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# INTERIOR FEEDBACK STABILIZATION OF WAVE EQUATIONS WITH TIME DEPENDENT DELAY

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Abstract. We study the stabilization problem by interior damping of the wave equation with boundary or internal time-varying delay feedback in a bounded and smooth domain. By introducing suitable Lyapunov functionals exponential stability estimates are obtained if the delay effect is appropriately compensated by the internal damping.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with boundary $\Gamma$ of class $C^2$. We assume that $\Gamma$ is divided into two parts $\Gamma_0$ and $\Gamma_1$; i.e., $\Gamma = \Gamma_0 \cup \Gamma_1$, with $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ and $\operatorname{meas} \Gamma_0 \neq 0$.

We consider the problem

$$u_{tt}(x,t) - \Delta u(x,t) - a\Delta u_t(x,t) = 0 \quad \text{in } \Omega \times (0,+\infty), \tag{1.1}$$

$$u(x,t) = 0 \quad \text{on } \Gamma_0 \times (0,+\infty), \tag{1.2}$$

$$\mu u_{tt}(x,t) = -\frac{\partial(u + a u_t)}{\partial \nu}(x,t) - k u_t(x, t-\tau(t)) \quad \text{on } \Gamma_1 \times (0,+\infty), \tag{1.3}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \tag{1.4}$$

$$u_t(x,t) = f_0(x,t) \quad \text{in } \Gamma_1 \times (-\tau(0), 0), \tag{1.5}$$

where $\nu(x)$ denotes the outer unit normal vector to the point $x \in \Gamma$ and $\frac{\partial u}{\partial \nu}$ is the normal derivative. Moreover, $\tau = \tau(t)$ is the time delay, $\mu, a, k$ are real numbers, with $\mu \geq 0, a > 0$, and the initial datum $(u_0, u_1, f_0)$ belongs to a suitable space.

It is well-known that the above model is exponentially stable in absence of delay; that is, if $\tau(t) \equiv 0$. We refer to [2, 18, 17, 19, 14, 16, 15, 30] for the more studied case $a = 0$, $\mu = 0$ and to [11, 22, 7, 28, 9] in the case $a, \mu > 0$.

In presence of a constant delay, when $\mu = 0$ and the condition (1.3) is substituted by

$$\frac{\partial u}{\partial \nu}(x,t) = -k u_t(x, t-\tau), \quad \Gamma_1 \times (0,+\infty),$$

the system becomes unstable for arbitrarily small delays (see [5]).

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Then, the 1-d version of the above model with $\mu = 0$ in the boundary condition (1.3) has been considered by Morgül [22] who proposed a class of dynamic boundary controllers to solve the stability robustness problem.

In the case $\mu > 0$, (1.3) is a so-called dynamic boundary condition. Dynamic boundary conditions arise in many physical applications, in particular they occur in elastic models. For instance, these conditions appear in modelling dynamic vibrations of linear viscoelastic rods and beams which have attached tip masses at their free ends. See [1, 3, 21, 9] and the references therein for more details. The above model without delay (e.g. $\tau = 0$) has been proposed in one dimension by Pellicer and Sòla-Morales [28] as an alternative model for the classical spring-mass damper system, the case $k = 0$ being treated in [11]. In both cases, no rates of convergence are proved. In dimension higher than 1, we refer to Gerbi and Said-Houari [9] where a nonlinear boundary feedback is even considered and the exponential growth of the energy is proved if the initial data are large enough. A different problem with a dynamic boundary condition (without delay), motivated by the study of flows of gas in a channel with porous walls, is analyzed in [7] where exponential decay is proved.

On the function $\tau$ we assume that there exist positive constants $\tau_0, \overline{\tau}$ such that

$$0 < \tau_0 \le \tau(t) \le \overline{\tau}, \quad \forall t > 0. \tag{1.6}$$

Moreover, we assume

$$\tau \in W^{2,\infty}([0,T]), \quad \forall T > 0, \tag{1.7}$$

$$\tau'(t) \le d < 1 \quad \forall t > 0. \tag{1.8}$$

Under the above assumptions on the time-delay function $\tau(t)$ we will prove that an exponential stability result holds under a suitable assumption between the coefficients $a$ and $k$ (namely condition (2.56) below).

We consider also the problem with interior delay

$$u_{tt}(x,t) - \Delta u(x,t) + a_0 u_t(x,t) + a_1 u_t(x, t-\tau(t)) = 0 \quad \text{in } \Omega \times (0,+\infty) \tag{1.9}$$

$$u(x,t) = 0 \quad \text{on } \Gamma \times (0,+\infty) \tag{1.10}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega \tag{1.11}$$

$$u_t(x,t) = g_0(x,t) \quad \text{in } \Omega \times (-\tau(0), 0), \tag{1.12}$$

where $\tau(t) > 0$ is the time-varying delay, $a_0$ and $a_1$ are real numbers with $a_0 > 0$, and the initial datum $(u_0, u_1, g_0)$ belongs to a suitable space.

The above model, with $a_0 > 0$, $a_1 > 0$ and a constant delay $\tau(t) \equiv \tau$ has been studied by the authors [23] in the case of mixed homogeneous Dirichlet-Neumann boundary conditions. Assuming that

$$0 \le a_1 < a_0 \tag{1.13}$$

a stabilization result is given, by using a suitable observability estimate. This is done by applying inequalities obtained from Carleman estimates for the wave equation by Lasiecka, Triggiani and Yao in [20] and by using compactness-uniqueness arguments. Instability phenomena when (1.13) is not satisfied are also illustrated. We refer to [6, 4] for instability examples of related problems in one dimension.

The analogous problem with boundary feedback has been introduced and studied by Xu, Yung, Li [29] in one-space dimension using a fine spectral analysis and in higher space dimension by the authors [23].

The case of time-varying delay has been already studied in [26] in one space dimension and in general dimension, with a possibly degenerate delay, in [25]. Both these papers deal with boundary feedback. See also [8] for abstract problems also under the assumption of non-degeneracy of $\tau(t)$.

Here, we will give an exponential stability result for problem (1.9)-(1.12) under the condition

$$|a_1| < \sqrt{1-d}\, a_0 \,, \tag{1.14}$$

where $d$ is the constant in (1.8).

The outline of the paper is the following. In section 2 we study well-posedness and exponential stability of the problem (1.1)–(1.5) with structural damping and boundary delay in both cases $\mu > 0$ and $\mu = 0$. In section 3 we analyze the problem with internal delay feedback (1.9)–(1.12).

## 2. Boundary delay feedback

In this section we concentrate on the problem with boundary delay (1.1)–(1.5).

Let $C_P$ be a Poincaré's type constant defined as the smallest positive constant such that

$$\int_{\Gamma_1} |v|^2 \, d\Gamma \le C_P \int_\Omega |\nabla v|^2 \, dx, \ \forall v \in H^1_{\Gamma_0}(\Omega), \tag{2.1}$$

where, as usual,

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0\}.$$

First of all we will give a well-posedness result under the assumption

$$|k| \le \frac{a}{C_P}\sqrt{1-d}, \tag{2.2}$$

where $d$ is the positive constant of assumption (1.8). We have to distinguish the two cases $\mu > 0$ and $\mu = 0$.

2.1. **Well-posedness in the case of dynamic boundary condition.** First we study the well-posedness of (1.1)–(1.5) for $\mu > 0$. We introduce the auxiliary unknown

$$z(x, \rho, t) = u_t(x, t - \tau(t)\rho), \quad x \in \Gamma_1, \ \rho \in (0, 1), \ t > 0. \tag{2.3}$$

Then, problem (1.1)–(1.5) is equivalent to

$$u_{tt}(x, t) - \Delta u(x, t) - a\Delta u_t(x, t) = 0 \quad \text{in } \Omega \times (0, +\infty), \tag{2.4}$$

$$\tau(t)z_t(x, \rho, t) + (1 - \tau'(t)\rho)z_\rho(x, \rho, t) = 0 \quad \text{in } \Gamma_1 \times (0, 1) \times (0, +\infty), \tag{2.5}$$

$$u(x, t) = 0 \quad \text{on } \Gamma_0 \times (0, +\infty), \tag{2.6}$$

$$\mu u_{tt}(x, t) = -\frac{\partial(u + au_t)}{\partial\nu}(x, t) - kz(x, 1, t) \quad \text{on } \Gamma_1 \times (0, +\infty), \tag{2.7}$$

$$z(x, 0, t) = u_t(x, t) \quad \text{on } \Gamma_1 \times (0, \infty), \tag{2.8}$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega, \tag{2.9}$$

$$z(x, \rho, 0) = f_0(x, -\rho\tau(0)) \quad \text{in } \Gamma_1 \times (0, 1). \tag{2.10}$$

Let us denote

$$U := (u, u_t, \gamma_1 u_t, z)^T,$$

where $\gamma_1$ is the trace operator on $\Gamma_1$. Then the previous problem is formally equivalent to

$$U' := (u_t, u_{tt}, \gamma_1 u_{tt}, z_t)^T$$

$$= \Big(u_t, \Delta u + a\Delta u_t, -\mu^{-1}\big(\frac{\partial(u+au_t)}{\partial\nu}(x,t) + kz(\cdot,1,\cdot)\Big), \frac{\tau'(t)\rho-1}{\tau(t)}z_\rho\Big)^T.$$

Therefore, problem (1.1)–(1.5) can be rewritten as

$$\begin{gathered} U' = \mathcal{A}(t)U, \\ U(0) = \big(u_0, u_1, \gamma_1 u_1, f_0(\cdot, -\cdot\,\tau)\big)^T, \end{gathered} \tag{2.11}$$

where the time varying operator $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t)\begin{pmatrix} u \\ v \\ v_1 \\ z \end{pmatrix} := \begin{pmatrix} v \\ \Delta(u+av) \\ -\mu^{-1}\left(\frac{\partial(u+av)}{\partial\nu} + kz(\cdot,1)\right) \\ \frac{\tau'(t)\rho-1}{\tau(t)}z_\rho \end{pmatrix},$$

with domain

$$\begin{aligned} \mathcal{D}(\mathcal{A}(t)) := \Big\{&(u,v,v_1,z)^T \in H^1_{\Gamma_0}(\Omega)^2 \times L^2(\Gamma_1) \times L^2(\Gamma_1; H^1(0,1)) : \\ &u + av \in E(\Delta, L^2(\Omega)), \frac{\partial(u+av)}{\partial\nu} \in L^2(\Gamma_1), \\ &v = v_1 = z(\cdot,0) \text{on } \Gamma_1\Big\}, \end{aligned} \tag{2.12}$$

where

$$E(\Delta, L^2(\Omega)) = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}.$$

Recall that for a function $u \in E(\Delta, L^2(\Omega))$, $\frac{\partial u}{\partial\nu}$ belongs to $H^{-1/2}(\Gamma_1)$ and the next Green formula is valid (see section 1.5 of [10])

$$\int_\Omega \nabla u \nabla w dx = -\int_\Omega \Delta u w dx + \langle \frac{\partial u}{\partial\nu}; w\rangle_{\Gamma_1} \forall w \in H^1_{\Gamma_0}(\Omega), \tag{2.13}$$

where $\langle\cdot;\cdot\rangle_{\Gamma_1}$ means the duality pairing between $H^{-1/2}(\Gamma_1)$ and $H^{1/2}(\Gamma_1)$.

Observe that the domain of $\mathcal{A}(t)$ is independent of the time $t$; i.e.,

$$\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0)), \quad t > 0. \tag{2.14}$$

$\mathcal{A}(t)$ is an unbounded operator in $\mathcal{H}$, the Hilbert space defined by

$$\mathcal{H} := H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1 \times (0,1)), \tag{2.15}$$

equipped with the standard inner product

$$\begin{aligned} \Big\langle \begin{pmatrix} u \\ v \\ v_1 \\ z \end{pmatrix}, \begin{pmatrix} \tilde u \\ \tilde v \\ \tilde v_1 \\ \tilde z \end{pmatrix} \Big\rangle_{\mathcal{H}} &:= \int_\Omega \{\nabla u(x)\nabla\tilde u(x) + v(x)\tilde v(x)\}dx \\ &\quad + \int_{\Gamma_1} v_1(x)\tilde v_1(x) d\Gamma + \int_{\Gamma_1}\int_0^1 z(x,\rho)\tilde z(x,\rho)\, d\rho\, d\Gamma. \end{aligned} \tag{2.16}$$

We can obtain a well-posedness result using semigroup arguments by Kato [12, 13, 27]. The following result is proved in [12, Theorem 1.9].

**Theorem 2.1.** *Assume that*

(i) $\mathcal{D}(\mathcal{A}(0))$ *is a dense subset of* $\mathcal{H}$,
(ii) $\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0))$ *for all* $t > 0$,

(iii) *for all* $t \in [0,T]$, $\mathcal{A}(t)$ *generates a strongly continuous semigroup on* $\mathcal{H}$ *and the family* $\mathcal{A} = \{\mathcal{A}(t) : t \in [0,T]\}$ *is stable with stability constants* $C$ *and* $m$ *independent of* $t$ *(i.e. the semigroup* $(S_t(s))_{s\geq 0}$ *generated by* $\mathcal{A}(t)$ *satisfies* $\|S_t(s)u\|_{\mathcal{H}} \leq Ce^{ms}\|u\|_{\mathcal{H}}$, *for all* $u \in \mathcal{H}$ *and* $s \geq 0$*),*

(iv) $\partial_t \mathcal{A}$ *belongs to* $L^{\infty}_{*}([0,T], B(\mathcal{D}(\mathcal{A}(0)),\mathcal{H}))$*, the space of equivalent classes of essentially bounded, strongly measurable functions from* $[0,T]$ *into the set* $B(\mathcal{D}(\mathcal{A}(0)),\mathcal{H})$ *of bounded operators from* $\mathcal{D}(\mathcal{A}(0))$ *into* $\mathcal{H}$.

*Then, problem* (2.11) *has a unique solution* $U \in C([0,T],\mathcal{D}(\mathcal{A}(0))) \cap C^1([0,T],\mathcal{H})$ *for any initial datum in* $\mathcal{D}(\mathcal{A}(0))$.

Therefore, we will check the above assumptions for problem (2.11).

**Lemma 2.2.** $D(\mathcal{A}(0))$ *is dense in* $\mathcal{H}$.

*Proof.* Let $(f,g,g_1,h)^T \in \mathcal{H}$ be orthogonal to all elements of $D(\mathcal{A}(0))$; that is,

$$
\begin{aligned}
0 &= \Big\langle \begin{pmatrix} u \\ v \\ v_1 \\ z \end{pmatrix}, \begin{pmatrix} f \\ g \\ g_1 \\ h \end{pmatrix} \Big\rangle_{\mathcal{H}} \\
&= \int_{\Omega} \{\nabla u(x)\nabla f(x) + v(x)g(x)\}dx + \int_{\Gamma_1} v_1 g_1 d\Gamma + \int_{\Gamma_1}\int_0^1 z(x,\rho)h(x,\rho)d\rho d\Gamma,
\end{aligned}
$$

for all $(u,v,v_1,z)^T \in D(\mathcal{A}(0))$. We first take $u=0$ and $v=0$ (then $v_1=0$) and $z \in \mathcal{D}(\Gamma_1 \times (0,1))$. As $(0,0,0,z)^T \in D(\mathcal{A}(0))$, we obtain

$$\int_{\Gamma_1}\int_0^1 z(x,\rho)h(x,\rho)d\rho d\Gamma = 0.$$

Since $\mathcal{D}(\Gamma_1 \times (0,1))$ is dense in $L^2(\Gamma_1 \times (0,1)$, we deduce that $h=0$.

In the same way, by taking $u=0$, $z=0$ and $v \in \mathcal{D}(\Omega)$ (then $v_1=0$) we see that $g=0$. Therefore, for $u=0$, $z=0$ we deduce also

$$\int_{\Gamma_1} g_1 v_1 d\Gamma = 0, \quad \forall v_1 \in \mathcal{D}(\Gamma_1)$$

and so $g_1 = 0$.

The above orthogonality condition is then reduced to

$$0 = \int_{\Omega} \nabla u \nabla f dx, \quad \forall (u,v,v_1,z)^T \in D(\mathcal{A}(0)).$$

By restricting ourselves to $v=0$ and $z=0$, we obtain

$$\int_{\Omega} \nabla u(x)\nabla f(x)dx = 0, \quad \forall (u,0,0,0)^T \in D(\mathcal{A}(0)).$$

But we easily see that $(u,0,0,0)^T \in D(\mathcal{A}(0))$ if and only if $u \in E(\Delta, L^2(\Omega)) \cap H^1_{\Gamma_0}(\Omega)$. This set is dense in $H^1_{\Gamma_0}(\Omega)$ (equipped with the inner product $\langle .,. \rangle_{H^1_{\Gamma_0}(\Omega)}$), thus we conclude that $f=0$. □

Assuming (2.2) we will show that $\mathcal{A}(t)$ generates a $C_0$ semigroup on $\mathcal{H}$ and using the variable norm technique of Kato from [13] and Theorem 2.1, that problem (2.11) has a unique solution.

Let $\xi$ be a positive constant that satisfies

$$\frac{|k|}{\sqrt{1-d}} \le \xi \le \frac{2a}{C_P} - \frac{|k|}{\sqrt{1-d}}. \tag{2.17}$$

Note that this choice of $\xi$ is possible from assumption (2.2).

We define on the Hilbert space $\mathcal{H}$ the time dependent inner product

$$\Big\langle \begin{pmatrix} u \\ v \\ v_1 \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{v}_1 \\ \tilde{z} \end{pmatrix} \Big\rangle_t := \int_\Omega \{\nabla u(x)\nabla\tilde{u}(x) + v(x)\tilde{v}(x)\}dx \\ + \mu\int_{\Gamma_1} v_1(x)\tilde{v}_1(x)d\Gamma + \xi\tau(t)\int_{\Gamma_1}\int_0^1 z(x,\rho)\tilde{z}(x,\rho)\,d\rho\,d\Gamma. \tag{2.18}$$

Using this time dependent inner product and Theorem 2.1, we can deduce a well-posedness result.

**Theorem 2.3.** *For any initial datum $U_0 \in \mathcal{D}(\mathcal{A}(0))$ there exists a unique solution*

$$U \in C([0,+\infty), \mathcal{D}(\mathcal{A}(0))) \cap C^1([0,+\infty), \mathcal{H})$$

*of system* (2.11).

*Proof.* We first observe that

$$\frac{\|\phi\|_t}{\|\phi\|_s} \le e^{\frac{c}{2\tau_0}|t-s|}, \quad \forall t,s \in [0,T], \tag{2.19}$$

where $\phi = (u,v,v_1,z)^T$ and $c$ is a positive constant. Indeed, for all $s,t \in [0,T]$, we have

$$\|\phi\|_t^2 - \|\phi\|_s^2 e^{\frac{c}{\tau_0}|t-s|} = \Big(1 - e^{\frac{c}{\tau_0}|t-s|}\Big)\Big\{\int_\Omega (|\nabla u(x)|^2 + v^2)dx + \mu\int_{\Gamma_1} v_1^2 d\Gamma\Big\} \\ + \xi\Big(\tau(t) - \tau(s)e^{\frac{c}{\tau_0}|t-s|}\Big)\int_{\Gamma_N}\int_0^1 z^2(x,\rho)\,d\rho\,d\Gamma.$$

We notice that $1 - e^{\frac{c}{\tau_0}|t-s|} \le 0$. Moreover $\tau(t) - \tau(s)e^{\frac{c}{\tau_0}|t-s|} \le 0$ for some $c > 0$. Indeed, $\tau(t) = \tau(s) + \tau'(a)(t-s)$, where $a \in (s,t)$, and thus,

$$\frac{\tau(t)}{\tau(s)} \le 1 + \frac{|\tau'(a)|}{\tau(s)}|t-s|.$$

By (1.7), $\tau'$ is bounded on $[0,T]$ and therefore, recalling also (1.6),

$$\frac{\tau(t)}{\tau(s)} \le 1 + \frac{c}{\tau_0}|t-s| \le e^{\frac{c}{\tau_0}|t-s|},$$

which proves (2.19).

Now we calculate $\langle \mathcal{A}(t)U, U\rangle_t$ for a fixed $t$. Take $U = (u,v,v_1,z)^T \in \mathcal{D}(\mathcal{A}(t))$. Then,

$$\langle \mathcal{A}(t)U, U\rangle_t = \Bigg\langle \begin{pmatrix} v \\ \Delta(u+av) \\ -\mu^{-1}\Big(\frac{\partial(u+av)}{\partial\nu} + kz(\cdot,1)\Big) \\ \frac{\tau'(t)\rho-1}{\tau(t)} z_\rho \end{pmatrix}, \begin{pmatrix} u \\ v \\ v_1 \\ z \end{pmatrix} \Bigg\rangle_t$$

$$
\begin{aligned}
&= \int_\Omega \{\nabla v(x)\nabla u(x) + v(x)\Delta(u(x)+av(x))\}dx \\
&\quad - \xi\int_{\Gamma_1}\int_0^1 (1-\tau'(t)\rho)z_\rho(x,\rho)z(x,\rho)d\rho d\Gamma \\
&\quad - \int_{\Gamma_1}\Big(\frac{\partial(u+av)}{\partial\nu}(x) + kz(x,1)\Big)v(x)\,d\Gamma.
\end{aligned}
$$

So, by Green's formula,

$$
\begin{aligned}
\langle \mathcal{A}(t)U, U\rangle_t &= -k\int_{\Gamma_1} z(x,1)v(x)d\Gamma - a\int_\Omega |\nabla v(x)|^2dx \\
&\quad - \xi\int_{\Gamma_1}\int_0^1 (1-\tau'(t)\rho)z_\rho(x,\rho)z(x,\rho)d\rho d\Gamma.
\end{aligned} \tag{2.20}
$$

Integrating by parts in $\rho$, we obtain

$$
\begin{aligned}
&\int_{\Gamma_1}\int_0^1 z_\rho(x,\rho)z(x,\rho)(1-\tau'(t)\rho)\,d\rho\,d\Gamma \\
&= \int_{\Gamma_1}\int_0^1 \frac{1}{2}\frac{\partial}{\partial\rho}z^2(x,\rho)(1-\tau'(t)\rho)d\rho\,d\Gamma \\
&= \frac{\tau'(t)}{2}\int_{\Gamma_1}\int_0^1 z^2(x,\rho)d\rho d\Gamma + \frac{1}{2}\int_{\Gamma_1}\{z^2(x,1)(1-\tau'(t)) - z^2(x,0)\}d\Gamma.
\end{aligned} \tag{2.21}
$$

Therefore, from (2.20) and (2.21),

$$
\begin{aligned}
&\langle \mathcal{A}(t)U, U\rangle_t \\
&= -k\int_{\Gamma_1} z(x,1)v(x)d\Gamma - a\int_\Omega |\nabla v(x)|^2dx \\
&\quad - \frac{\xi}{2}\int_{\Gamma_1}\{z^2(x,1)(1-\tau'(t)) - z^2(x,0)\}d\Gamma - \frac{\xi\tau'(t)}{2}\int_{\Gamma_1}\int_0^1 z^2(x,\rho)d\rho d\Gamma \\
&= -k\int_{\Gamma_1} z(x,1)v(x)d\Gamma - a\int_\Omega |\nabla v(x)|^2dx - \frac{\xi}{2}\int_{\Gamma_1} z^2(x,1)(1-\tau'(t))d\Gamma \\
&\quad + \frac{\xi}{2}\int_{\Gamma_1} v^2(x)d\Gamma - \frac{\xi\tau'(t)}{2}\int_{\Gamma_1}\int_0^1 z^2(x,\rho)d\rho d\Gamma,
\end{aligned}
$$

from which, using Cauchy-Schwarz's inequality, a trace estimate and Poincaré's Theorem, it follows that

$$
\begin{aligned}
\langle \mathcal{A}(t)U, U\rangle_t &\le -\Big(a - \frac{|k|C_P}{2\sqrt{1-d}} - \frac{\xi}{2}C_P\Big)\int_\Omega |\nabla v(x)|^2dx \\
&\quad - \Big(\frac{\xi}{2}(1-d) - \frac{|k|}{2}\sqrt{1-d}\Big)\int_{\Gamma_1} z^2(x,1)d\Gamma + \kappa(t)\langle U, U\rangle_t,
\end{aligned} \tag{2.22}
$$

where

$$
\kappa(t) = \frac{(\tau'(t)^2+1)^{\frac{1}{2}}}{2\tau(t)}. \tag{2.23}
$$

Now, observe that from (2.17),

$$
\langle \mathcal{A}(t)U, U\rangle_t - \kappa(t)\langle U, U\rangle_t \le 0, \tag{2.24}
$$

which means that the operator $\tilde{\mathcal{A}}(t) = \mathcal{A}(t) - \kappa(t)I$ is dissipative.

Moreover,

$$\kappa'(t) = \frac{\tau''(t)\tau'(t)}{2\tau(t)(\tau'(t)^2+1)^{\frac{1}{2}}} - \frac{\tau'(t)(\tau'(t)^2+1)^{\frac{1}{2}}}{2\tau(t)^2}$$

is bounded on $[0, T]$ for all $T > 0$ (by (1.6) and (1.7)) and we have

$$\frac{d}{dt}\mathcal{A}(t)U = \begin{pmatrix} 0 \\ 0 \\ \frac{\tau''(t)\tau(t)\rho-\tau'(t)(\tau'(t)\rho-1)}{\tau(t)^2} z_\rho \end{pmatrix}$$

with $\frac{\tau''(t)\tau(t)\rho-\tau'(t)(\tau'(t)\rho-1)}{\tau(t)^2}$ bounded on $[0, T]$. Thus

$$\frac{d}{dt}\tilde{\mathcal{A}}(t) \in L^\infty_*([0, T], B(D(\mathcal{A}(0)), \mathcal{H})), \tag{2.25}$$

the space of equivalence classes of essentially bounded, strongly measurable functions from $[0, T]$ into $B(D(\mathcal{A}(0)), \mathcal{H})$.

Now, we show that $\lambda I - \mathcal{A}(t)$ is surjective for fixed $t > 0$ and $\lambda > 0$. Given $(f, g, g_1, h)^T \in \mathcal{H}$, we seek $U = (u, v, v_1, z)^T \in \mathcal{D}(\mathcal{A}(t))$ solution of

$$(\lambda I - \mathcal{A}(t)) \begin{pmatrix} u \\ v \\ v_1 \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ g_1 \\ h \end{pmatrix},$$

that is verifying

$$\begin{gathered} \lambda u - v = f \\ \lambda v - \Delta(u + av) = g \\ \lambda v_1 + \mu^{-1}\left(\frac{\partial(u+av)}{\partial\nu}(x) + kz(x, 1)\right) = g_1 \\ \lambda z + \frac{1-\tau'(t)\rho}{\tau(t)} z_\rho = h. \end{gathered} \tag{2.26}$$

Suppose that we have found $u$ with the appropriate regularity. Then

$$v := \lambda u - f \tag{2.27}$$

and we can determine $z$. Indeed, by (2.12),

$$z(x, 0) = v(x), \quad \text{for } \ x \in \Gamma_1, \tag{2.28}$$

and, from (2.26),

$$\lambda z(x, \rho) + \frac{1-\tau'(t)\rho}{\tau(t)} z_\rho(x, \rho) = h(x, \rho), \quad \text{for } x \in \Gamma_1,\ \rho \in (0, 1). \tag{2.29}$$

Then, by (2.28) and (2.29), we obtain

$$z(x, \rho) = v(x)e^{-\lambda\rho\tau(t)} + \tau(t)e^{-\lambda\rho\tau(t)} \int_0^\rho h(x, \sigma)e^{\lambda\sigma\tau(t)} d\sigma,$$

if $\tau'(t) = 0$, and

$$\begin{aligned} z(x, \rho) &= v(x)e^{\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\rho)} \\ &\quad + e^{\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\rho)} \int_0^\rho \frac{h(x, \sigma)\tau(t)}{1-\tau'(t)\sigma} e^{-\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\sigma)} d\sigma, \end{aligned}$$

otherwise. So, from (2.27),

$$
\begin{aligned}
z(x,\rho) &= \lambda u(x)e^{-\lambda\rho\tau(t)} - f(x)e^{-\lambda\rho\tau(t)} \\
&\quad + \tau(t)e^{-\lambda\rho\tau(t)}\int_0^\rho h(x,\sigma)e^{\lambda\sigma\tau(t)}d\sigma, \quad \text{on } \Gamma_1\times(0,1),
\end{aligned} \tag{2.30}
$$

if $\tau'(t)=0$, and

$$
\begin{aligned}
z(x,\rho) &= \lambda u(x)e^{\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\rho)} - f(x)e^{\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\rho)} \\
&\quad + e^{\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\rho)}\int_0^\rho \frac{h(x,\sigma)\tau(t)}{1-\tau'(t)\sigma}e^{-\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\sigma)}d\sigma,
\end{aligned} \tag{2.31}
$$

on $\Gamma_1\times(0,1)$ otherwise.

In particular, if $\tau'(t)=0$,

$$
z(x,1) = \lambda u(x)e^{-\lambda\tau(t)} + z_0(x), \quad x\in\Gamma_1, \tag{2.32}
$$

with $z_0\in L^2(\Gamma_1)$ defined by

$$
z_0(x) = -f(x)e^{-\lambda\tau(t)} + \tau(t)e^{-\lambda\tau(t)}\int_0^1 h(x,\sigma)e^{\lambda\sigma\tau(t)}d\sigma, \quad x\in\Gamma_1, \tag{2.33}
$$

and, if $\tau'(t)\neq 0$,

$$
z(x,1) = \lambda u(x)e^{\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t))} + z_0(x), \quad x\in\Gamma_1, \tag{2.34}
$$

with $z_0\in L^2(\Gamma_1)$ defined by

$$
\begin{aligned}
z_0(x) &= -f(x)e^{\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t))} \\
&\quad + e^{\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t))}\int_0^1 \frac{h(x,\sigma)\tau(t)}{1-\tau'(t)\sigma}e^{-\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\sigma)}d\sigma,
\end{aligned} \tag{2.35}
$$

for $x\in\Gamma_1$. Then, we have to find $u$. In view of the equation $\lambda v - \Delta(u+av) = g$, we set $s = u+av$ and look at $s$. Now according to (2.27), we may write

$$
v = \lambda u - f = \lambda s - f - \lambda a v,
$$

or equivalently

$$
v = \frac{\lambda}{1+\lambda a}s - \frac{1}{1+\lambda a}f. \tag{2.36}
$$

Hence once $s$ will be found, we will get $v$ by (2.36) and then $u$ by $u = s - av$, or equivalently

$$
u = \frac{1}{1+\lambda a}s + \frac{a}{1+\lambda a}f. \tag{2.37}
$$

By (2.36) and (2.26), the function $s$ satisfies

$$
\frac{\lambda^2}{1+\lambda a}s - \Delta s = g + \frac{\lambda}{1+\lambda a}f \quad \text{in } \Omega, \tag{2.38}
$$

with the boundary conditions

$$
s = 0 \quad \text{on } \Gamma_0, \tag{2.39}
$$

as well as (at least formally)

$$
\frac{\partial s}{\partial \nu} = \mu g_1 - \mu\lambda v_1 - kz(\cdot,1) \quad \text{on } \Gamma_1,
$$

which becomes due to (2.36), (2.37), (2.32), (2.34) and the requirement that $v_1 = \gamma_1 v$ on $\Gamma_1$:

$$\frac{\partial s}{\partial \nu} = -\frac{\lambda(ke^{-\lambda\tau(t)} + \mu\lambda)}{1+\lambda a} s + l \quad \text{on } \Gamma_1, \tag{2.40}$$

where

$$l = \mu g_1 + \frac{\lambda(\mu - kae^{-\lambda\tau(t)})}{1+\lambda a} f - kz_0 \quad \text{on } \Gamma_1$$

if $\tau'(t) = 0$, otherwise

$$\frac{\partial s}{\partial \nu} = -\frac{\lambda(ke^{-\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t))} + \mu\lambda)}{1+\lambda a} s + \tilde{l} \quad \text{on } \Gamma_1, \tag{2.41}$$

where

$$\tilde{l} = \mu g_1 + \frac{\lambda(\mu - kae^{-\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t))})}{1+\lambda a} f - kz_0 \quad \text{on } \Gamma_1.$$

From (2.38), integrating by parts, and using (2.39), (2.40), (2.41) we find the variational problem

$$\begin{aligned} &\int_\Omega (\frac{\lambda^2}{1+\lambda a} sw + \nabla s \cdot \nabla w) dx + \int_{\Gamma_1} \frac{\lambda(ke^{-\lambda\tau} + \mu\lambda)}{1+\lambda a} swd\Gamma \\ &= \int_\Omega (g + \frac{\lambda}{1+\lambda a} f) wdx + \int_{\Gamma_1} lwd\Gamma \quad \forall w \in H^1_{\Gamma_0}(\Omega) \end{aligned} \tag{2.42}$$

if $\tau'(t) = 0$, otherwise

$$\begin{aligned} &\int_\Omega (\frac{\lambda^2}{1+\lambda a} sw + \nabla s \cdot \nabla w) dx + \int_{\Gamma_1} \frac{\lambda(ke^{-\lambda\frac{\tau}{\tau'}\ln(1-\tau')} + \mu\lambda)}{1+\lambda a} swd\Gamma \\ &= \int_\Omega (g + \frac{\lambda}{1+\lambda a} f) wdx + \int_{\Gamma_1} \tilde{l} wd\Gamma \quad \forall w \in H^1_{\Gamma_0}(\Omega). \end{aligned} \tag{2.43}$$

As the left-hand side of (2.42), (2.43) is coercive on $H^1_{\Gamma_0}(\Omega)$, the Lax-Milgram lemma guarantees the existence and uniqueness of a solution $s \in H^1_{\Gamma_0}(\Omega)$ of (2.42), (2.43).

If we consider $w \in \mathcal{D}(\Omega)$ in (2.42), (2.43), we have that $s$ solves (2.38) in $\mathcal{D}'(\Omega)$ and thus $s = u + av \in E(\Delta, L^2(\Omega))$.

Using Green's formula (2.13) in (2.42) and using (2.38), we obtain

$$\int_{\Gamma_1} \frac{\lambda(ke^{-\lambda\tau} + \mu\lambda)}{1+\lambda a} swd\Gamma + \langle \frac{\partial s}{\partial \nu}; w\rangle_{\Gamma_1} = \int_{\Gamma_1} lw\, d\Gamma,$$

leading to (2.40) and then to the third equation of (2.26) due to the definition of $l$ and the relations between $u$, $v$ and $s$. We find the same result if $\tau'(t) \neq 0$.

In conclusion, we have found $(u, v, v_1, z)^T \in \mathcal{D}(\mathcal{A})$, which verifies (2.26), and thus $\lambda I - \mathcal{A}(t)$ is surjective for some $\lambda > 0$ and $t > 0$. Again as $\kappa(t) > 0$, this proves that

$$\lambda I - \tilde{\mathcal{A}}(t) = (\lambda + \kappa(t))I - \mathcal{A}(t) \quad \text{is surjective} \tag{2.44}$$

for any $\lambda > 0$ and $t > 0$.

Then, (2.19), (2.24) and (2.44) imply that the family $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}(t) : t \in [0, T]\}$ is a stable family of generators in $\mathcal{H}$ with stability constants independent of $t$, by [13, Proposition 1.1]. Therefore, the assumptions (i)-(iv) of Theorem 2.1 are satisifed by (2.14), (2.19), (2.24), (2.25), (2.44) and Lemma 2.2, and thus, the problem

$$\begin{gathered}\tilde{U}' = \tilde{\mathcal{A}}(t)\tilde{U}\\ \tilde{U}(0) = U_0\end{gathered}$$

has a unique solution $\tilde{U} \in C([0,+\infty), D(\mathcal{A}(0))) \cap C^1([0,+\infty), \mathcal{H})$ for $U_0 \in D(\mathcal{A}(0))$. The requested solution of (2.52) is then given by

$$U(t) = e^{\beta(t)}\tilde{U}(t)$$

with $\beta(t) = \int_0^t \kappa(s)ds$, because

$$\begin{aligned}U'(t) &= \kappa(t)e^{\beta(t)}\tilde{U}(t) + e^{\beta(t)}\tilde{U}'(t)\\ &= \kappa(t)e^{\beta(t)}\tilde{U}(t) + e^{\beta(t)}\tilde{\mathcal{A}}(t)\tilde{U}(t)\\ &= e^{\beta(t)}(\kappa(t)\tilde{U}(t) + \tilde{\mathcal{A}}(t)\tilde{U}(t))\\ &= e^{\beta(t)}\mathcal{A}(t)\tilde{U}(t) = \mathcal{A}(t)e^{\beta(t)}\tilde{U}(t)\\ &= \mathcal{A}(t)U(t).\end{aligned}$$

This concludes the proof. □

**Theorem 2.4.** *Assume that* (1.6)–(1.7) *and* (2.2) *hold. Then for any initial datum* $U_0 \in \mathcal{H}$ *there exists a unique solution* $U \in C([0,+\infty), \mathcal{H})$ *of problem* (2.11)*. Moreover, if* $U_0 \in \mathcal{D}(\mathcal{A}(0))$*, then*

$$U \in C([0,+\infty), \mathcal{D}(\mathcal{A}(0))) \cap C^1([0,+\infty), \mathcal{H}).$$

2.2. **Well-posedness in the case $\mu = 0$.** As before, we use the auxiliary unknown (2.3). Then, problem (1.1)–(1.5), with $\mu = 0$, is equivalent to

$$u_{tt}(x,t) - \Delta u(x,t) - a\Delta u_t(x,t) = 0 \quad \text{in } \Omega \times (0,+\infty), \tag{2.45}$$

$$\tau(t)z_t(x,\rho,t) + (1-\tau'(t)\rho)z_\rho(x,\rho,t) = 0 \quad \text{in } \Gamma_1 \times (0,1) \times (0,+\infty), \tag{2.46}$$

$$u(x,t) = 0 \quad \text{on } \Gamma_0 \times (0,+\infty), \tag{2.47}$$

$$\frac{\partial(u+au_t)}{\partial\nu}(x,t) = -kz(x,1,t) \quad \text{on } \Gamma_1 \times (0,+\infty), \tag{2.48}$$

$$z(x,0,t) = u_t(x,t) \quad \text{on } \Gamma_1 \times (0,\infty), \tag{2.49}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \tag{2.50}$$

$$z(x,\rho,0) = f_0(x,-\rho\tau(0)) \quad \text{in } \Gamma_1 \times (0,1). \tag{2.51}$$

If we denote $U := (u, u_t, z)^T$, then

$$U' := (u_t, u_{tt}, z_t)^T = \Big(u_t, \Delta u + a\Delta u_t, \frac{\tau'(t)\rho - 1}{\tau(t)} z_\rho\Big)^T.$$

Therefore, problem (2.45)–(2.51) can be rewritten as

$$\begin{gathered}U' = \mathcal{A}^0(t)U,\\ U(0) = (u_0, u_1, f_0(\cdot, -\cdot\,\tau(0)))^T,\end{gathered} \tag{2.52}$$

where the time dependent operator $\mathcal{A}^0(t)$ is defined by

$$\mathcal{A}^0(t)\begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ \Delta(u+av) \\ \frac{\tau'(t)\rho-1}{\tau(t)} z_\rho \end{pmatrix},$$

with domain

$$\begin{aligned} \mathcal{D}(\mathcal{A}^0(t)) := \Big\{ &(u,v,z)^T \in H^1_{\Gamma_0}(\Omega)^2 \times L^2(\Gamma_1; H^1(0,1)) : u+av \in E(\Delta, L^2(\Omega)), \\ &\frac{\partial(u+av)}{\partial \nu} = -kz(\cdot,1) \text{ on } \Gamma_1; v = z(\cdot,0) \text{ on } \Gamma_1 \Big\}. \end{aligned} \tag{2.53}$$

Note that for $(u,v,z)^T \in \mathcal{D}(\mathcal{A}^0(t))$, $\frac{\partial(u+av)}{\partial\nu}$ belongs to $L^2(\Gamma_1)$ since $z(\cdot,1)$ is in $L^2(\Gamma_1)$.

Finally, as above, observe that domain of $\mathcal{A}^0(t)$ is independent of the time $t$; i.e.,

$$\mathcal{D}(\mathcal{A}^0(t)) = \mathcal{D}(\mathcal{A}^0(0)), \quad t > 0.$$

Denote by $\mathcal{H}^0$ the Hilbert space

$$\mathcal{H}^0 := H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1 \times (0,1)), \tag{2.54}$$

equipped with the scalar product

$$\begin{aligned} \Big\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} \tilde u \\ \tilde v \\ \tilde z \end{pmatrix} \Big\rangle_{\mathcal{H}^0} &:= \int_\Omega \{\nabla u(x)\nabla\tilde u(x) + v(x)\tilde v(x)\}\,dx \\ &\quad + \int_{\Gamma_1}\int_0^1 z(x,\rho)\tilde z(x,\rho)d\rho d\Gamma, \end{aligned} \tag{2.55}$$

Arguing analogously to the case $\mu > 0$ we can deduce an existence and uniqueness result.

**Theorem 2.5.** *Assume that (1.6)–(1.7) and (2.2) hold. Then, for any initial datum $U_0 \in \mathcal{H}^0$ there exists a unique solution $U \in C([0,+\infty), \mathcal{H}^0)$ of problem (2.52). Moreover, if $U_0 \in \mathcal{D}(\mathcal{A}^0(0))$, then*

$$U \in C([0,+\infty), \mathcal{D}(\mathcal{A}^0(0))) \cap C^1([0,+\infty), \mathcal{H}^0).$$

**Remark 2.6.** This well-posedness theorem can be also deduced from the abstract framework of [8] (see Theorem 2.2 in [8]) for second order evolution equations. On the contrary, the case $\mu > 0$ is not covered by this abstract result.

2.3. **Stability result.** Now, we show that problem (1.1)–(1.5) is uniformly exponentially stable under the assumption

$$|k| < \frac{a}{C_P}\sqrt{1-d}\,. \tag{2.56}$$

We define the energy of system (1.1)–(1.5) as

$$F(t) := \frac{1}{2}\int_\Omega \{u_t^2 + |\nabla u|^2\}dx + \frac{\xi}{2}\int_{t-\tau(t)}^t \int_{\Gamma_1} e^{\lambda(s-t)} u_t^2(x,s)d\Gamma ds + \frac{\mu}{2}\int_{\Gamma_1} u_t^2(x,t)d\Gamma, \tag{2.57}$$

where $\xi, \lambda$ are suitable positive constants. We fix $\xi$ such that

$$\frac{|k|}{\sqrt{1-d}} < \xi < \frac{2a}{C_P} - \frac{|k|}{\sqrt{1-d}}. \tag{2.58}$$

Note that (2.56) ensures that this choice is possible. Moreover, the parameter $\lambda$ is fixed satisfying

$$\lambda < \frac{1}{\overline{\tau}} \Big| \log \frac{|k|}{\xi\sqrt{1-d}} \Big| \,. \tag{2.59}$$

Remark that in the case of a constant delay, we can take $\lambda = 0$ and in that case $F(t)$ corresponds to the natural energy of $(u, u_t, z)$ (up to the factor $\frac{1}{2}$), see [24]. Here the time dependence of the delay implies that our system is no more invariant by translation and therefore we have to replace the arguments from [24] by the use of an appropriate Lyapunov functional. We start with the following estimate.

**Proposition 2.7.** *Assume* (1.6)–(1.7) *and* (2.56). *Then, for any regular solution of problem* (1.1)–(1.5) *the energy is decreasing and, for a suitable positive constant* $C$, *we have*

$$\begin{aligned} F'(t) \le -C\Big\{ \int_\Omega |\nabla u_t(x,t)|^2 dx + \int_{\Gamma_1} u_t^2(x, t-\tau(t)) d\Gamma \Big\} \\ - C \int_{t-\tau(t)}^t \int_{\Gamma_1} e^{\lambda(s-t)} u_t^2(x,s) d\Gamma ds. \end{aligned} \tag{2.60}$$

*Proof.* Differentiating (2.57), we obtain

$$\begin{aligned} F'(t) = \int_\Omega \{u_t u_{tt} + \nabla u \nabla u_t\} dx + \frac{\xi}{2} \int_{\Gamma_1} u_t^2(x,t) d\Gamma + \mu \int_{\Gamma_1} u_t(t) u_{tt}(t)\, d\Gamma \\ - \frac{\xi}{2} \int_{\Gamma_1} e^{-\lambda\tau(t)} u_t^2(x, t-\tau(t))(1-\tau'(t)) d\Gamma \\ - \lambda \frac{\xi}{2} \int_{t-\tau(t)}^t \int_{\Gamma_1} e^{-\lambda(t-s)} u_t^2(x,s) d\Gamma ds, \end{aligned}$$

and then, applying Green's formula,

$$\begin{aligned} F'(t) = \int_\Omega a u_t(x,t) \Delta u_t(x,t) dx + \int_{\Gamma_1} u_t(t) \frac{\partial u}{\partial \nu}(t) d\Gamma \\ - \frac{\xi}{2} \int_{\Gamma_1} e^{-\lambda\tau(t)} u_t^2(x, t-\tau(t))(1-\tau'(t)) d\Gamma + \frac{\xi}{2} \int_{\Gamma_1} u_t^2(x,t) d\Gamma \\ - \lambda \frac{\xi}{2} \int_{t-\tau(t)}^t \int_{\Gamma_1} e^{-\lambda(t-s)} u_t^2(x,s) d\Gamma ds + \mu \int_{\Gamma_1} u_t(t) u_{tt}(t) d\Gamma. \end{aligned} \tag{2.61}$$

Integrating once more by parts and using the boundary conditions we obtain

$$\begin{aligned} F'(t) = -a \int_\Omega |\nabla u_t(x,t)|^2 dx - k \int_{\Gamma_1} u_t(t) u_t(t-\tau(t)) d\Gamma \\ - \frac{\xi}{2} \int_{\Gamma_1} e^{-\lambda\tau(t)} u_t^2(x, t-\tau(t))(1-\tau'(t)) d\Gamma + \frac{\xi}{2} \int_{\Gamma_1} u_t^2(x,t) d\Gamma \\ - \lambda \frac{\xi}{2} \int_{t-\tau(t)}^t \int_{\Gamma_1} e^{-\lambda(t-s)} u_t^2(x,s) d\Gamma\, ds. \end{aligned} \tag{2.62}$$

Now, applying Cauchy-Schwarz's inequality and recalling the assumptions (1.6) and (1.8), we obtain

$$F'(t) \le -a \int_\Omega |\nabla u_t(x,t)|^2 dx + \frac{\xi}{2} \int_{\Gamma_1} u_t^2(x,t) d\Gamma + \frac{|k|}{2\sqrt{1-d}} \int_{\Gamma_1} u_t^2(x,t) d\Gamma$$

$$
\begin{aligned}
&+\frac{|k|}{2}\sqrt{1-d}\int_{\Gamma_1} u_t^2(t-\tau(t))d\Gamma \\
&-\frac{\xi}{2}(1-d)e^{-\lambda\overline{\tau}}\int_{\Gamma_1} u_t^2(x,t-\tau(t))d\Gamma-\lambda\frac{\xi}{2}\int_{t-\tau(t)}^t\int_{\Gamma_1} e^{-\lambda(t-s)}u_t^2(x,s)d\Gamma\,ds \\
\le &-\Big(a-\frac{|k|C_P}{2\sqrt{1-d}}-\frac{\xi}{2}C_P\Big)\int_\Omega |\nabla u_t(x,t)|^2dx \\
&-\Big(e^{-\lambda\overline{\tau}}\frac{\xi}{2}(1-d)-\frac{|k|}{2}\sqrt{1-d}\Big)\int_{\Gamma_1} u_t^2(x,t-\tau(t))d\Gamma \\
&-\lambda\frac{\xi}{2}\int_{t-\tau(t)}^t\int_{\Gamma_1} e^{-\lambda(t-s)}u_t^2(x,s)d\Gamma ds,
\end{aligned}
$$

where in the last inequality we also use a trace estimate and Poincaré's Theorem. Therefore, (2.60) immediately follows recalling (2.58) and (2.59). □

Now, let us define the Lyapunov functional

$$
\hat{F}(t)=F(t)+\gamma\Big[\int_\Omega u(x,t)u_t(x,t)dx+\mu\int_{\Gamma_1} u(x,t)u_t(x,t)d\Gamma\Big], \tag{2.63}
$$

where $\gamma$ is a positive small constant that we will choose later on.

Note that, from Poincaré's Theorem, the functional $\hat{F}$ is equivalent to the energy $F$, that is, for $\gamma$ small enough, there exist two positive constant $\beta_1^0,\beta_2^0$ such that

$$
\beta_1^0\hat{F}(t)\le F(t)\le\beta_2^0\hat{F}(t),\quad \forall t\ge 0. \tag{2.64}
$$

**Lemma 2.8.** *For any regular solution of problem* (1.1)–(1.5),

$$
\begin{aligned}
&\frac{d}{dt}\Big\{\int_\Omega u(x,t)u_t(x,t)dxdt+\mu\int_{\Gamma_1} u(x,t)u_t(x,t)d\Gamma\Big\} \\
&\le C\Big\{\int_\Omega|\nabla u_t(x,t)|^2dx+\int_{\Gamma_1} u_t^2(x,t-\tau(t))d\Gamma\Big\}-\frac{1}{2}\int_\Omega|\nabla u(x,t)|^2dx,
\end{aligned} \tag{2.65}
$$

*for a suitable positive constant $C$ (that is different from the one from Proposition 2.7).*

*Proof.* Differentiating and integrating by parts we have

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega uu_tdx &= \int_\Omega u_t^2(x,t)dx+\int_\Omega u(\Delta u+a\Delta u_t)dx \\
&= \int_\Omega u_t^2(x,t)dx-\int_\Omega|\nabla u(x,t)|^2dx-a\int_\Omega\nabla u(x,t)\cdot\nabla u_t(x,t)dx \\
&\quad+\int_{\Gamma_1} u(t)\frac{\partial(u+au_t)}{\partial\nu}(t)d\Gamma.
\end{aligned} \tag{2.66}
$$

From (2.66), using the boundary condition on $\Gamma_1$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big\{\int_\Omega uu_t dx + \mu\int_{\Gamma_1} u(x,t)u_t(x,t)d\Gamma\Big\}\\
&= \int_\Omega u_t^2(x,t)dx + \int_\Omega u(\Delta u + a\Delta u_t)dx + \mu\int_{\Gamma_1} u_t^2(x,t)d\Gamma + \mu\int_{\Gamma_1} u(x,t)u_{tt}(x,t)d\Gamma\\
&= \int_\Omega u_t^2(x,t)dx - \int_\Omega |\nabla u(x,t)|^2dx - a\int_\Omega \nabla u(x,t)\cdot\nabla u_t(x,t)dx\\
&\quad - k\int_{\Gamma_1} u(t)u_t(t-\tau(t))d\Gamma + \mu\int_{\Gamma_1} u_t^2(x,t)d\Gamma.
\end{aligned} \tag{2.67}$$

We can conclude by using Young's inequality, a trace estimate and Poincaré's Theorem. □

Finally using the above results we can deduce an exponential stability estimate.

**Theorem 2.9.** *Assume* (1.6)–(1.7) *and* (2.56). *Then there exist positive constants* $C_1, C_2$ *such that for any solution of problem* (1.1)-(1.5),

$$F(t) \le C_1F(0)e^{-C_2t}, \quad \forall t \ge 0. \tag{2.68}$$

*Proof.* From Lemma 2.8, taking $\gamma$ sufficiently small in the definition of the Lyapunov functional $\hat{F}$, we have

$$\begin{aligned}
\frac{d}{dt}\hat{F}(t) \le -C\Big\{\int_\Omega |\nabla u_t(x,t)|^2dx + \int_{\Gamma_1} u_t^2(x,t-\tau(t))dx\Big\}\\
- C\int_{t-\tau(t)}^t e^{-\lambda(t-s)}\int_{\Gamma_1} u_t^2(x,s)d\Gamma\, ds - \frac{\gamma}{2}\int_\Omega |\nabla u(x,t)|^2dx,
\end{aligned} \tag{2.69}$$

for a suitable positive constant $C$ that is different from the one in (2.65). Poincaré's Theorem implying

$$\int_\Omega |u_t(x,t)|^2dx + \int_{\Gamma_1} |u_t(x,t)|^2ds \le C_{P1}\int_\Omega |\nabla u_t(x,t)|^2dx,$$

for some $C_{P1} > 0$, we obtain

$$\frac{d}{dt}\hat{F}(t) \le -C'F(t), \tag{2.70}$$

for a suitable positive constant $C'$. This clearly implies the exponential estimate (2.68) recalling (2.64). □

**Remark 2.10.** Note that in the case of a constant delay the exponential stability result holds under the condition $|k| < a/C_P$ (corresponding to (2.56) since, in this case, $d = 0$), see [24]. On the contrary, if this condition is no more valid, then some instabilities may occur, we refer to [24] for some illustrations.

## 3. Internal delay feedback

3.1. **Well-posedness.** First of all we formulate a well-posedness result under the assumption

$$|a_1| \le a_0\sqrt{1-d}\,. \tag{3.1}$$

Let us set

$$z(x,\rho,t) = u_t(x,t-\tau(t)\rho), \quad x\in\Omega,\ \rho\in(0,1),\ t>0. \tag{3.2}$$

Then, problem (1.9)–(1.12) is equivalent to

$$u_{tt}(x,t) - \Delta u(x,t) + a_0 u_t(x,t) + a_1 z(x,1,t) = 0 \quad \text{in } \Omega \times (0,+\infty) \tag{3.3}$$

$$\tau(t) z_t(x,\rho,t) + (1-\tau'(t)\rho) z_\rho(x,\rho,t) = 0 \quad \text{in } \Omega \times (0,1) \times (0,+\infty) \tag{3.4}$$

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,+\infty) \tag{3.5}$$

$$z(x,0,t) = u_t(x,t) \quad \text{on } \Omega \times (0,\infty) \tag{3.6}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega \tag{3.7}$$

$$z(x,\rho,0) = g_0(x,-\rho\tau(0)) \quad \text{in } \Omega \times (0,1). \tag{3.8}$$

Let us denote $U := (u, u_t, z)^T$, then

$$U' := (u_t, u_{tt}, z_t)^T = \Big( u_t, \Delta u - a_0 u_t - a_1 z(\cdot,1,\cdot), \frac{\tau'(t)\rho - 1}{\tau(t)} z_\rho \Big)^T.$$

Therefore, problem (3.3)–(3.8) can be rewritten as

$$\begin{gathered} U' = \mathcal{A}^1(t) U \\ U(0) = (u_0, u_1, g_0(\cdot, -\cdot\,\tau(0)))^T \end{gathered} \tag{3.9}$$

where the time dependent operator $\mathcal{A}^1(t)$ is defined by

$$\mathcal{A}^1(t) \begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ \Delta u - a_0 v - a_1 z(\cdot,1) \\ \frac{\tau'(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix},$$

with domain

$$\begin{aligned} \mathcal{D}(\mathcal{A}^1(t)) := \Big\{ & (u,v,z)^T \in \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \times H^1(\Omega) \times L^2(\Omega; H^1(0,1)) : \\ & v = z(\cdot,0) \text{ in } \Omega \Big\}. \end{aligned} \tag{3.10}$$

Note that the domain of $\mathcal{A}^1(t)$ is independent of the time $t$; i.e.,

$$\mathcal{D}(\mathcal{A}^1(t)) = \mathcal{D}(\mathcal{A}^1(0)), \quad t > 0.$$

Let us introduce the Hilbert space

$$\mathcal{H}^1 := H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0,1)), \tag{3.11}$$

equipped with the inner product

$$\begin{aligned} & \Big\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} \tilde u \\ \tilde v \\ \tilde z \end{pmatrix} \Big\rangle_{\mathcal{H}^1} \\ & := \int_\Omega \{ \nabla u(x) \nabla \tilde u(x) + v(x) \tilde v(x) \} dx + \int_\Omega \int_0^1 z(x,\rho) \tilde z(x,\rho) d\rho dx. \end{aligned} \tag{3.12}$$

Next we state the well-posedness result then follows from [8, Theorem 2.2] that extends the well-posedness result of [23] for wave equations with constant delays to an abstract second order evolution equation with time-varying delay.

**Theorem 3.1.** *Assume* (1.6)–(1.7) *and* (3.1). *Then, for any initial datum $U_0 \in \mathcal{H}^1$ there exists a unique solution $U \in C([0,+\infty), \mathcal{H}^1)$ of problem* (3.9). *Moreover, if $U_0 \in \mathcal{D}(\mathcal{A}^1(0))$, then*

$$U \in C([0,+\infty), \mathcal{D}(\mathcal{A}^1(0))) \cap C^1([0,+\infty), \mathcal{H}^1).$$

**Remark 3.2.** In [25] the authors considered a wave equation with boundary time-varying delay feedback without the assumption $\tau(t) > \tau_0 > 0$ of non degeneracy of $\tau$, but in less general spaces. We expect that a well-posedness result holds also for problem (1.9)–(1.12) without this restriction on $\tau$. However, we preferred to consider non degenerate $\tau$ in order to avoid technicalities.

3.2. **Stability result.** We will give an exponential stability result for problem (1.9)–(1.12) under assumption (1.14). We define the energy of system (1.9)–(1.12) as

$$E(t) := \frac{1}{2}\int_\Omega \{u_t^2 + |\nabla u|^2\}dx + \frac{\xi}{2}\int_{t-\tau(t)}^t \int_\Omega e^{\lambda(s-t)}u_t^2(x,s)dx\, ds, \tag{3.13}$$

where $\xi, \lambda$ are suitable positive constants. We will fix $\xi$ such that

$$2a_0 - \frac{a_1}{\sqrt{1-d}} - \xi > 0, \quad \xi - \frac{a_1}{\sqrt{1-d}} > 0, \tag{3.14}$$

$$\lambda < \frac{1}{\overline{\tau}}\Big| \log \frac{|a_1|}{\xi\sqrt{1-d}}\Big|\,. \tag{3.15}$$

Note that assumption (1.14) guarantees the existence of such a constant $\xi$. We have the following estimate.

**Proposition 3.3.** *Assume* (1.6)-(1.7) *and* (1.14). *Then, for any regular solution of problem* (1.9)-(1.12) *the energy decays and there exists a positive constant $C$ such that*

$$E'(t) \le -C\int_\Omega \{u_t^2(x,t) + u_t^2(x,t-\tau(t))\} - C\int_{t-\tau(t)}^t \int_\Omega e^{\lambda(s-t)}u_t^2(x,s)dxds. \tag{3.16}$$

*Proof.* Differentiating (3.13), we obtain

$$\begin{aligned}
E'(t) = &\int_\Omega \{u_t u_{tt} + \nabla u \nabla u_t\}dx + \frac{\xi}{2}\int_\Omega u_t^2(x,t)\,dx\\
&- \frac{\xi}{2}\int_\Omega e^{-\lambda\tau(t)}u_t^2(x,t-\tau(t))(1-\tau'(t))dx\\
&- \lambda\frac{\xi}{2}\int_{t-\tau(t)}^t \int_\Omega e^{-\lambda(t-s)}u_t^2(x,s)dxds,
\end{aligned}$$

and then, applying Green's formula,

$$\begin{aligned}
E'(t) = &-a_0\int_\Omega u_t^2(x,t)dx - \int_\Omega a_1 u_t(t)u_t(x,t-\tau(t))dx\\
&- \frac{\xi}{2}\int_\Omega e^{-\lambda\tau(t)}u_t^2(x,t-\tau(t))(1-\tau'(t))dx + \frac{\xi}{2}\int_\Omega u_t^2(x,t)dx\\
&- \lambda\frac{\xi}{2}\int_{t-\tau(t)}^t \int_\Omega e^{-\lambda(t-s)}u_t^2(x,s)dxds.
\end{aligned} \tag{3.17}$$

Now, applying Cauchy-Schwarz's inequality and recalling the assumptions (1.6) and (1.8), we obtain

$$\begin{aligned}
E'(t) \le &-a_0\int_\Omega u_t^2(x,t)dx - a_1\int_\Omega u_t(t)u_t(t-\tau(t))dx + \frac{\xi}{2}\int_\Omega u_t^2(x,t)dx\\
&- \frac{\xi}{2}(1-d)e^{-\lambda\overline{\tau}}\int_\Omega u_t^2(x,t-\tau(t))dx - \lambda\frac{\xi}{2}\int_{t-\tau(t)}^t \int_\Omega e^{-\lambda(t-s)}u_t^2(x,s)dxds
\end{aligned}$$

$$
\begin{aligned}
&\le -\Big(a_0 - \frac{|a_1|}{2\sqrt{1-d}} - \frac{\xi}{2}\Big)\int_\Omega u_t^2(x,t)dx\\
&\quad - \Big(e^{-\lambda\overline{\tau}}\frac{\xi}{2}(1-d) - \frac{|a_1|}{2}\sqrt{1-d}\Big)\int_\Omega u_t^2(x,t-\tau(t))\,dx\\
&\quad - \lambda\frac{\xi}{2}\int_{t-\tau(t)}^t\int_\Omega e^{-\lambda(t-s)}u_t^2(x,s)\,dx\,ds
\end{aligned}
$$

from which easily follows (3.16) recalling (3.14) and (3.15). □

Now, let us introduce the Lyapunov functional

$$
\hat{E}(t) = E(t) + \gamma\int_\Omega u(x,t)u_t(x,t)dx, \tag{3.18}
$$

where $\gamma$ is a suitable small positive constant.

Note that, from Poincaré's Theorem, the functional $\hat{E}$ is equivalent to the energy $E$, that is, for $\gamma$ small enough, there exist two positive constant $\beta_1, \beta_2$ such that

$$
\beta_1\hat{E}(t) \le E(t) \le \beta_2\hat{E}(t), \quad \forall t \ge 0. \tag{3.19}
$$

**Lemma 3.4.** *For any regular solution of problem* (1.9)–(1.12)*,*

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega u(x,t)u_t(x,t)\,dx\,dt\\
&\le C\int_\Omega [u_t^2(x,t) + u_t^2(x,t-\tau(t))]dx - \frac{1}{2}\int_\Omega |\nabla u(x,t)|^2dx,
\end{aligned} \tag{3.20}
$$

*for a suitable positive constants $C$.*

*Proof.* Differentiating and integrating by parts

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega uu_tdx &= \int_\Omega u_t^2(x,t)dx + \int_\Omega u(\Delta u - a_0u_t(t) - a_1u_t(t-\tau(t))\,dx\\
&= \int_\Omega u_t^2(x,t)dx - \int_\Omega |\nabla u(x,t)|^2\,dx - \int_\Omega a_0u(t)u_t(t)dx\\
&\quad + \int_\Omega a_1u(t)u_t(t-\tau(t))\,dx.
\end{aligned} \tag{3.21}
$$

We can conclude by using Young's inequality and Poincaré's Theorem. □

Therefore, analogously to the case of boundary delay feedback, we can now obtain a uniform exponential decay estimate.

**Theorem 3.5.** *Assume* (1.6)–(1.7) *and* (1.14)*. Then there exist positive constants $C_1, C_2$ such that for any solution of problem* (1.9)–(1.12)*,*

$$
E(t) \le C_1E(0)e^{-C_2t}, \quad \forall t \ge 0. \tag{3.22}
$$

**Remark 3.6.** Using Lemma 3.4 this stability result can be deduced with the help of Theorem 4.3 of [8]. The difference with [8] relies on the choice of a simpler Lyapunov functional that renders the proof of the exponential decay more simple.

**Remark 3.7.** Note that in [23] we have assumed that the coefficient $a_1$ of the delay term is positive. But this assumption is not necessary. The results of [23] are valid, with analogous proofs, also for $a_1$ of arbitrary sign satisfying $|a_1| < a_0$.

**Remark 3.8.** Note that in the proof of the stability estimate we did not use the condition of non degeneracy of the delay $\tau(t) \geq \tau_0 > 0$. So, if $u$ is a regular solution of problem (1.9)–(1.12) the exponential stability result holds for $u$ also in presence of a possibly degenerate $\tau$, (cf. Remark 3.2). The same is true for solutions to problem (1.1)–(1.5).

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