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ASYMPTOTIC BEHAVIOR OF SECOND-ORDER IMPULSIVE DIFFERENTIAL EQUATIONS

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ABSTRACT. In this article, we study the asymptotic behavior of all solutions of 2-th order nonlinear delay differential equation with impulses. Our main tools are impulsive differential inequalities and the Riccati transformation. We illustrate the results by an example.

1. Introduction

Consider the impulsive differential equation

$$\big(r(t)(x'(t))^{\alpha}\big)' + p(t)(x'(t))^{\alpha} + f(t, x(t-\delta)) = 0, \quad t \geq t_0,\ t \neq t_k, \tag{1.1}$$

$$x(t_k^+) = J_k(x(t_k)), \quad x'(t_k^+) = I_k(x'(t_k)), \quad k = 1, 2, 3 \dots, \tag{1.2}$$

where α is the quotient of positive odd integers.

The theory of impulsive differential/difference equations is emerging as an important area of investigation, since it is much richer than the corresponding theory of differential/difference equations without impulsive effects. Moreover, such equations may model several real world phenomena [4]. There are many papers devoted to the oscillation criteria of differential equations with impulses [2, 5, 6] and to the asymptotic behavior of all solutions of differential equations without impulses [8].

Recently, Tang [7] studied the equation

$$\begin{gathered}(r(t)x'(t))' + p(t)x'(t) + f(t, x(t-\delta)) = 0, \quad t \neq t_k,\\ x(t_k^+) = J_k(x(t_k)), \quad k = 1, 2, 3 \dots,\\ x'(t_k^+) = I_k(x'(t_k)), \quad k = 1, 2, 3 \dots.\end{gathered}$$

He obtained sufficient conditions of asymptotic behavior of all solutions of the equation.

Motivated by [7], using impulsive differential inequality and the Riccati transformation, we study the asymptotic behavior of solutions of (1.1), (1.2).

Definition 1.1. For $\phi \in C([t_0 - \delta, t_0], \mathbb{R})$, a function $x : [t_0 - \delta, +\infty) \to \mathbb{R}$ is called a solution of (1.1), (1.2) satisfying the initial value condition

$$x(t) = \phi(t), \quad t \in [t_0 - \delta, t_0]$$

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if the following conditions are satisfied:

(i) $x(t)=\phi(t)$ for $t\in[t_0-\delta,t_0]$,

(ii) x,x' are continuously differentiable for $t>t_0$, $t\neq t_k$ $(k=1,2,\dots)$ and satisfy (1.1),

(iii) $x(t_k^-)=x(t_k), x'(t_k^-)=x'(t_k)$, $k=1,2,\dots$ and satisfy (1.2).

As is customary, a solution of (1.1), (1.2) is said to be non-oscillatory if it is eventually positive or eventually negative. Otherwise, it will be called oscillatory.

2. Main results

In this paper, we assume that the following conditions hold:

(H1) f is continuous on $[t_0,+\infty)\times\mathbb{R}$, $xf(t,x)>0$ for $x\neq 0$, and $\frac{f(t,x)}{g(x)}\geq h(t)$ for $x\neq 0$, where $g(\gamma x)\geq\gamma g(x)$ for $\gamma>0$, $x'g'(x)>0$, and h,r' are continuous on $[t_0,+\infty)$, $h(t)\geq 0, r(t)>0$.

(H2) p,J_k,I_k are continuous on $\mathbb{R}$ and there exist positive numbers a_k^*,a_k,b_k^*,b_k such that $a_k^*\leq\frac{I_k(x)}{x}\leq a_k, b_k^*\leq\frac{J_k(x)}{x}\leq b_k$.

(H3) $\lim_{t\to\infty}\int_{t_j}^t\prod_{t_j<t_k<s}\frac{a_k^*}{b_k}\exp(-\int_{t_j}^s\frac{r'(\sigma)+p(\sigma)}{\alpha r(\sigma)}d\sigma)ds=+\infty$.

(H4)
$$\begin{aligned}&\sum_{m=1}^{n-1}\prod_{k=m}^{n-1}\prod_{l=0}^{m-1}b_{j+k}a_{j+l}^*\int_{t_{j+m-1}}^{t_{j+m}}\exp\Big(-\int_{t_j}^u\frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)du\\&+\prod_{k=0}^{n-1}a_{j+k}^*\int_{t_{j+n-1}}^{t_{j+n}}\exp\Big(-\int_{t_j}^u\frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)du\to+\infty,\quad\text{as } n\to\infty.\end{aligned}$$

(H5)
$$\lim_{t\to\infty}\int_{t_0}^t\prod_{t_0<t_k<s}\frac{1}{c_k}\exp(\int_{t_0}^s\frac{p(\sigma)}{r(\sigma)}d\sigma)h(s)ds=+\infty,$$
where
$$c_k=\begin{cases}a_k^\alpha, & t_k-\delta\neq t_j,\\ \frac{a_k^\alpha}{b_j^*}, & t_k-\delta=t_j.\end{cases}$$

In the following, we also assume that solutions to (1.1), (1.2) exist on $[t_0,+\infty)$.

Lemma 2.1 ([1]). *Let the function $m\in PC^1(\mathbb{R}_+,\mathbb{R})$ satisfy the inequalities*
$$m'(t)\leq p(t)m(t)+q(t),\quad t\neq t_k,$$
$$m(t_k^+)\leq d_km(t_k)+b_k,\quad k=1,2,\dots,$$
where $p,q\in PC(R_+,R)$ and $d_k\geq 0,b_k$ are constants, then
$$\begin{aligned}m(t)\leq m(t_0)&\prod_{t_0<t_k<t}d_k\exp\Big(\int_{t_0}^tp(s)ds\Big)+\sum_{t_0<t_k<t}\Big(\prod_{t_k<t_j<t}d_j\exp\Big(\int_{t_k}^tp(s)ds\Big)\Big)b_k\\&+\int_{t_0}^t\prod_{s<t_k<t}d_k\exp\Big(\int_s^tp(\sigma)d\sigma\Big)q(s)ds,\quad t\geq t_0.\end{aligned}\tag{2.1}$$

Lemma 2.2. *Let x be a solution of (1.1), (1.2). Suppose that there exist some $T\geq t_0$ such that $x(t)>0,t\geq T$. If (H1)–(H3) are satisfied, then $x'(t_k)>0$ and $x'(t)>0$ for $t\in(t_k,t_{k+1}]$, where $t_k\geq T$, $k=1,2,\dots$.*

Proof. We first prove that $x'(t_k) > 0$ for any $t_k \geq T$. If not, there must exist some j such that $x'(t_j) < 0$, $t_j \geq T$ and $x'(t_j^+) = I_j(x'(t_j)) \leq a_j^* x'(t_j) < 0$. Let

$$x'(t_j) \exp\Big(\int_{t_0}^{t_j} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big) =: \beta < 0.$$

From (1.1), it is clear that

$$\Big(x'(t) \exp\Big(\int_{t_0}^{t} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big)\Big)' = -\frac{f(t, x(t-\delta))}{\alpha r(t)(x'(t))^{\alpha-1}} \exp\Big(\int_{t_0}^{t} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big).$$

Since α is the quotient of positive odd integers, $(x'(t))^{\alpha-1} > 0$, we obtain

$$\Big(x'(t) \exp\Big(\int_{t_0}^{t} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big)\Big)' < 0. \tag{2.2}$$

Hence, the function $x'(t)\exp(\int_{t_0}^{t} \frac{r'(s)+p(s)}{\alpha r(s)} ds)$ is decreasing on $(t_j, t_{j+1}]$,

$$x'(t_{j+1}) \exp\Big(\int_{t_0}^{t_{j+1}} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big) \leq x'(t_j^+) \exp\Big(\int_{t_0}^{t_j} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big);$$

i.e.,

$$x'(t_{j+1}) \exp\Big(\int_{t_0}^{t_{j+1}} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big) \leq a_j^* \beta$$

and

$$\begin{aligned} x'(t_{j+2}) \exp\Big(\int_{t_0}^{t_{j+2}} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big) &\leq x'(t_{j+1}^+) \exp\Big(\int_{t_0}^{t_{j+1}} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big) \\ &\leq a_{j+1}^* a_j^* \beta. \end{aligned}$$

By induction, we obtain

$$x'(t_{j+n}) \exp\Big(\int_{t_0}^{t_{j+n}} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big) \leq \prod_{k=0}^{n-1} a_{j+k}^* \beta,$$

while for $t \in (t_{j+n}, t_{j+n+1}]$, we have

$$x'(t) \leq \prod_{t_j \leq t_k < t} a_k^* \beta \exp\Big(-\int_{t_0}^{t} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big). \tag{2.3}$$

From the condition $x(t_n^+) \leq b_n x(t_n)$, we have the impulsive differential inequality

$$\begin{gathered} x'(t) \leq \prod_{t_j \leq t_k < t} a_k^* \beta \exp\Big(-\int_{t_0}^{t} \frac{r'(s)+p(s)}{\alpha r(s)} ds\Big), \quad t \neq t_k,\ k = j+1, j+2, \dots, \\ x(t_k^+) \leq b_k x(t_k), \quad t = t_k,\ t \geq t_j. \end{gathered}$$

Applying Lemma 2.1, we have

$$\begin{aligned} x(t) &\leq x(t_j^+) \prod_{t_j < t_k < t} b_k + a_j^* \beta \int_{t_j}^{t} \prod_{s < t_k < t} b_k \prod_{t_j < t_i < s} a_i^* \exp\Big(-\int_{t_0}^{s} \frac{r'(\sigma)+p(\sigma)}{\alpha r(\sigma)} d\sigma\Big) ds \\ &\leq \prod_{t_j < t_k < t} b_k \Big\{x(t_j^+) + a_j^* \beta \int_{t_j}^{t} \prod_{t_j < t_i < s} \frac{a_i^*}{b_i} \exp\Big(-\int_{t_0}^{s} \frac{r'(\sigma)+p(\sigma)}{\alpha r(\sigma)} d\sigma\Big) ds\Big\}. \end{aligned}$$

Since $x(t_k) > 0$ for $t_k \geq T$, one can find that the above inequality contradicts (H3) as $t \to \infty$, therefore, $x'(t_k) \geq 0 (t \geq T)$.

By condition (H2), we have $x'(t_k^+) \geq a_k^* x'(t_k)$ for any $t_k \geq T$. Because the function $x'(t)\exp(\int_{t_0}^t \frac{r'(s)+p(s)}{\alpha r(s)} ds)$ is decreasing on $(t_{j+i-1}, t_{j+i}]$, we obtain

$$x'(t)\exp(\int_{t_0}^t \frac{r'(s)+p(s)}{\alpha r(s)} ds) > 0$$

for any $t \in (t_{j+i-1}, t_{j+i}]$, which implies $x'(t) \geq 0$ for $t \geq T$. The proof is complete. $\square$

Theorem 2.3. *If* (H1)-(H3), (H5) *are satisfied, then every solution* x *of* (1.1), (1.2) *satisfies* $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. Let x be a solution of (1.1)-(1.2), and by contradiction assume that

$$\liminf_{t\to\infty} |x(t)| > 0.$$

Without loss of generality, we may assume that $x(t) > 0$ on $(t_0, +\infty)$. By Lemma 2.2, $x'(t) > 0$ for all $t \geq t_0$. We use a Riccati transformation of the form

$$V(t) = \frac{r(t)(x'(t))^\alpha}{g(x(t-\delta))}. \tag{2.4}$$

Differentiating $V(t)$, we obtain

$$\begin{aligned}
V'(t) &= \frac{(r(t)(x'(t))^\alpha)' g(x(t-\delta)) - r(t)(x'(t))^\alpha g'(x(t-\delta))x'(t-\delta)}{g^2(x(t-\delta))} \\
&= \frac{-p(t)(x'(t))^\alpha - f(t, x(t-\delta))}{g(x(t-\delta))} - \frac{x'(t-\delta)g'(x(t-\delta))}{r(t)(x'(t))^\alpha} V^2(t) \\
&\leq -p(t)\frac{V(t)}{r(t)} - h(t).
\end{aligned}$$

From (2.4) and (H1), it is clear that

$$\begin{aligned}
V(t_k^+) &= \frac{r(t_k^+)(x'(t_k^+))^\alpha}{g(x(t_k^+ - \delta))} \\
&\leq \begin{cases} \frac{r(t_k)(x'(t_k))^\alpha a_k^\alpha}{g(x(t_k-\delta))} = a_k^\alpha V(t_k) = c_k V(t_k), & t_k - \delta \neq t_j, \\ \frac{r(t_k)(x'(t_k))^\alpha a_k^\alpha}{g(x(t_j^+))} \leq \frac{a_k^\alpha}{b_j^*} V(t_k) = c_k V(t_k), & t_k - \delta = t_j, \end{cases}
\end{aligned}$$

where c_k's are defined in (H5). Applying Lemma 2.1, we have

$$\begin{aligned}
V(t) \leq& \prod_{t_0<t_k<t} c_k \exp\Big(-\int_{t_0}^t \frac{p(s)}{r(s)} ds\Big) \\
&\times \Big[V(t_0) - \int_{t_0}^t \prod_{t_0<t_k<s} \frac{1}{c_k} \exp\Big(\int_{t_0}^s \frac{p(\sigma)}{r(\sigma)} d\sigma\Big) h(s) ds\Big].
\end{aligned}$$

By (H5), the above inequality is impossible. The proof is complete. $\square$

Lemma 2.4. *Let* x *be a solution of* (1.1), (1.2). *Suppose that there exist some* $T \geq t_0$ *such that* $x(t) > 0$, $t \geq T$. *If (H1), (H2), (H4) are satisfied, then* $x'(t_k) > 0$ *and* $x'(t) > 0$ *for* $t \in (t_k, t_{k+1}]$, *where* $t_k \geq T, k = 1, 2, \dots$.

Proof. Firstly, for $x(t)>0$, $t\geq T$, we will prove that $x'(t_k)>0$, for any $t_k\geq T$, $T\geq t_0$. If not, there exist some j such that $x'(t_j)<0$, $t_j\geq T$ and $x'(t_j^+)=I_j(x'(t_j))\leq a_j^* x'(t_j)<0$. From (1.1), it is clear that

$$\Big(x'(t)\exp\Big(\int_{t_0}^t \frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)\Big)'=-\frac{f(t,x(t-\delta))}{\alpha r(t)(x'(t))^{\alpha-1}}\exp\Big(\int_{t_0}^t \frac{r'(s)+p(s)}{\alpha r(s)}ds\Big).$$

Since α is the quotient of positive odd integers, $(x'(t))^{\alpha-1}>0$, we obtain

$$\Big(x'(t)\exp\Big(\int_{t_0}^t \frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)\Big)'<0.$$

Hence, the function $x'(t)\exp\Big(\int_{t_0}^t \frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)$ is decreasing on $(t_j,t_{j+1}]$,

$$x'(t_{j+1})\exp\Big(\int_{t_0}^{t_{j+1}} \frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)\leq x'(t_j^+)\exp\Big(\int_{t_0}^{t_j} \frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)),$$

i.e.,

$$x'(t_{j+1})\leq a_j^* x'(t_j)\exp\Big(-\int_{t_j}^{t_{j+1}} \frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)$$

and

$$x'(t_{j+2})\leq a_{j+1}^* a_j^* x'(t_j)\exp\Big(-\int_{t_j}^{t_{j+2}} \frac{r'(s)+p(s)}{\alpha r(s)}ds\Big).$$

By induction, we obtain

$$x'(t_{j+n})\leq \prod_{k=0}^{n-1} a_{j+k}^* x'(t_j)\exp\Big(-\int_{t_j}^{t_{j+n}} \frac{r'(s)+p(s)}{\alpha r(s)}ds\Big).$$

Because the function $x'(t)\exp(\int_{t_0}^t \frac{r'(s)+p(s)}{\alpha r(s)}ds)$ is decreasing on $(t_j,t_{j+1}]$, we have

$$x'(t)\leq a_j^* x'(t_j)\exp\Big(-\int_{t_j}^{t} \frac{r'(s)+p(s)}{\alpha r(s)}ds\Big),\quad t\in(t_j,t_{j+1}]. \tag{2.5}$$

Integrating (2.5) from m to t, we have

$$x(t)\leq x(m)+a_j^* x'(t_j)\int_m^t \exp\Big(-\int_{t_j}^{u} \frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)du,\quad t_j<m<t_{j+1}.$$

Let $t\to t_{j+1}$, $m\to t_j^+$. We have

$$\begin{aligned}
x(t_{j+1})&\leq x(t_j^+)+a_j^* x'(t_j)\int_{t_j}^{t_{j+1}} \exp\Big(-\int_{t_j}^{u} \frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)du\\
&\leq b_j x(t_j)+a_j^* x'(t_j)\int_{t_j}^{t_{j+1}} \exp\Big(-\int_{t_j}^{u} \frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)du,
\end{aligned}$$

and

$$\begin{aligned}
x(t_{j+2})&\leq x(t_{j+1}^+)+a_{j+1}^* x'(t_{j+1})\int_{t_{j+1}}^{t_{j+2}} \exp\Big(-\int_{t_{j+1}}^{u} \frac{r'(s)+p(s)}{\alpha r(s)}ds\Big))du\\
&\leq b_{j+1}b_j x(t_j)+a_j^* b_{j+1} x'(t_j)\int_{t_j}^{t_{j+1}} \exp\Big(-\int_{t_j}^{u} \frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)du\\
&\quad + a_{j+1}^* a_j^* x'(t_j)\int_{t_{j+1}}^{t_{j+2}} \exp\Big(-\int_{t_j}^{u} \frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)du.
\end{aligned}$$

By induction, we have

$$\begin{aligned} x(t_{j+n}) \le x'(t_j)\Big[\sum_{m=1}^{n-1}\prod_{k=m}^{n-1}\prod_{l=0}^{m-1} b_{j+k}a^*_{j+l}\int_{t_{j+m-1}}^{t_{j+m}}\exp\Big(-\int_{t_j}^{u}\frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)du \\ +\prod_{k=0}^{n-1}a^*_{j+k}\int_{t_{j+n-1}}^{t_{j+n}}\exp\Big(-\int_{t_j}^{u}\frac{r'(s)+p(s)}{\alpha r(s)}ds\Big)du\Big]+\prod_{k=0}^{n-1}b_{j+k}x(t_j). \end{aligned}$$

Since $x(t_k) > 0\,(t_k \ge T)$, we find that the above inequality contradicts condition (H4), therefore $x'(t_k) \ge 0$ for $t \ge T$. Further, for $t \in (t_j, t_{j+1}]$, we obtain

$$x'(t)\exp\Big(\int_{t_0}^{t}\frac{r'(s)+p(s)}{\alpha r(s)}ds\Big) \ge x'(t_{j+1})\exp\Big(\int_{t_0}^{t_{j+1}}\frac{r'(s)+p(s)}{\alpha r(s)}ds\Big) > 0,$$

which implies $x'(t) > 0$ for $t \ge T$. This completes the proof. □

Using Lemma 2.4, we have the following Theorem.

Theorem 2.5. *If* (H1), (H2), (H4), (H5) *are satisfied, then every solution x of* (1.1), (1.2) *satisfies* $\liminf_{t\to\infty}|x(t)| = 0$.

Example. Consider

$$\begin{gathered} \Big(t(x'(t))^3\Big)' - \big(x'(t)\big)^3 + \frac{1}{t^2}x(t-\frac{1}{3}) = 0, \quad t \ne k,\ t \ge \frac{1}{2}, \\ x'(k^+) = \frac{k}{k+1}x'(k), \quad x(k^+) = x(k), \quad k = 1,2,\dots . \end{gathered}$$

Comparing with (1.1), (1.2), we see that $r(t) = t$, $p(t) = -1$, $\alpha = 3$, $\delta = 1/3$, $t_{k+1} - t_k > 1/3$ and $a_k = a^*_k = k/(k+1)$, $b_k = b^*_k = 1$. Obviously (H1), (H2) are satisfied,

$$\begin{aligned} &\lim_{t\to\infty}\int_{t_j}^{t}\prod_{t_j<t_k<s}\frac{a^*_k}{b_k}\exp\Big(-\int_{t_j}^{s}\frac{r'(\sigma)+p(\sigma)}{3r(\sigma)}d\sigma\Big)ds \\ &> (j+1)\lim_{t\to\infty}\int_{t_j}^{t}\frac{ds}{s+1} = +\infty, \end{aligned}$$

and

$$\begin{aligned} &\lim_{t\to\infty}\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{1}{c_k}\exp\Big(\int_{t_0}^{s}\frac{p(\sigma)}{r(\sigma)}d\sigma\Big)h(s)ds \\ &= \lim_{t\to\infty}\int_{t_0}^{t}\prod_{t_0<t_k<s}(\frac{1}{a_k})^{\alpha}\exp(-\ln s+\ln t_0)\frac{1}{s^2}ds \\ &> \frac{1}{2}\lim_{t\to\infty}\int_{t_0}^{t}ds = +\infty. \end{aligned}$$

So (H3) and (H5) are satisfied. By Theorem 2.3, it is clear that every solution of this equation satisfies $\liminf_{t\to\infty}|x(t)| = 0$.

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