infty}^2\|\nabla\Theta_t\|^2 &\le c_{117}\big(\|\Theta\|_{W^{2,3}}^2 + \|\nabla\Theta_t\|^2\big)\\ &\le c_{118}\big(\|\Theta_t\|_{L^3}^2 + \|U\cdot\nabla\Theta\|_{L^3}^2 + \|\nabla\Theta_t\|^2\big)\\ &\le c_{119}\big(\|\Theta_t\|_{H^1}^2 + \|U\|_{L^\infty}^2\|\nabla\Theta\|_{H^1}^2 + \|\nabla\Theta_t\|^2\big)\\ &\le c_{120}\big(\|\Theta_t\|_{H^1}^2 + \|\nabla\Theta\|^2\big),\end{aligned} \tag{4.36}$$

where we have used (3.55) for $\|\nabla\Theta\|_{H^1}^2$. So we update (4.35) as

$$\kappa\frac{d}{dt}\|\nabla\Theta_t\|^2 + \|\Theta_{tt}\|^2 \le c_{121}\Big(\|\Theta_t\|_{H^1}^2 + \|\nabla\Theta\|^2\Big). \tag{4.37}$$

Then it is clear that, by absorbing the RHS of (4.37) into the LHS of (4.33), it holds that

$$\frac{d}{dt}\big(J_5(t)\big) + K_5(t) \le 0, \tag{4.38}$$

where

$$J_5(t) \cong \|\Theta\|_{H^2}^2 + \|\Theta_t\|_{H^1}^2, \quad K_5(t) \cong \|\Theta\|_{H^2}^2 + \|\Theta_t\|_{H^1}^2 + \|\Theta_{tt}\|^2. \tag{4.39}$$

Therefore,

$$\|\Theta(\cdot,t)\|_{H^2}^2 + \|\Theta_t(\cdot,t)\|_{H^1}^2 \le c_{122}e^{-c_{123}t}, \quad \forall\, t \ge 0. \tag{4.40}$$

which, together with (4.28), implies that

$$\|\Theta(\cdot,t)\|_{H^3}^2 \le c_{124}e^{-c_{123}t}, \quad \forall\, t \ge 0. \tag{4.41}$$

4.3. **Uniform estimates of U and ω.** In this section, we show uniform estimates for U and ω as indicated in Theorem 1.2. For this purpose, we observe, for any $p \ge 2$, it holds that

$$\begin{aligned}\frac{d}{dt}\|\omega\|_{L^p} &\le 2\|\nabla\mu\|_{L^{2p}}\|\nabla\Phi\|_{L^{2p}} + \|\nabla\Theta\|_{L^p}\\ &\le 2\|\nabla\mu\|_{L^\infty}|\Omega|^{\frac{1}{2p}}\|\nabla\Phi\|_{L^\infty}|\Omega|^{\frac{1}{2p}} + \|\nabla\Theta\|_{L^\infty}|\Omega|^{1/p}\\ &\le c_{127}\max\{1,|\Omega|\}\big(\|\nabla\mu\|_{H^2}\|\nabla\Phi\|_{H^2} + \|\nabla\Theta\|_{H^2}\big).\end{aligned} \tag{4.42}$$

Using the decay estimates for Φ, μ and Θ, we update (4.42) as

$$\frac{d}{dt}\|\omega\|_{L^p} \le c_{128}e^{-c_{129}t}, \quad \forall\, t \ge 0, \tag{4.43}$$

where the constants c_{128} and c_{129} are independent of $p \ge 2$. Upon integrating (4.43) in time, we have

$$\|\omega(\cdot,t)\|_{L^p} \le c_{130} + \|\omega_0\|_{L^\infty}\max\{1,|\Omega|\}, \quad \forall\, t \ge 0. \tag{4.44}$$

Letting $p \to \infty$ in (4.44) we have

$$\|\omega(\cdot,t)\|_{L^\infty} \le c_{131}, \quad \forall\, t \ge 0. \tag{4.45}$$

Moreover, by Lemma 2.3,

$$\begin{aligned}\|U(\cdot,t)\|_{W^{1,p}} &\le c_7(p)\big(\|\omega(\cdot,t)\|_{L^p} + \|U(\cdot,t)\|_{L^p}\big)\\ &\le c_7(p)\big(\|\omega(\cdot,t)\|_{L^\infty} + \|U(\cdot,t)\|_{L^\infty}\big)\max\{1,|\Omega|\}.\end{aligned}$$

By (4.1) and (4.45), we have

$$\|U(\cdot,t)\|_{W^{1,p}} \le c_{132}(p), \quad \forall\, t \ge 0,\ \forall\, 1 \le p < \infty.$$

This estimate, together with (4.27), (4.41) and (4.45), completes the proof of Theorem 1.2.

We complete this section with the following remark.

Remark 4.1. Using the arguments in this paper one can show that Theorem 1.2 still holds if the Dirichlet boundary condition for θ is replaced by the Neumann boundary condition $\nabla\theta \cdot \mathbf{n}|_{\partial\Omega} = 0$. In this case, the asymptotic state of θ is $\tilde{\theta} = \frac{1}{|\Omega|}\int_\Omega \theta_0(\mathbf{x})d\mathbf{x}$, which is a constant.

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