Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 38, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

FEEDBACK STABILIZATION OF PARABOLIC SYSTEMS WITH BILINEAR CONTROLS

MOHAMED OUZAHRA

Abstract. In this article, we study infinite-bilinear systems, and consider a decomposition of the state space via the spectral properties of the systems. Then we apply this approach to strong and exponential stabilization problem using quadratic and constrained feedbacks. We present also some applications.

1. Introduction

In this work, we study the stabilization of bilinear systems governed by the abstract equation

$$\frac{dz(t)}{dt} = Az(t) + v(t)Bz(t), \quad z(0) = z_0, \tag{1.1}$$

on a separable Hilbert space H with the inner product $\langle\cdot,\cdot\rangle$ and corresponding norm $\|\cdot\|$, where $A : D(A) \subset H \to H$ is a linear operator with domain $D(A)$, the operator A generates a C_0-semigroup $S(t)$ on H, the operator $B : H \to H$ is linear, and the scalar valued function $t \mapsto v(t)$ represents the control.

In [1], the quadratic control

$$v(t) = -\langle z(t), Bz(t)\rangle, \tag{1.2}$$

was proposed to study the feedback stabilization of (1.1), and a weak stabilization result was established under the condition

$$\langle BS(t)y, S(t)y\rangle = 0,\ \forall t \geq 0 \Rightarrow y = 0\,. \tag{1.3}$$

In [2, 4], it has been proved that under (1.3), the same quadratic control (1.2) ensures the strong stabilization for a class of semilinear systems.

If the assumption (1.3) is replaced by

$$\int_0^T |\langle BS(t)y, S(t)y\rangle| dt \geq \delta\|y\|^2,\ \forall\, y \in H, \text{(for some } T, \delta > 0), \tag{1.4}$$

then we have strong stability of (1.1) with the decay estimate $\|z(t)\| = O(1/\sqrt{t})$, (see [2, 10]).

2000 *Mathematics Subject Classification.* 93D15, 93D09.
Key words and phrases. Bilinear systems; constrained controls; feedback stabilization; robustness.

Submitted January 14, 2011. Published March 7, 2011.

Recently, the exponential stabilization problem of distributed bilinear systems has been resolved (see [12]). Then it has been proved that under the assumption (1.4), the feedback defined by

$$v(t) = \begin{cases} -\frac{\langle Bz(t), z(t)\rangle}{\|z(t)\|^2}, & \text{if } z(t) \neq 0 \\ 0, & \text{if } z(t) = 0, \end{cases} \tag{1.5}$$

guarantees the exponential stabilization. The finite-dimensional case has been treated in [5].

It is interesting to investigate the relation between the stability of a distributed parameter system and that of a finite-dimensional system. In [6], it has been showed that if the spectrum $\sigma(A)$ of A can be decomposed into $\sigma_u(A) = \{\lambda : \mathrm{Re}(\lambda) \geq -\eta\}$ and $\sigma_s(A) = \{\lambda : \mathrm{Re}(\lambda) < -\eta\}$ for some $\eta > 0$, such that $\sigma_u(A)$ can be separated from $\sigma_s(A)$ by a simple and closed curve C, then the state space H can be decomposed according to

$$H = H_u \oplus H_s, \tag{1.6}$$

where $H_u = P_u H$, $H_s = P_s H$, and P_u is the projection operator

$$P_u = \frac{1}{2\pi i} \int_C (\lambda I - A)^{-1} d\lambda$$

and $P_s = I - P_u$. Then the operator A can be decomposed as $A = A_u \oplus A_s$ with $A_u = P_u A$ and $A_s = P_s A$. For linear systems and based on the above decomposition (1.6), it has been showed that the whole system can be divided into two uncoupled subsystems, one of which is exponentially stable without applying controls, while another one is unstable. Then under the spectrum growth assumption:

$$\lim_{t \to +\infty} \frac{\ln \|S_s(t)\|}{t} = \sup \mathrm{Re}(\sigma(A_s)), \tag{1.7}$$

where $S_s(t)$ denotes the semigroup generated by A_s, it has been proved that stabilizing a linear system turns out to stabilizing its unstable part (see [9]). This technique has been used to study weak and strong stabilization of (1.1) using the quadratic control

$$v_u(t) = -\langle z(t), P_u B P_u z(t)\rangle, \tag{1.8}$$

where it has been assumed that the operator B can be decomposed as

$$B = B_u \oplus B_s, \tag{1.9}$$

with $B_u = P_u B P_u$ and $B_s = P_s B P_s$ (see [11]). The aim of this work consists on exploring the decomposition (1.6) of the state space, to study the strong and exponential stabilization of the system (1.1) using quadratic and constrained controls. In the second section, we show that one can achieve the strong stabilization of (1.1) under the condition (1.3). If in addition (1.9) holds, then under a weaker version of (1.4), one obtains exponential stabilization. In the third section, we study the question of robustness. The last section concerns some situations that illustrate the established results.

2. Stabilization results

2.1. **Strong stabilization.** The next result concerns the strong stabilization of (1.1).

Theorem 2.1. *Let (i) A generate a linear C_0-semigroup $S(t)$ of contractions on H, (ii) A allow the decomposition* (1.6) *of H with* $\dim H_u < +\infty$ *such that* (1.7) *holds, and (iii) B be compact such that*

$$\langle BS(t)y, S(t)y\rangle = 0,\ \forall t \geq 0 \Rightarrow y = 0\,. \tag{2.1}$$

Then the system (1.1) *is strongly stabilizable by the feedbacks*

$$v(t) = -\rho\langle z(t), Bz(t)\rangle, \tag{2.2}$$

and

$$v(t) = -\rho\frac{\langle z(t), Bz(t)\rangle}{\|z(t)\|^2}\mathbf{1}_E \tag{2.3}$$

where $\rho > 0$ and $\mathbf{1}_E$ is the characteristic function of the set $E = \{t \geq 0 : z(t) \neq 0\}$.

Proof. System (1.1), controlled by (2.2) or (2.3), possesses a unique mild solution $z(t)$ defined on a maximal interval $[0, t_{max}[$ and given by the variation of constants formula

$$z(t) = S(t)z_0 - \int_0^t S(t-s)F(z(s))ds, \tag{2.4}$$

where $F(z) = \rho\langle BZ, z\rangle Bz$ corresponds to (2.2) and $F(z) = \rho\frac{\langle BZ,z\rangle}{\|z\|^2}Bz$, for all $z \neq 0$, $F(0) = 0$ corresponds to (2.3). Since $S(t)$ is a semigroup of contractions,

$$\frac{d\|z(t)\|^2}{dt} \leq -2\langle F(z(t)), z(t)\rangle, \quad \forall z_0 \in D(A)\,. \tag{2.5}$$

It follows that

$$\|z(t)\| \leq \|z_0\|\,. \tag{2.6}$$

Based on (2.4) and using the fact that $S(t)$ is a semigroup of contractions and Gronwall inequality, we deduce that the map $z_0 \to z(t)$ is continuous from H to H. We deduce that (2.6) holds for all $z_0 \in H$ and hence $t_{\max} = +\infty$ (see [13]).

Now let us show that $z(t) \rightharpoonup 0$, as $t \to +\infty$. Let $t_n \to +\infty$ such that $z(t_n)$ weakly converge in H, and let $y \in H$ such that $z(t_n) \rightharpoonup y$, as $n \to \infty$. (The existence of such (t_n) and y is ensured by (2.6) and by the fact that H is reflexive). □

Let us recall the following existing result.

Lemma 2.2 ([10, 12])**.** *Let A generate a semigroup of contractions $S(t)$ on H and let B be linear and bounded. Then* (1.1)*, controlled by* (2.2) *or* (2.3)*, possesses a unique mild solution $z(t)$ on $\mathbb{R}^+$ for each $z_0 \in H$ which satisfies*

$$\int_0^T |\langle BS(t)z_0, S(t)z_0\rangle|dt \leq \mathcal{C}\|z_0\|\Big\{\int_0^T \langle F(z(t)), z(t)\rangle dt\Big\}^{1/2}, \tag{2.7}$$

for all $T > 0$ and for some $\mathcal{C} = \mathcal{C}(T, \|z_0\|) > 0$.

Taking $z(t_n)$ as initial state in (2.7) and using superposition property of the solution we obtain, via the dominated convergence theorem, $\langle BS(t)y, S(t)y\rangle = 0$, for all $t \geq 0$. It follows from (2.1) that $y = 0$. Hence $z(t) \rightharpoonup 0$, as $t \to +\infty$, and since $\dim H_u < +\infty$, we have

$$z_u(t) \to 0, \quad \text{as } t \to +\infty.$$

For the component $z_s(t)$ of $z(t)$, we have

$$z_s(t) = S_s(t)z_{0s} - \int_0^t S_s(t-\tau)F(z(\tau))d\tau\,. \tag{2.8}$$

The semigroup $S_s(t)$ satisifes, by (1.7), the inequality

$$\|S_s(t)\| \le \alpha e^{-\eta t}, \quad \forall t \ge 0, \quad \text{(for some } \alpha, \eta > 0). \tag{2.9}$$

Then for all $0 \le t_1 \le t$, we have $z_s(t) = S_s(t-t_1)z_s(t_1) - \int_{t_1}^t S_s(t-\tau)F(z(\tau))d\tau$. It follows that

$$\|z_s(t)\| \le \alpha e^{-\eta(t-t_1)}\|z_s(t_1)\| + \alpha \int_{t_1}^t e^{-\eta(t-\tau)}\|F(z(\tau))\|d\tau\,. \tag{2.10}$$

Since F is sequentially continuous,

$$F(z(t)) \to 0, \quad \text{as } t \to +\infty\,.$$

Let $\varepsilon > 0$ and let $t_1 > 0$ such that $\|F(z(t))\| < \varepsilon$, for all $t \ge t_1$. It follows that

$$\|z_s(t)\| \le \alpha e^{-\eta(t-t_1)}\|z_s(t_1)\| + \frac{\alpha\varepsilon}{\eta}, \quad \forall\, t \ge t_1\,.$$

Hence $z(t) = z_u(t) + z_s(t) \to 0, \quad \text{as } t \to +\infty$.

Remark 2.3. (1) Note that the feedback (2.3) is a bounded function in time and is uniformly bounded with respect to initial states

$$|v(t)| \le \rho\|B\|, \quad \forall t \ge 0,\ \forall z_0 \in H\,.$$

(2) For finite-dimensional systems, the conditions (1.3) and (1.4) are equivalent (see [5, 14]). However, in infinite-dimensional case and if B is compact, then the condition (1.4) is impossible. Indeed, if (φ_j) is an orthonormal basis of H, then applying (1.4) for $y = \varphi_j$ and using the fact that $\varphi_j \rightharpoonup 0$, as $j \to +\infty$, we obtain the contradiction $\delta = 0$.

(3) In [1] (resp. [12]), the case $H_u = H$ has been considered and it has been shown that (1.2) (resp. (1.5)) guarantees the weak stability of (1.1).

(4) Since $\|z(t)\|$ decreases, then we have $\exists t_0 \ge 0;\ z(t_0) = 0 \Leftrightarrow z(t) = 0,\ \forall t \ge t_0$. In this case we have $v(t) = 0,\ \forall t \ge t_0$.

(5) Note that for finite-dimensional bilinear systems, unlike the linear case, the strong stability is not equivalent to the exponential one (see [11]).

(6) We can extend the estimate (2.7) to the case where B is nonlinear and locally Lipschitz, so we can obtain a semi-linear version of the above theorem.

2.2. **Exponential stabilization.** In this section, we will associate between the exponential stabilizability of the whole system (1.1) and the one of its unstable part. In the sequel we suppose that (1.9) holds, so that the system (1.1) can be decomposed in the following two subsystems

$$\frac{dz_u(t)}{dt} = A_u z_u(t) + v(t)B_u z_u(t), \quad z_u(0) = z_{0u} \in H_u, \tag{2.11}$$

$$\frac{dz_s(t)}{dt} = A_s z_s(t) + v(t)B_s z_s(t), \quad z_s(0) = z_{0s} \in H_s, \tag{2.12}$$

in the state spaces H_u and H_s respectively.

In the following result, we study the exponential stabilizability of system (1.1).

Theorem 2.4. *Let (i) A generate a linear C_0-semigroup $S(t)$ such that $S_u(t)$ is of isometries and (1.7) holds, (ii) A allow the decomposition (1.6) of H with $\dim H_u < +\infty$, (iii) $B \in \mathcal{L}(H)$ such that for all $y_u \in H_u$, we have*

$$\langle B_u e^{tA_u} y_u, e^{tA_u} y_u\rangle = 0, \quad \forall t \ge 0 \Rightarrow y_u = 0\,. \tag{2.13}$$

Then there exists $\rho > 0$ *such that the feedback*

$$v_u(t) = -\rho \frac{\langle z_u(t), B_u z_u(t)\rangle}{\|z_u(t)\|^2} \mathbf{1}_{E_u}, \tag{2.14}$$

where $E_u = \{t \geq 0 : z_u(t) \neq 0\}$*, exponentially stabilizes* (1.1).

Proof. Let us consider the system

$$\frac{dz_u(t)}{dt} = A_u z_u(t) + f_\rho(z_u(t)) B_u z_u(t),\ z_u(0) = z_{0u}, \tag{2.15}$$

where

$$f_\rho(z_u) = \begin{cases} -\rho \frac{\langle z_u, B_u z_u\rangle}{\|z_u\|^2}, & z_u \neq 0 \\ 0, & z_u = 0 \end{cases}$$

The system (2.15) possesses a unique mild solution $z_u(t)$ defined for all $t \geq 0$, and the map $T_u(t) z_{0u} = z_u(t)$ defines a nonlinear semigroup on H_u.

Integrating the inequality over the interval $[k, k+1]$, $k \in \mathbb{N}$,

$$\frac{d\|z_u(t)\|^2}{dt} \leq -2|f_\rho(z_u(t))|^2 \|z_u(t)\|^2, \tag{2.16}$$

we obtain

$$\|z_u(k+1)\|^2 - \|z_u(k)\|^2 \leq -2 \int_k^{k+1} |f_\rho(z_u(\tau))|^2 \|z_u(\tau)\|^2 d\tau$$

and since $\|z_u(t)\|$ decreases, we deduce that

$$\|z_u(k+1)\|^2 - \|z_u(k)\|^2 \leq -2\|z_u(k+1)\|^2 \int_k^{k+1} |f_\rho(z_u(\tau))|^2 d\tau\,. \tag{2.17}$$

Let $\delta := 2 \inf_{\|z_{0u}\|=1} \int_0^1 |f_\rho(T_u(\tau) z_{0u})|^2 d\tau \geq 0$. Since B_u is linear, then $f_\rho(\lambda z_u) = f_\rho(z_u), \forall \lambda \in \mathbb{C}, z_u \in H_u$. Then by an argument of uniqueness of the mild solution, we deduce that $T_u(t)(\lambda z_u) = \lambda T_u(t) z_u,\ \forall t \geq 0, \lambda \in \mathbb{C}, z_u \in H_u$. It follows that $2 \int_0^1 |f_\rho(T_u(\tau) z_{0u})|^2 d\tau \geq \delta,\ \forall z_{0u} \in H_u - \{0\}$. Using the superposition property of the semigroup $T_u(t)$ and the fact that $\|T_u(t) z_{0u}\| \leq \|z_{0u}\|$ we deduce that $2 \int_k^{k+1} |f_\rho(T_u(\tau) z_{0u})|^2 d\tau \geq \delta,\ \forall k \in \mathbb{N}$. Then (2.17) implies

$$\|z_u(k+1)\|^2 - \|z_u(k)\|^2 \leq -\delta \|z_u(k+1)\|^2\,.$$

It follows that

$$\|z_u(k)\| \leq \frac{\|z_{0u}\|}{(1+\delta)^{\frac{k}{2}}}\,.$$

Finally, using the fact that $\|z_u(t)\|$ decreases, we deduce that

$$\|z_u(t)\| \leq \|z_u(k)\| \leq M e^{-\sigma t} \|z_{0u}\|,$$

where $M = e^{\frac{\ln(1+\delta)}{2}}$ and $\sigma = \frac{\ln(1+\delta)}{2}$. Let us show that $\delta > 0$. Assume that $\delta = 0$. Since $\int_0^1 |f_\rho(T_u(\tau) z_{0u})|^2 d\tau$ depends continuously on z_{0u} and $\dim H_u < +\infty$, then there exists $y_u \in H_u$ such that $\|y_u\| = 1$ and $\int_0^1 |f_\rho(T_u(\tau) y_u)|^2 d\tau = 0$. Then $f_\rho(T_u(t) y_u) = 0,\ \forall 0 \leq t \leq 1$, which imply that $T_u(t) y_u = S_u(t) y_u,\ \forall 0 \leq t \leq 1$ and hence $< B_u S_u(t) y_u, S_u(t) y_u\rangle = 0,\ \forall 0 \leq t \leq 1$. Now from [5, 14], we have

$$\int_0^1 | < B_u S_u(t) z_u, S_u(t) z_u > | dt \geq \delta \|z_u\|^2,\ \forall z_u \in H_u,$$

which gives the contradiction $y_u = 0$.

For the component $z_s(t)$, we shall show that $z_s(t)$ is defined for all $t \geq 0$ and exponentially converges to 0, as $t \to +\infty$. The system (1.1), excited by the control (2.14), admits a unique mild solution defined for all t in a maximal interval $[0, t_{\max}[$ by

$$z(t) = S(t)z_0 + \int_0^t v_u(\tau)S(t-\tau)Bz(\tau)d\tau\,.$$

Thus

$$z_s(t) = S_s(t)z_{0s} + \int_0^t v_u(\tau)S_s(t-\tau)B_s z_s(\tau)d\tau, \quad \forall t \in [0, t_{\max}[\,. \tag{2.18}$$

It follows from (2.9) and (2.18) that

$$\|z_s(t)\| \leq \alpha\, e^{-\eta t}\|z_{0s}\| + \alpha L \int_0^t e^{-\eta(t-\tau)}|v_u(\tau)|\|z_s(\tau)\|d\tau$$

for all $t \in [0, t_{\max}[$ with $L = \|B\|$. Then the scalar function $y(t) = \|z_s(t)\|\, e^{\eta t}$ satisfies

$$y(t) \leq \alpha\|z_{0s}\| + \rho\alpha L^2 \int_0^t y(\tau)d\tau\,.$$

Gronwall inequality then yields $y(t) \leq \alpha\|z_{0s}\|e^{\rho\alpha L^2 t}$. In other words,

$$\|z_s(t)\| \leq \alpha\|z_{0s}\|e^{(\rho\alpha L^2-\eta)t}. \tag{2.19}$$

Taking $\rho > 0$ such that $\rho\alpha L^2 - \eta < 0$, it follows that $z_s(t)$ is bounded on $[0, t_{\max}[$ so that $t_{\max} = +\infty$, and the estimate (2.19) holds for all $t \geq 0$. We conclude that

$$\|z(t)\| \leq Ne^{-\beta t}\|z_0\|, \quad \forall t \geq 0, \tag{2.20}$$

where $N > 0$ and $\beta = \min(\sigma, \eta - \rho\alpha L^2) > 0$. □

Remark 2.5. (1) The feedback (2.14) depends only on the unstable part $z_u(t)$ and is uniformly bounded with respect to initial states.

(2) The quadratic control (1.8) does not guarantee the exponential stability.

(3) We note that (2.13) is weaker than both (1.3) and (1.4) .

(4) The rate of exponential convergence β (given in (2.20)) can be explicitly expressed, and from [12], one can calculate the parameter ρ corresponding to the optimal value of β.

(5) If $H = H_u$ is of finite-dimension, then we retrieve the result of [5].

(6) In the case $\dim H_u = +\infty$ or if B is nonlinear and locally Lipschitz, then following the techniques used in [12], we can obtain the result of Theorem 2.4 if (2.13) is changed to

$$\int_0^T |\langle B_u e^{tA_u}y_u, e^{tA_u}y_u\rangle|dt \geq \delta\|y_u\|^2, \quad \forall y_u \in H_u,\ (T, \delta > 0)\,.$$

(7) It is easily verified that the condition (1.9) holds in the case of commutative systems (i.e. B commutes with A).

(8) Note that the condition (1.9) also holds in the case $H_u = H$. This special case has been treated in [12].

3. Robustness

In this section, we study the robustness of the controls (2.2), (2.3) and (2.14) under a class of perturbations on the dynamic A of (1.1).

3.1. **Strong robustness.** In this part, we consider the strong robustness of the feedbacks (2.2) and (2.3). We show that the stability property of the system (1.1) remains invariant under a certain class of bounded perturbations. Consider the perturbed system

$$\frac{dz(t)}{dt} = (A+E)z(t) + v(t)Bz(t), \quad z(0) = z_0 \,. \tag{3.1}$$

Where E is a perturbation of A. Let us define

$$\begin{aligned}\Lambda = \big\{ &E \in \mathcal{L}(H); E \text{ commutes with } P_u,\ E = FB \text{ for some } F \in \mathcal{L}(H),\\ &A+E \text{ is dissipative and } \|E_s\| < \frac{\eta}{\alpha} \big\}.\end{aligned}$$

where α, η are given by (2.9).

Proposition 3.1. *Let*

- (i) *A generate a linear C_0-semigroup $S(t)$ of contractions on H such that* (1.7) *holds,*
- (ii) *B be self-adjoint, positive and compact such that* (2.1) *holds, and*
- (iii) *$E \in \Lambda$.*

Then both the feedbacks (2.2) *and* (2.3) *strongly stabilize the system* (3.1)*.*

Proof. Since $A+E$ is dissipative, the operator $A+E$ generates a semigroup of contractions $T(t)$ (see [13]). Let us show that (2.1) $\Rightarrow$ ($\langle BT(t)y, T(t)y\rangle = 0$ for all $t \geq 0 \Longrightarrow y = 0$). Suppose that $\langle BT(t)y, T(t)y\rangle = 0$ for all $t \geq 0$. Since $B \geq 0$, we have $B^{1/2}T(t)y = 0$ and so $BT(t)y = 0$ for all $t \geq 0$. Using the variation of constant formula, we obtain

$$T(t)y = S(t)y + \int_0^t S(t-s)ET(s)y ds, \quad \forall y \in H, \tag{3.2}$$

which implies that $T(t)y = S(t)y$, and so $\langle BS(t)y, S(t)y\rangle >= 0$ for all $t \geq 0$. It follows from (2.1) that $y = 0$.

Taking $y = z_{0s}$ in (3.2) and using the fact that E commutes with P_u, we obtain the formula

$$T_s(t)z_{0s} = S_s(t)z_{0s} + \int_0^t S_s(t-s)E_sT_s(s)z_{0s} ds,$$

where $T_s(t)$ is the restriction of $T(t)$ to H_s. Then, using the fact that $S_s(t)$ verifies (2.9), we obtain

$$\|T_s(t)z_{0s}\| \leq \alpha\|z_{0s}\| \exp(-\eta t) + \alpha\|E_s\| \int_0^t \exp(-\eta(t-s))\|T_s(s)z_{0s}\| ds \,.$$

Applying the Gronwall's inquality, we deduce that

$$\|T_s(t)z_{0s}\| \leq \alpha \exp((\alpha\|E_s\| - \eta)t)\|z_{0s}\| \,.$$

Taking $\|E_s\| < \frac{\eta}{\alpha}$, we conclude that $T_s(t)$ satisifes (2.9). The result of Theorem 2.1 completes the proof. ∎

3.2. **Exponential robustness.** Consider the perturbed system

$$\frac{dz(t)}{dt} = (A+E)z(t) + v(t)Bz(t), \quad z(0) = z_0, \tag{3.3}$$

where E is a perturbation of A. In this part, we consider the robustness of the feedback control law (2.14).

Let us define the set of admissible operator of perturbations

$$\begin{aligned}\Lambda = \big\{ &E \in \mathcal{L}(H); E \text{ commutes with } P_u,\ E_u = B_u F_u,\ \text{for some } F_u \in \mathcal{L}(H_u),\\ &A_u + E_u \text{ is dissipative and } \|E_s\| \le \epsilon \frac{\eta}{\alpha},\ (0 < \epsilon < 1) \big\}.\end{aligned}$$

Proposition 3.2. *Let*

(i) *A_u generates a linear C_0-semigroup of isometries $S_u(t)$ on H such that* (1.7) *holds,*
(ii) *$B \in \mathcal{L}(H)$ and B_u is self-adjoint, positive such that* (2.13) *holds, and*
(iii) *$E \in \Lambda$.*

Then there exists $\rho > 0$ such that the feedback (2.14) *is exponentially robust.*

Proof. Since E commutes with P_u, the perturbed system (3.3) can be decomposed as

$$\frac{dz_u(t)}{dt} = (A_u + E_u)z_u(t) + v(t)B_u z_u(t), \quad z_u(0) = z_{0u} \in H_u,$$

and

$$\frac{dz_s(t)}{dt} = (A_s + E_s)z_s(t) + v(t)B_s z_s(t), \quad z_s(0) = z_{0s} \in H_s. \tag{3.4}$$

Using the same techniques as in the proof of Proposition 3.1, we can show that

$$\langle B_u T_u(t) y_u, T_u(t) y_u \rangle = 0,\ \forall\, t \ge 0 \Longrightarrow y_u = 0$$

and

$$\|T_s(t) z_{0s}\| \le \alpha \exp((\alpha \|E_s\| - \eta)t) \|z_{0s}\|\,.$$

We conclude that $T_s(t)$ verifies (2.9) and hence the solution of (3.4) verifies the estimate (2.19). Taking $\rho > 0$ such that $\rho < (1-\epsilon)\frac{\eta}{\alpha\|B\|^2}$, we deduce that the feedback law (2.14) exponentially stabilizes the perturbed system. □

4. Applications

Let us consider the system defined by

$$\begin{gathered}\frac{\partial z(x,t)}{\partial t} = \frac{\partial^2 z(x,t)}{\partial x^2} + v(t)Bz(t), \quad \forall x \in]0,1[,\ \forall t > 0,\\ z'(0,t) = z'(1,t) = 0, \quad \forall t > 0\end{gathered} \tag{4.1}$$

where the state space is $H = L^2(0,1)$, the operator is $Az = \frac{\partial^2 z}{\partial x^2}$, for $z \in \mathcal{D}(A) = \{z \in H^2(0,1) : z'(0) = z'(1) = 0\}$. The spectrum of A is given by the simple eigenvalues $\lambda_j = -\pi^2(j-1)^2$, $j \in \mathbb{N}^*$ and eigenfunctions $\varphi_1(x) = 1$ and $\varphi_j(x) = \sqrt{2}\cos((j-1)\pi x)$ for all $j \ge 2$.

For the operator of control B, we consider two situations:

Case 1: $B = I$. This case has been considered in [11] to obtain strong stabilizability of (4.1) using the quadratic feedback (1.8). However, in this way one does not obtain a better convergence than of order of $1/\sqrt{t}$. In the following, we show that the feedback (2.14) ensures the exponential stability. In the case $\int_0^1 z(x,t)dx = 0$, we have $z_u(t) = 0$ for all $t \ge 0$, then $v_u(t) = 0$ and $z(t) = z_s(t) = S_s(t)z_{0s}$.

Let us suppose that $\int_0^1 z(x,t)dx \neq 0$. Here we can take $\eta = \alpha = 1$. Then applying the result of Theorem 2.4, we deduce that for all $0 < \rho < 1$, the control $v_u(t) = -\rho$ achieves the exponential stabilization of (4.1) with the rate of convergence $\beta = \min(\frac{1}{2}\ln(\frac{11}{9}), 1-\rho)$.

Note that $z_u(t)$ can be directly expressed $z_u(t) = e^{-t} z_{0u},\ \forall t \geq 0$. This shows that the rate of exponential convergence β can be improved since we have $1 > \min(\frac{1}{2}\ln(\frac{11}{9}), 1-\rho)$.

Now let us examine the robustness of the control (2.14). Let us reconsider the above system with the perturbation $Ez = \epsilon(z - \int_0^1 z(x)dx),\ 0 < \epsilon < 1$. The perturbed open-loop system remains unstable. However, for all $0 < \rho < 1-\epsilon$, the control $v(t) = -\rho$ exponentially stabilizes the perturbed system.

Case 2: $Bz = \sum_{j=1}^{+\infty} \alpha_j < z, \varphi_j > \varphi_j$, where $\alpha_j \geq 0$ for all $j \geq 1$ and $\sum_{j=1}^{+\infty} \alpha_j^2 < \infty$. This case was considered in [12], where it has been showed that if, $\alpha_j > 0$ for all $j \geq 1$ then (1.5) weakly stabilizes (4.1). Here we show that the stability is in the strong sense. Clearly B is a linear and compact operator, and from the relation

$$\langle BS(t)y, S(t)y\rangle = \sum_{j=1}^{+\infty} \alpha_j |\langle z, \varphi_j\rangle|^2,$$

we can see that (2.1) holds if $\alpha_j > 0$ for all $j \geq 1$. In this case, Theorem 2.1 applies and the system (4.1) is strongly stabilizable using the controls (2.2) and (2.3); i.e.,

$$v_1(t) = -\rho \sum_{j=1}^{+\infty} \alpha_j |\langle z(\cdot,t), \varphi_j\rangle|^2$$

and

$$v_2(t) = \frac{-\rho}{\|z(\cdot,t)\|^2} \sum_{j=1}^{+\infty} \alpha_j |\langle z(\cdot,t), \varphi_j\rangle|^2, \quad \forall z_0 \neq 0\,.$$

Note that (1.4) does not hold, so the existing result of [12] is not applicable to establish exponential stabilization by the control (1.5).

It is clear that we can apply the result of Theorem 2.4 to deduce that if $\alpha_1 > 0$ and $0 < \rho < 1/\alpha_1$, then the control defined by $v_u(t) = -\alpha_1\rho$, for all z_0 such that $z_{0u} \neq 0$ ensures the exponential stabilizability of (4.1) with the rate of convergence $\beta = \min(\ln(11/9)/2, 1-\rho)$. With the perturbation

$$Ez = \epsilon(z - \int_0^1 z(x)dx), \quad 0 < \epsilon \leq 1$$

on the open-loop system, both controls (2.2) and (2.3) are strongly robust, and for $0 < \epsilon < 1$, the control (2.14) is exponentially robust.

Conclusion. In this work we have considered the problem of strong stabilization of a constrained parabolic bilinear system under the conventional ad-condition (1.3). Under a weaker condition than (1.4), an exponential stabilization result has been established. Also the question of robustness of the stabilizing controls is discussed.

References

[1] J. Ball and M. Slemrod; *Feedback stabilization of distributed semilinear control systems*, Appl. Math. Opt., 5 (1979), 169-179.

[2] L. Berrahmoune; *Stabilization and decay estimate for distributed bilinear systems.* Systems and Control Letters., 36 (1999), 167-171.
[3] L. Berrahmoune; *Stabilization of bilinear control systems in Hilbert space with nonquadratic feedback*, Rend. Circ. Mat. Palermo., 58 (2009), 275282.
[4] L. Berrahmoune, Y. Elboukfaoui, M. Erraoui; *Remarks on the feedback stabilization of systems affine in control*, Eur. J. Control 7 (2001), 17-28.
[5] Min-Shin Chen; *Exponential stabilization of a constrained bilinear system*, Automatica., 34 (1998), 989-992.
[6] T. Kato; *Perturbation theory for linear operators*, New York., Springer, 1980.
[7] T. I. Seidman, H. Li; *A note on stabilization with saturating feedback*, Discrete Contin. Dyn. Syst. 7 (2001) 319328.
[8] Rob Luesink and Henk Nijmeijer; *On the Stabilization of Bilinear Systems via Constant Feedback*, Linear Algebra and its Applications., 122/123/124: (1989), 457-474.
[9] R. Triggiani; *On the stabilizability problem in Banach space.* J. Math. Anal. Appl, 52 (1975), 383-403.
[10] M. Ouzahra; *Strong stabilization with decay estimate of semilinear systems*, Systems and Control Letters, 57 (2008), 813-815.
[11] M. Ouzahra; *Stabilization of infinite-dimensional bilinear systems using a quadratic feedback control*, International Journal of Control., 82 (2009), 1657-1664.
[12] M. Ouzahra; *Exponential and Weak Stabilization of Constrained Bilinear Systems*, SIAM J. Control Optim., 48, Issue 6 (2010), 3962-3974.
[13] A. Pazy; *Semi-groups of linear operators and applications to partial differential equations*, Springer Verlag, New York, (1983).
[14] J. P. Quinn; *Stabilization of bilinear systems by quadratic feedback control*, J. Math. Anal. Appl, 75 (1980), 66-80.

Mohamed Ouzahra
Department of Mathematics and informatics, P.O. Box 5206, ENS, Fes, Morocco
E-mail address: `m.ouzahra@yahoo.fr`