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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF NONLINEAR MIXED INTEGRODIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITION IN BANACH SPACES

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Abstract. In this article, we study the existence and uniqueness of mild and strong solutions of a nonlinear mixed Volterra-Fredholm integrodifferential equation with nonlocal condition in Banach spaces. Furthermore, we study continuous dependence of mild solutions. Our analysis is based on semigroup theory and Banach fixed point theorem.

1. Introduction

Let X be a Banach space with norm $\|\cdot\|$. Let $B_r = \{x \in X : \|x\| \le r\} \subset X$ be a closed ball in X and $E = C([t_0, t_0+\beta]; B_r)$ denote the complete metric space with metric

$$d(x, y) = \|x - y\|_E = \sup_{t\in[t_0,t_0+\beta]} \{\|x(t) - y(t)\| : x, y \in E\}.$$

Motivated by the work in [3, 7], we consider the nonlinear mixed Volterra-Fredholm integrodifferential equation

$$x'(t) + Ax(t) = f(t, x(t), \int_{t_0}^{t} k(t, s, x(s))ds, \int_{t_0}^{t_0+\beta} h(t, s, x(s))ds), \quad t \in [t_0, t_0 + \beta] \tag{1.1}$$

$$x(t_0) + g(t_1, t_2, \dots, t_p, x(\cdot)) = x_0, \tag{1.2}$$

where $0 \le t_0 < t_1 < t_2 < \cdots < t_p \le t_0 + \beta$, $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$, $t \ge 0$, in a Banach space X and the nonlinear functions $f : [t_0, t_0 + \beta] \times X \times X \times X \to X$, $g : [t_0, t_0 + \beta]^p \times X \to X$, $k, h : [t_0, t_0 + \beta] \times [t_0, t_0 + \beta] \times X \to X$ and x_0 is a given element of X.

The notion of "nonlocal condition" has been introduced to extend the study of the classical initial value problems and it is more precise for describing nature phenomena than the classical condition since more information is taken into account,

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thereby decreasing the negative effects incurred by a possibly erroneous single measurement taken at the initial value. The importance of nonlocal conditions in many applications is discussed in [1, 4, 5, 8, 9, 10]. For example, in [10], the author used

$$g(t_1, t_2, \dots, t_p, x(\cdot)) = \sum_{i=1}^{p} c_i x(t_i), \tag{1.3}$$

where $c_i, (i = 1, 2, \dots, p)$ are given constants and $t = 0 < t_1 < \dots < t_p \leq b$ to describe, for instance, the diffusion phenomenon of a small amount of gas in a transparent tube can give better result than using the usual local Cauchy problem with $x(0) = x_0$. In this case, (1.3) allows the additional measurements at t_i, $i = 1, 2, \dots p$. Subsequently, several authors are devoted to studying of nonlocal problems by using different techniques, see [2, 6, 11, 13, 14, 16, 17] and the references given therein.

The objective of the present paper is to study the existence, uniqueness and other properties of solutions of the problem (1.1)–(1.2). The main tool employed in our analysis is based on the Banach fixed point theorem and the theory of semigroups. Our results extend and improve the correspondence results in [12]. We indicate that the method used in this paper is different from that in [12].

This article is organized as follows. In section 2, we present the preliminaries and the statement of our main results. Section 3 deals with proof of the theorems. Finally in section 4 we give example to illustrate the application of our results.

2. Preliminaries and Main Results

Before proceeding to the statement of our main results, we shall setforth some preliminaries and hypotheses that will be used in our subsequent discussion.

Definition 2.1. A continuous solution x of the integral equation

$$\begin{aligned} x(t) &= T(t-t_0)x_0 - T(t-t_0)g(t_1, t_2, \dots, t_p, x(\cdot)) \\ &\quad + \int_{t_0}^{t} T(t-s) f\Big(s, x(s), \int_{t_0}^{s} k(s,\tau,x(\tau))d\tau, \int_{t_0}^{t_0+\beta} h(s,\tau,x(\tau))d\tau\Big) ds, \end{aligned} \tag{2.1}$$

with $t \in [t_0, t_0+\beta]$, is said to be a mild solution of (1.1)–(1.2) on $[t_0, t_0+\beta]$.

Definition 2.2. A function x is said to be a strong solution of (1.1)–(1.2) on $[t_0, t_0+\beta]$ if x is differentiable almost everywhere on $[t_0, t_0+\beta]$, $x' \in L^1([t_0, t_0+\beta], X)$ and satisfying (1.1)–(1.2) a.e. on $[t_0, t_0+\beta]$.

We list the following hypotheses for our convenience.

(H1) There exists a constant $G > 0$ such that
$$\|g(t_1, t_2, \dots, t_p, x_1(\cdot)) - g(t_1, t_2, \dots, t_p, x_2(\cdot))\| \leq G\|x_1 - x_2\|_E$$
for $x_1, x_2 \in E$.

(H2) $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$, $t \geq 0$ in X such that
$$\|T(t)\| \leq M,$$
for some $M \geq 1$.

(H3) There are constants L_1, K_1, H_1 and G_1 such that
$$L_1 = \max_{t_0 \leq t \leq t_0+\beta} \|f(t, 0, 0, 0)\|,$$

$$K_1 = \max_{t_0 \le s \le t \le t_0+\beta} \|k(t,s,0)\|,$$
$$H_1 = \max_{t_0 \le s,t \le t_0+\beta} \|h(t,s,0)\|,$$
$$G_1 = \max_{x \in E} \|g(t_1,t_2,\dots,t_p,x(\cdot))\|.$$

(H4) The constants $\|x_0\|, M, G_1, L, K, K_1, H, H_1, \beta$ and r satisfy the following two inequalities:
$$M[\|x_0\| + G_1 + Lr\beta + LKr\beta^2 + LK_1\beta^2 + LHr\beta^2 + LH_1\beta^2 + L_1\beta] \le r,$$
$$[MG + ML\beta + MLK\beta^2 + MLH\beta^2] < 1.$$

With these preparations we are now in a position to state our main results to be proved in the present paper.

Theorem 2.3. *Assume that*

(i) *hypotheses* (H1)–(H4) *hold,*

(ii) $f : [t_0, t_0+\beta] \times X \times X \times X \to X$ *is continuous in* t *on* $[t_0, t_0+\beta]$ *and there exists a constant* $L > 0$ *such that*
$$\|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)\| \le L(\|x_1-x_2\| + \|y_1-y_2\| + \|z_1-z_2\|),$$
for $x_i, y_i, z_i \in B_r$, $i = 1,2$.

(iii) $k, h : [t_0, t_0+\beta] \times [t_0, t_0+\beta] \times X \to X$ *are continuous in* s, t *on* $[t_0, t_0+\beta]$ *and there exist positive constants* K, H *such that*
$$\|k(t,s,x_1) - k(t,s,x_2)\| \le K(\|x_1-x_2\|),$$
$$\|h(t,s,x_1) - h(t,s,x_2)\| \le H(\|x_1-x_2\|),$$
for $x_i, y_i \in B_r$, $i = 1,2$.

Then problem (1.1)–(1.2) *has a unique mild solution on* $[t_0, t_0+\beta]$.

Theorem 2.4. *Assume that*

(i) *hypotheses* (H1)–(H4) *hold,*

(ii) X *is a reflexive Banach space with norm* $\|\cdot\|$ *and* $x_0 \in D(A)$,*the domain of* A,

(iii) $g(t_1,t_2,\dots,t_p,x(\cdot)) \in D(A)$,

(iv) *There exists a constant* $L > 0$ *such that*
$$\|f(t_1,x_1,y_1,z_1) - f(t_2,x_2,y_2,z_2)\| \le L(|t_1-t_2| + \|x_1-x_2\| + \|y_1-y_2\| + \|z_1-z_2\|),$$

(v) *There exist constants* $K, H > 0$ *such that*
$$\|k(t_1,s,x_1) - k(t_2,s,x_2)\| \le K(|t_1-t_2| + \|x_1-x_2\|),$$
$$\|h(t_1,s,x_1) - h(t_2,s,x_2)\| \le H(|t_1-t_2| + \|x_1-x_2\|),$$

Then x *is a unique strong solution of* (1.1)–(1.2) *on* $[t_0, t_0+\beta]$.

Theorem 2.5. *Suppose that the functions* f, g, k *and* h *satisfy hypotheses* (H1)-(H4) *and assumptions (ii), (iii) of Theorem 2.3. Then, for each pair of elements* $x_0^*, x_0^{**} \in X$, *and for the corresponding mild solutions* x_1, x_2 *of problem* (1.1) *with* $x_1(t_0) + g(t_1,t_2,\dots,t_p,x_1(\cdot)) = x_0^*$ *and* $x_2(t_0) + g(t_1,t_2,\dots,t_p,x_2(\cdot)) = x_0^{**}$, *the inequality*
$$\|x_1 - x_2\|_E \le \frac{M}{(1-MG)} \|x_0^* - x_0^{**}\| \exp\big(\frac{ML\beta}{(1-MG)}(1 + K\beta + H\beta)\big)$$

is true, whenever $G < 1/M$.

3. Proofs of theorems

Proof of Theorem 2.3. Define an operator $F : E \to E$ by

$$\begin{aligned}(Fz)(t) &= T(t-t_0)x_0 - T(t-t_0)g(t_1,t_2,\dots,t_p,z(\cdot))\\ &\quad + \int_{t_0}^{t} T(t-s)f(s,z(s),\int_{t_0}^{s} k(s,\tau,z(\tau))d\tau, \int_{t_0}^{t_0+\beta} h(s,\tau,z(\tau))d\tau)ds,\end{aligned}$$

for $t \in [t_0, t_0+\beta]$. Now, we show that F maps E into itself. For $z \in E, t \in [t_0, t_0+\beta]$ and using hypotheses (H2)-(H4) and assumptions (ii), (iii), we have

$$\begin{aligned}&\|(Fz)(t)\|\\ &\le \|T(t-t_0)x_0\| + \|T(t-t_0)g(t_1,t_2,\dots,t_p,z(\cdot))\|\\ &\quad + \|\int_{t_0}^{t} T(t-s)f(s,z(s),\int_{t_0}^{s} k(s,\tau,z(\tau))d\tau, \int_{t_0}^{t_0+\beta} h(s,\tau,z(\tau))d\tau)ds\|\\ &\le M\|x_0\| + MG_1 + M\int_{t_0}^{t}[\|f(s,z(s),\int_{t_0}^{s} k(s,\tau,z(\tau))d\tau,\\ &\quad \int_{t_0}^{t_0+\beta} h(s,\tau,z(\tau))d\tau) - f(s,0,0,0)\| + \|f(s,0,0,0)\|]ds\\ &\le M\|x_0\| + MG_1 + M\int_{t_0}^{t}[L(\|z(s)-0\| + \|\int_{t_0}^{s} k(s,\tau,z(\tau))d\tau - 0\|\\ &\quad + \|\int_{t_0}^{t_0+\beta} h(s,\tau,z(\tau))d\tau - 0\|) + \|f(s,0,0,0)\|]ds\\ &\le M\|x_0\| + MG_1 + M\int_{t_0}^{t}[Lr + L\int_{t_0}^{s}\|k(s,\tau,z(\tau)) - k(s,\tau,0) + k(s,\tau,0)\|d\tau\\ &\quad + L\int_{t_0}^{t_0+\beta}\|h(s,\tau,z(\tau)) - h(s,\tau,0) + h(s,\tau,0)\|d\tau + L_1]ds\\ &\le M\|x_0\| + MG_1 + M\int_{t_0}^{t}[Lr + L\beta(Kr+K_1) + L\beta(Hr+H_1) + L_1]ds\\ &\le M[\|x_0\| + G_1 + Lr\beta + LKr\beta^2 + LK_1\beta^2 + LHr\beta^2 + LH_1\beta^2 + L_1\beta] \le r.\end{aligned}$$

Thus, F maps E into itself.

Now, for every $z_1, z_2 \in E$, $t \in [t_0, t_0+\beta]$ and using hypotheses (H1), (H2), (H4) and assumptions (ii), (iii), we obtain

$$\begin{aligned}&\|(Fz_1)(t) - (Fz_2)(t)\|\\ &\le \|T(t-t_0)\|\|g(t_1,t_2,\dots,t_p,z_1(\cdot)) - g(t_1,t_2,\dots,t_p,z_2(\cdot))\|\\ &\quad + \int_{t_0}^{t}\|T(t-s)\|\|[f(s,z_1(s),\int_{t_0}^{s} k(s,\tau,z_1(\tau))d\tau, \int_{t_0}^{t_0+\beta} h(s,\tau,z_1(\tau))d\tau)\\ &\quad - f(s,z_2(s),\int_{t_0}^{s} k(s,\tau,z_2(\tau))d\tau, \int_{t_0}^{t_0+\beta} h(s,\tau,z_2(\tau))d\tau)]\|ds\\ &\le MG\|z_1 - z_2\|_E + \int_{t_0}^{t} ML[\|z_1(s) - z_2(s)\|\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_0}^{s}\|k(s,\tau,z_1(\tau))-k(s,\tau,z_2(\tau))\|d\tau\\
&+\int_{t_0}^{t_0+\beta}\|h(s,\tau,z_1(\tau))-h(s,\tau,z_2(\tau))\|d\tau]ds\\
&\le MG\|z_1-z_2\|_E+ML\|z_1-z_2\|_E\int_{t_0}^{t}[1+K\int_{t_0}^{s}d\tau+H\int_{t_0}^{t_0+\beta}d\tau]ds\\
&\le MG\|z_1-z_2\|_E+ML\|z_1-z_2\|_E\beta[1+K\beta+H\beta]\\
&\le q\|z_1-z_2\|_E,
\end{aligned}
$$

where $q=MG+ML\beta+MLK\beta^2+MLH\beta^2$ and hence, we obtain

$$\|Fz_1-Fz_2\|_E\le q\|z_1-z_2\|_E,$$

with $0<q<1$. This shows that the the operator F is a contraction on the complete metric space E. By the Banach fixed point theorem, the function F has a unique fixed point in the space E and this point is the mild solution of problem (1.1)–(1.2) on $[t_0,t_0+\beta]$. This completes the proof of the Theorem 2.3. □

Proof of Theorem 2.4. All the assumptions of Theorem 2.3 are being satisfied, then problem (1.1)–(1.2) has a unique mild solution belonging to E. Now we will show that x is unique strong solution of (1.1)–(1.2) on $[t_0,t_0+\beta]$. Take

$$
\begin{aligned}
L_2&=\max_{t_0\le t\le t_0+\beta}\|f(t,x(t),0,0)\|,\\
K_2&=\max_{t_0\le s\le t\le t_0+\beta}\|k(t,s,x(s))\|,\\
H_2&=\max_{t_0\le s,t\le t_0+\beta}\|h(t,s,x(s))\|.
\end{aligned}
$$

For $0<\theta<t-t_0$ and $t\in[t_0,t_0+\beta]$, we have

$$
\begin{aligned}
&x(t+\theta)-x(t)\\
&=[T(t+\theta-t_0)-T(t-t_0)]x_0\\
&\quad-[T(t+\theta-t_0)-T(t-t_0)]g(t_1,t_2,\dots,t_p,x(\cdot))\\
&\quad+\int_{t_0}^{t_0+\theta}T(t+\theta-s)f(s,x(s),\int_{t_0}^{s}k(s,\tau,x(\tau))d\tau,\int_{t_0}^{t_0+\beta}h(s,\tau,x(\tau))d\tau)ds\\
&\quad+\int_{t_0+\theta}^{t+\theta}T(t+\theta-s)f(s,x(s),\int_{t_0}^{s}k(s,\tau,x(\tau))d\tau,\int_{t_0}^{t_0+\beta}h(s,\tau,x(\tau))d\tau)ds\\
&\quad-\int_{t_0}^{t}T(t-s)f(s,x(s),\int_{t_0}^{s}k(s,\tau,x(\tau))d\tau,\int_{t_0}^{t_0+\beta}h(s,\tau,x(\tau))d\tau)ds\\
&=T(t-t_0)[T(\theta)-I]x_0-T(t-t_0)[T(\theta)-I]g(t_1,t_2,\dots,t_p,x(\cdot))\\
&\quad+\int_{t_0}^{t_0+\theta}T(t+\theta-s)[f(s,x(s),\int_{t_0}^{s}k(s,\tau,x(\tau))d\tau,\int_{t_0}^{t_0+\beta}h(s,\tau,x(\tau))d\tau)\\
&\quad-f(s,x(s),0,0)+f(s,x(s),0,0)]ds\\
&\quad+\int_{t_0}^{t}T(t-s)[f(s+\theta,x(s+\theta),\int_{t_0}^{s+\theta}k(s+\theta,\tau,x(\tau))d\tau,\\
&\quad\int_{t_0}^{t_0+\beta}h(s+\theta,\tau,x(\tau))d\tau)-f(s,x(s),
\end{aligned}
$$

$$\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau, \int_{t_0}^{t_0+\beta} h(s,\tau,x(\tau))d\tau)]ds.$$

Using the assumptions and the fact $\|[T(\theta)-I]x\| = \theta\|Ax\| + o(\theta)$, we obtain

$$\begin{aligned}
&\|x(t+\theta)-x(t)\| \\
&\le M[\theta\epsilon_1 + \theta\|Ax_0\|] + M[\theta\epsilon_2 + \theta\|Ag(t_1,t_2,\dots,t_p,x(\cdot))\|] \\
&\quad + \int_{t_0}^{t_0+\theta} M[\|f(s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau, \\
&\quad \int_{t_0}^{t_0+\beta} h(s,\tau,x(\tau))d\tau) - f(s,x(s),0,0)\| + \|f(s,x(s),0,0)\|]ds \\
&\quad + \int_{t_0}^{t} M[\|f(s+\theta,x(s+\theta),\int_{t_0}^{s+\theta} k(s+\theta,\tau,x(\tau))d\tau, \int_{t_0}^{t_0+\beta} h(s+\theta,\tau,x(\tau))d\tau) \\
&\quad - f(s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau, \int_{t_0}^{t_0+\beta} h(s,\tau,x(\tau))d\tau)\|]ds \\
&\le M[\theta\epsilon_1 + \theta\|Ax_0\|] + M[\theta\epsilon_2 + \theta\|Ag(t_1,t_2,\dots,t_p,x(\cdot))\|] \\
&\quad + \int_{t_0}^{t_0+\theta} ML[\int_{t_0}^{s} K_2 d\tau + \int_{t_0}^{t_0+\beta} H_2 d\tau]ds \\
&\quad + M\int_{t_0}^{t_0+\theta} L_2 ds + \int_{t_0}^{t} ML[\theta + \|x(s+\theta)-x(s)\| \\
&\quad + \int_{t_0}^{s} K(s+\theta-s| + \|x(\tau)-x(\tau)\|)d\tau \\
&\quad + \int_{s}^{s+\theta} K_2 d\tau + \int_{t_0}^{t_0+\beta} H(|s+\theta-s| + \|x(\tau)-x(\tau)\|)d\tau]ds \\
&\le M[\theta\epsilon_1 + \theta\|Ax_0\|] + M[\theta\epsilon_2 + \theta\|Ag(t_1,t_2,\dots,t_p,x(\cdot))\|] + MLK_2\theta\beta + MLH_2\theta\beta \\
&\quad + ML_2\theta + ML\theta\beta + ML\int_{t_0}^{t} \|x(s+\theta)-x(s)\|ds \\
&\quad + MLK\theta\beta^2 + MLK_2\theta\beta + MLH\theta\beta^2 \\
&\le P\theta + ML\int_{t_0}^{t} \|x(s+\theta)-x(s)\|ds,
\end{aligned}$$

where $\epsilon_1, \epsilon_2 > 0$ and

$$\begin{aligned}
P = M[&\epsilon_1 + \|Ax_0\| + \epsilon_2 + \|Ag(t_1,t_2,\dots,t_p,x(\cdot))\| + LK_2\beta + LH_2\beta \\
&+ L_2 + L\beta + LK\beta^2 + LH\beta^2 + LK_2\beta,
\end{aligned}$$

which is independent of θ and $t \in [t_0, t_0+\beta]$.

Thanks to Gronwall's inequality, we obtain

$$\|x(t+\theta)-x(t)\| \le P\theta e^{ML\beta}, \quad \text{for } t \in [t_0, t_0+\beta].$$

Therefore, x is Lipschitz continuous on $[t_0, t_0+\beta]$. The Lipschitz continuity of x on $[t_0, t_0+\beta]$ combined with (iv) and (v) of Theorem 2.4 implies

$$t \to f(t,x(t),\int_{t_0}^{t} k(t,s,x(s))ds, \int_{t_0}^{t_0+\beta} h(t,s,x(s))ds)$$

is Lipschitz continuous on $[t_0, t_0+\beta]$. By [15, Corollary 4.2.11], we observe that the equation

$$y'(t) + Ay(t) = f(t, x(t), \int_{t_0}^{t} k(t,s,x(s))ds, \int_{t_0}^{t_0+\beta} h(t,s,x(s))ds), \quad t \in [t_0, t_0+\beta]$$
$$y(t_0) = x_0 - g(t_1, t_2, \dots, t_p, x(\cdot))$$

has a unique strong solution $y(t)$ on $[t_0, t_0+\beta]$ satisfying the equation

$$\begin{aligned} y(t) &= T(t-t_0)x_0 - T(t-t_0)g(t_1, t_2, \dots, t_p, x(\cdot)) \\ &\quad + \int_{t_0}^{t} T(t-s)f(s, x(s), \int_{t_0}^{s} k(s,\tau,x(\tau))d\tau, \int_{t_0}^{t_0+\beta} h(s,\tau,x(\tau))d\tau)ds \\ &= x(t), \quad t \in [t_0, t_0+\beta]. \end{aligned}$$

Consequently, $x(t)$ is the strong solution of initial value problem (1.1)–(1.2) on $[t_0, t_0+\beta]$. This completes the proof of Theorem 2.4. □

Proof of Theorem 2.5. Suppose that $x_1(t)$ and $x_2(t)$ satisfy (1.1) on $[t_0, t_0+\beta]$ with $x_1(t_0) + g(t_1, t_2, \dots, t_p, x_1(\cdot)) = x_0^*$ and $x_2(t_0) + g(t_1, t_2, \dots, t_p, x_2(\cdot)) = x_0^{**}$, respectively and $x_1, x_2 \in E$. Using the equation (2.1), hypotheses (H1)–(H4) and assumptions (ii), (iii), we obtain

$$\begin{aligned} &\|x_1(t) - x_2(t)\| \\ &\le M\|x_0^* - x_0^{**}\| + MG\|x_1 - x_2\|_E + \int_{t_0}^{t} ML\Big[\|x_1(s) - x_2(s)\| \\ &\quad + \int_{t_0}^{s} K\|x_1(\tau) - x_2(\tau)\|d\tau + \int_{t_0}^{t_0+\beta} H\|x_1(\tau) - x_2(\tau)\|d\tau\Big]ds \\ &\le M\|x_0^* - x_0^{**}\| + MG\|x_1 - x_2\|_E \\ &\quad + \int_{t_0}^{t} ML\Big[\|x_1(s) - x_2(s)\| + \int_{t_0}^{s} K \sup_{\tau\in[t_0,s]} \|x_1(\tau) - x_2(\tau)\|d\tau \\ &\quad + \int_{t_0}^{t_0+\beta} H \sup_{\tau\in[t_0,t_0+\beta]} \|x_1(\tau) - x_2(\tau)\|d\tau\Big]ds \\ &\le M\|x_0^* - x_0^{**}\| + MG\|x_1 - x_2\|_E + \int_{t_0}^{t} ML\big[1 + \beta K + \beta H\big]\|x_1 - x_2\|_E ds. \end{aligned}$$

Therefore, we obtain

$$\|x_1 - x_2\|_E \le \frac{M}{(1-MG)}\|x_0^* - x_0^{**}\| + \int_{t_0}^{t} \frac{ML\beta}{(1-MG)}(1 + K\beta + H\beta)\|x_1 - x_2\|_E ds.$$

Using Gronwall's inequality, we obtain

$$\|x_1 - x_2\|_E \le \frac{M}{(1-MG)}\|x_0^* - x_0^{**}\| \exp\big(\frac{ML\beta}{(1-MG)}(1 + K\beta + H\beta)\big),$$

provided that $G < \frac{1}{M}$. From this inequality, it follows that the continuous dependence of solutions depends upon the initial data. This completes the proof of the Theorem 2.5. □

4. Application

To illustrate the applications of some of our main results, we consider the nonlinear mixed Volterra- Fredholm partial integrodifferential equation

$$w_t(u,t) - w_{uu}(u,t) = P(t, w(u,t), \int_0^t k_1(t,s,w(u,s))ds, \int_0^\beta h_1(t,s,w(u,s))ds), \\ 0 < u < 1, \quad 0 \le t \le \beta \tag{4.1}$$

with initial and boundary conditions

$$w(0,t) = w(1,t) = 0, \quad 0 \le t \le \beta, \tag{4.2}$$

$$w(u,0) + \sum_{i=1}^{p} w(u,t_i) = w_0(u), \quad 0 < t_1 < t_2 < \cdots < t_p \le \beta. \tag{4.3}$$

where $P : [0,\beta] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $k_1, h_1 : [0,\beta] \times [0,\beta] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. We assume that the functions P, k_1 and h_1 in (4.1)–(4.3) satisfy the following conditions:

(1) There exists a constant $G^* > 0$ such that
$$|\sum_{i=1}^{p} w(u,t_i) - \sum_{i=1}^{p} w(v,t_i)| \le G^* \sup_{t\in[0,\beta]} |u(t) - v(t)|$$
for $u, v \in E_1 = C([0,\beta]; B^*_{r^*})$, where $B^*_{r^*} = \{x \in R : |x| \le r^*\}$.

(2) There are constants L_1^*, K_1^*, H_1^* and G_1^* such that
$$L_1^* = \max_{0\le t\le\beta} |P(t,0,0,0)|,$$
$$K_1^* = \max_{t_0\le s\le t\le t_0+\beta} |k_1(t,s,0)|,$$
$$H_1^* = \max_{t_0\le s,t\le t_0+\beta} |h_1(t,s,0)|,$$
$$G_1^* = \max_{x\in E_1} |\sum_{i=1}^{p} w(u,t_i)|, \quad 0 < u < 1.$$

(3) $P : [0,\beta] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous in t on $[0,\beta]$ and there exists a constant $L^* > 0$ such that
$$|P(t,x_1,y_1,z_1) - P(t,x_2,y_2,z_2)| \le L^*(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|),$$
for $x_i, y_i, z_i \in B^*_{r^*}$, $i = 1, 2$.

(4) $k, h : [0,\beta] \times [0,\beta] \times \mathbb{R} \to \mathbb{R}$ are continuous in s, t on $[0,\beta]$ and there exist respectively constants $K^*, H^* > 0$ such that
$$|k_1(t,s,x_1) - k_1(t,s,x_2)| \le K^*(|x_1 - x_2|),$$
$$|h_1(t,s,x_1) - h_1(t,s,x_2)| \le H^*(|x_1 - x_2|),$$
for $x_i, y_i \in B^*_{r^*}$, $i = 1, 2$.

(5) $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$, $t \ge 0$ in X such that
$$\|T(t)\| \le M^*,$$
for some $M^* \ge 1$.

(6) The constants $|w_0(u)|, M^*, G_1^*, L^*, K^*, K_1^*, H^*, H_1^*, \beta$ and r satisfy the following two inequalities:

$$\begin{aligned} &M^*[|w_0(u)| + G_1^* + L^* r\beta + L^* K^* r\beta^2 + L^* K_1^* \beta^2 \\ &+ L^* H^* r\beta^2 + L^* H_1^* \beta^2 + L_1^* \beta] \leq r^*, \end{aligned}$$

and

$$[M^* G^* + M^* L^* \beta + M^* L^* K^* \beta^2 + M^* L^* H^* \beta^2] < 1.$$

First, we reduce the equations (4.1)–(4.3) into (1.1)–(1.2) by making suitable choices of A, f, g, k and h. Let $X = L^2[0,1]$. Define the operator $A : X \to X$ by $Az = -z''$ with domain $D(A) = \{z \in X : z, z'$ are absolutely continuous, $z'' \in X$ and $z(0) = z(1) = 0\}$. Define the functions $f : [0,\beta] \times X \times X \times X \to X$, $k : [0,\beta] \times [0,\beta] \times X \to X$, $h : [0,\beta] \times [0,\beta] \times X \to X$ and $g : [0,\beta]^p \times X \to X$ as follows

$$\begin{gathered} f(t,x,y,z)(u) = P(t, x(u), y(u), z(u)), \\ k(t,s,x)(u) = k_1(t,s,x(u)), \\ h(t,s,x)(u) = h_1(t,s,x(u)), \\ g(t_1, t_2, \dots, t_p, x(\cdot)u = \sum_{i=1}^{p} w(u, t_i) \end{gathered}$$

for $t \in [0,\beta], x, y, z \in X$ and $0 < u < 1$. Then the above problem (4.1)–(4.3) can be formulated abstractly as nonlinear mixed Volterra-Fredholm integrodifferential equation in Banach space X:

$$x'(t) + Ax(t) = f(t, x(t), \int_{t_0}^{t} k(t,s,x(s))ds, \int_{t_0}^{t_0+\beta} h(t,s,x(s))ds), \quad t \in [t_0, t_0+\beta] \tag{4.4}$$

$$x(t_0) + g(t_1, t_2, \dots, t_p, x(\cdot) = x_0. \tag{4.5}$$

Since all the hypotheses of the Theorem 2.3 are satisfied, the Theorem 2.3 can be applied to guarantee the mild solution of the nonlinear mixed Volterra-Fredholm partial integrodifferential equations (4.1)–(4.3).

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