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CONSTRUCTION OF ALMOST PERIODIC FUNCTIONS WITH GIVEN PROPERTIES

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Abstract. We define almost periodic functions with values in a pseudometric space $\mathcal{X}$. We mention the Bohr and the Bochner definition of almost periodicity. We present one modifiable method for constructing almost periodic functions in $\mathcal{X}$. Applying this method, we prove that in any neighbourhood of an almost periodic skew-Hermitian linear differential system there exists a system which does not possess a nontrivial almost periodic solution.

1. Introduction

This paper is motivated by [35] where almost periodic *sequences* and linear *difference* systems are considered. Here we will consider almost periodic *functions* and linear *differential* systems. Our aim is to show a way one can generate almost periodic functions with several prescribed properties. Since our process can be used for generalizations of classical (complex valued) almost periodic functions, we introduce the almost periodicity in pseudometric spaces and we present our method for almost periodic functions with values in a pseudometric space $\mathcal{X}$.

Note that we obtain the most important case if $\mathcal{X}$ is a Banach space, and that the theory of almost periodic functions of real variable with values in a Banach space, given by Bochner [3], is in its essential lines similar to the theory of classical almost periodic functions which is due to Bohr [4], [5]. We introduce almost periodic functions in pseudometric spaces using a trivial extension of the Bohr concept, where the modulus is replaced by the distance. In the classical case, we refer to the monographs [8, 12]; for functions with values in Banach spaces, to [1], [8, Chapter VI]; for other extensions, to [13, 18].

Necessary and sufficient conditions for a continuous function with values in a Banach space to be almost periodic may be no longer valid for continuous functions in general metric spaces. For the approximation condition, it is seen that the completeness of the space of values is necessary and Tornehave [33] also requires the local connection by arcs of the space of values. In the Bochner condition, it suffices to replace the convergence by the Cauchy condition. Since we need the Bochner concept as well, we recall that the Bochner condition means that any sequence

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of translates of a given continuous function has a subsequence which converges, uniformly on the domain of the function. The fact, that this condition is equivalent with the Bohr definition of almost periodicity in Banach spaces, was proved by Bochner [3].

The above mentioned Bohr definition and Bochner condition are formulated in Section 2 (with some basic properties of almost periodic functions). In this section, processes from [8] are generalized. Analogously, the theory of almost periodic functions of real variable with fuzzy real numbers as values is developed in [2]. We remark that fuzzy real numbers form a complete metric space.

In Section 3, we mention the way one can construct almost periodic functions with prescribed properties in a pseudometric space. We present it in Theorems 3.1, 3.2, 3.3. Note that it is possible to obtain many modifications and generalizations of our process. A special construction of almost periodic functions with given properties is published (and applied) in [19].

Then we will analyse almost periodic solutions of almost periodic linear differential systems. Sometimes this field is called the Favard theory what is based on the Favard contributions in [11] (see also [6, Theorem 1.2], [12, Theorem 6.3] or [25, Theorem 1]; for homogeneous case, see [7], [10]). In this context, sufficient conditions for the existence of almost periodic solutions are mentioned in [16] (for generalizations, see [15], [17]; for other extensions and supplements of the Favard theorem, see [3], [6], [12] and the references cited therein). Certain sufficient conditions, under which *homogeneous* systems that have nontrivial bounded solutions also have nontrivial almost periodic solutions, are given in [26].

It is a corollary of the Favard and the Floquet theory that any bounded solution of an almost periodic linear differential system is almost periodic if the matrix valued function, which determines the system, is periodic (see [12, Corollary 6.5]; for a generalization in the homogeneous case, see [14]). This result is no longer valid for systems with almost periodic coefficients. There exist systems for which all solutions are bounded, but none of them is almost periodic (see, e.g., [20], [27]). Homogeneous systems have the zero solution which is almost periodic, but do not need to have an other almost periodic solution. We note that the existence of a homogeneous system, which has bounded solutions (separated from zero) and, at the same time, all systems from some neighbourhood of it do not have any nontrivial almost periodic solution, is proved in [29].

We will consider the set of systems of the form

$$x' = A(t)x, \tag{1.1}$$

where A is almost periodic and all matrices $A(t)$, $t \in \mathbb{R}$, are skew-Hermitian, with the uniform topology of matrix functions A on the real axis. In [28], it is proved that the systems (1.1), all of whose solutions are almost periodic, form a dense subset of the set of all considered systems. We add that special cases of this result are proved in [21, 22]. For systems whose solutions are not almost periodic, we refer to [30].

In Section 4, using the method for constructing almost periodic functions from Section 3, we will prove that, in any neighbourhood of a system of the form (1.1), there exists a system which does not possess an almost periodic solution other than the trivial one, not only with a fundamental matrix which is not almost periodic as in [30]. It means that, applying our method, we will get a stronger version of a statement from [30]. We remark that constructions of almost periodic homogeneous linear differential system with given properties are used in [23, 24] as well.

Let $\mathcal{X}$ be an arbitrary pseudometric space with a pseudometric ϱ; i.e., let

$$\varrho(x,x) = 0, \quad \varrho(x,y) = \varrho(y,x) \geq 0, \quad \varrho(x,y) \leq \varrho(x,z) + \varrho(z,y)$$

for any $x, y, z \in \mathcal{X}$. The symbol $\mathcal{O}_\varepsilon(x)$ will denote the ε-neighbourhood of x in $\mathcal{X}$ for arbitrary $\varepsilon > 0$, $x \in \mathcal{X}$ and $\mathbb{R}_0^+$ the set of all nonnegative real numbers. For the used notations, we can also refer to [35].

2. Almost periodic functions in pseudometric spaces

We introduce the Bohr almost periodicity in $\mathcal{X}$. Observe that we are not able to distinguish between $x \in \mathcal{X}$ and $y \in \mathcal{X}$ if $\varrho(x,y) = 0$.

Definition 2.1. A continuous function $\psi : \mathbb{R} \to \mathcal{X}$ is *almost periodic* if for any $\varepsilon > 0$, there exists a number $p(\varepsilon) > 0$ with the property that any interval of length $p(\varepsilon)$ of the real line contains at least one point s, such that

$$\varrho(\psi(t+s), \psi(t)) < \varepsilon, \quad t \in \mathbb{R}.$$

The number s is called an *ε-translation number* and the set of all ε-translation numbers of ψ is denoted by $\mathfrak{T}(\psi, \varepsilon)$.

If $\mathcal{X}$ is a Banach space, then a continuous function ψ is almost periodic if and only if any set of translates of ψ has a subsequence, uniformly convergent on $\mathbb{R}$ in the sense of the norm. See, e.g., [8, Theorem 6.6]. Evidently, this result cannot be longer valid if the space of values is not complete. Nevertheless, we prove the below given Theorem 2.4, where the convergence is replaced by the Cauchy condition. Before proving this statement, we mention two simple lemmas. Their proofs can be easily obtained by modifying the proofs of [8, Theorem 6.2] and [8, Theorem 6.5], respectively.

Lemma 2.2. *An almost periodic function with values in $\mathcal{X}$ is uniformly continuous on the real line.*

Lemma 2.3. *The set of all values of an almost periodic function $\psi : \mathbb{R} \to \mathcal{X}$ is totally bounded in $\mathcal{X}$.*

Now we can formulate the main result of this section, which we will apply in Section 4.

Theorem 2.4. *Let $\psi : \mathbb{R} \to \mathcal{X}$ be a continuous function. Then, ψ is almost periodic if and only if from any sequence of the form $\{\psi(t+s_n)\}_{n\in\mathbb{N}}$, where s_n are real numbers, one can extract a subsequence $\{\psi(t+r_n)\}_{n\in\mathbb{N}}$ satisfying the Cauchy uniform convergence condition on $\mathbb{R}$; i.e., for any $\varepsilon > 0$, there exists $l(\varepsilon) \in \mathbb{N}$ with the property that*

$$\varrho(\psi(t+r_i), \psi(t+r_j)) < \varepsilon, \quad t \in \mathbb{R}$$

for all $i, j > l(\varepsilon)$, $i, j \in \mathbb{N}$.

Proof. The sufficiency of the condition can be proved using a simple extension of the argument used in the proof of [8, Theorem 1.10]. In that proof, it is only supposed, by contradiction, that any sequence of translates of ψ has a subsequence which satisfies the Cauchy uniform convergence condition, and that ψ is not almost periodic. Thus, it suffices to replace the modulus by the distance in the proof of [8, Theorem 1.10].

To prove the converse implication, we will assume that ψ is an almost periodic function, and we will apply the well-known method of diagonal extraction and modify the proof of [8, Theorem 6.6].

Let $\{t_n; n \in \mathbb{N}\}$ be a dense subset of $\mathbb{R}$ and $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be an arbitrarily given sequence. From the sequence $\{\psi(t_1 + s_n)\}_{n\in\mathbb{N}}$, using Lemma 2.3, we choose a subsequence $\{\psi(t_1 + r_n^1)\}_{n\in\mathbb{N}}$ such that, for any $\varepsilon > 0$, there exists $l_1(\varepsilon) \in \mathbb{N}$ with the property that

$$\varrho(\psi(t_1 + r_i^1), \psi(t_1 + r_j^1)) < \varepsilon, \quad i, j > l_1(\varepsilon),\, i, j \in \mathbb{N}.$$

Such a subsequence exists because infinitely many values $\psi(t_1 + s_n)$ is in a neighbourhood of radius 2^{-i} for all $i \in \mathbb{N}$ (consider the method of diagonal extraction). Analogously, from the sequence $\{\psi(t_2 + r_n^1)\}_{n\in\mathbb{N}}$, we obtain $\{\psi(t_2 + r_n^2)\}_{n\in\mathbb{N}}$ such that, for any $\varepsilon > 0$, there exists $l_2(\varepsilon) \in \mathbb{N}$ for which

$$\varrho(\psi(t_2 + r_i^2), \psi(t_2 + r_j^2)) < \varepsilon, \quad i, j > l_2(\varepsilon),\, i, j \in \mathbb{N}.$$

We proceed further in the same way. We obtain $\{r_n^k\} \subseteq \cdots \subseteq \{r_n^1\}$, $k \in \mathbb{N}$.

Let $\varepsilon > 0$ be arbitrarily given, $p = p(\varepsilon/5)$ be from Definition 2.1, $\delta = \delta(\varepsilon/5)$ correspond to $\varepsilon/5$ from the definition of the uniform continuity of ψ (see Lemma 2.2) and let a finite set $\{t_1, \dots, t_j\} \subset \{t_n; n \in \mathbb{N}\}$ satisfy

$$\min_{i\in\{1,\dots,j\}} |t_i - t| < \delta, \quad t \in [0, p].$$

Obviously, there exists $l \in \mathbb{N}$ such that, for all integers $n_1, n_2 > l$, it is valid

$$\varrho(\psi(t_i + r_{n_1}^{n_1}), \psi(t_i + r_{n_2}^{n_2})) < \frac{\varepsilon}{5}, \quad i \in \{1, \dots, j\}.$$

Let $t \in \mathbb{R}$ be given, $s = s(t) \in [-t, -t + p]$ be an $\varepsilon/5$-translation number of ψ, and $t_i = t_i(s) \in \{t_1, \dots, t_j\}$ be such that $|t + s - t_i| < \delta$. Finally, we have

$$\begin{aligned}
&\varrho(\psi(t + r_{n_1}^{n_1}), \psi(t + r_{n_2}^{n_2})) \\
&\le \varrho(\psi(t + r_{n_1}^{n_1}), \psi(t + r_{n_1}^{n_1} + s)) + \varrho(\psi(t + r_{n_1}^{n_1} + s), \psi(t_i + r_{n_1}^{n_1})) \\
&\quad + \varrho(\psi(t_i + r_{n_1}^{n_1}), \psi(t_i + r_{n_2}^{n_2})) + \varrho(\psi(t_i + r_{n_2}^{n_2}), \psi(t + r_{n_2}^{n_2} + s)) \\
&\quad + \varrho(\psi(t + r_{n_2}^{n_2} + s), \psi(t + r_{n_2}^{n_2})).
\end{aligned}$$

Thus, we obtain

$$\varrho(\psi(t + r_{n_1}^{n_1}), \psi(t + r_{n_2}^{n_2})) < \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon \tag{2.1}$$

for all $t \in \mathbb{R}$, $n_1, n_2 > l$, $n_1, n_2 \in \mathbb{N}$. Evidently, (2.1) completes the proof of the theorem if we put $r_n := r_n^n$, $n \in \mathbb{N}$. □

Analogously as for almost periodic functions with values in a Banach space, one can prove many properties of almost periodic functions in a pseudometric space. For example, the limit of a uniformly convergent sequence of almost periodic functions is almost periodic (see [8, Theorem 6.4]).

Using Theorem 2.4 n-times, one can also obtain:

Corollary 2.5. *If $\mathcal{X}_1, \dots, X_n$ are pseudometric spaces and $\psi_1, \dots, \psi_n$ are arbitrary almost periodic functions with values in $\mathcal{X}_1, \dots, \mathcal{X}_n$, respectively, then the function ψ, with values in $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ given by $\psi := (\varphi_1, \dots, \varphi_n)$, is almost periodic as well.*

Moreover, from Corollary 2.5 it follows that the set

$$\mathfrak{T}(\psi_1, \varepsilon) \cap \mathfrak{T}(\psi_2, \varepsilon) \cap \dots \cap \mathfrak{T}(\psi_n, \varepsilon)$$

is relative dense in $\mathbb{R}$ for arbitrary almost periodic functions $\psi_1, \psi_2, \dots, \psi_n$ and any $\varepsilon > 0$. We add that one can use Corollary 2.5 to obtain simple modifications of the below presented method of constructions of almost periodic functions.

3. Construction of almost periodic functions

Now we present the way one can generate almost periodic functions with given properties in the next theorem.

Theorem 3.1. *For arbitrary $a > 0$, any continuous function $\psi : \mathbb{R} \to \mathcal{X}$ such that*

$$\begin{gathered}
\psi(t) \in \mathcal{O}_a(\psi(t-1)), \quad t \in (1, 2], \\
\psi(t) \in \mathcal{O}_a(\psi(t+2)), \quad t \in (-2, 0], \\
\psi(t) \in \mathcal{O}_{a/2}(\psi(t-4)), \quad t \in (2, 6], \\
\psi(t) \in \mathcal{O}_{a/2}(\psi(t+8)), \quad t \in (-10, -2], \\
\psi(t) \in \mathcal{O}_{a/4}(\psi(t-2^4)), \quad t \in (2+2^2, 2+2^2+2^4], \\
\psi(t) \in \mathcal{O}_{a/4}(\psi(t+2^5)), \quad t \in (-2^5-2^3-2, -2^3-2], \\
\dots \\
\psi(t) \in \mathcal{O}_{a\,2^{-n}}(\psi(t-2^{2n})), \quad t \in (2+2^2+\dots+2^{2n-2}, 2+2^2+\dots+2^{2n-2}+2^{2n}], \\
\psi(t) \in \mathcal{O}_{a\,2^{-n}}(\psi(t+2^{2n+1})), \quad t \in (-2^{2n+1}-\dots-2^3-2, -2^{2n-1}-\dots-2^3-2], \\
\dots
\end{gathered}$$

is almost periodic.

Proof. Let $\varepsilon > 0$ be arbitrary and $k = k(\varepsilon) \in \mathbb{N}$ be such that $2^k > 8a/\varepsilon$. We have to prove that the set of all ε-translation numbers of ψ is relative dense in $\mathbb{R}$. We will obtain this from the fact that $l \cdot 2^{2k}$ is an ε-translation number of ψ for any integer l. We see that

$$\begin{gathered}
\psi(t) \in \mathcal{O}_{\varepsilon/8}(\psi(t-2^{2k})), \quad t \in (2+2^2+\dots+2^{2k-2}, 2+2^2+\dots+2^{2k}], \\
\psi(t) \in \mathcal{O}_{\varepsilon/8}(\psi(t+2^{2k+1})),\ t \in (-2^{2k+1}-\dots-2^3-2, -2^{2k-1}-\dots-2^3-2], \\
\psi(t) \in \mathcal{O}_{\varepsilon/16}(\psi(t-2^{2k+2})), \quad t \in (2+2^2+\dots+2^{2k}, 2+2^2+\dots+2^{2k+2}], \\
\dots
\end{gathered}$$

In a pseudometric space $\mathcal{X}$, this implies

$$\begin{gathered}
\psi(t+2^{2k}) \in \mathcal{O}_{\varepsilon/8}(\psi(t)), \quad t \in [-2^{2k-1}-\dots-2^3-2, 2+2^2+\dots+2^{2k-2}], \\
\psi(t-2^{2k+1}) \in \mathcal{O}_{\varepsilon/8}(\psi(t)), \quad t \in [-2^{2k-1}-\dots-2^3-2, 2+2^2+\dots+2^{2k-2}], \\
\psi(t-2^{2k}) \in \mathcal{O}_{\varepsilon/8+\varepsilon/8}(\psi(t)), \quad t \in [-2^{2k-1}-\dots-2^3-2, 2+2^2+\dots+2^{2k-2}], \\
\psi(t+2^{2k+1}) \in \mathcal{O}_{\varepsilon/8+\varepsilon/16}(\psi(t)), \quad t \in [-2^{2k-1}-\dots-2^3-2, 2+2^2+\dots+2^{2k-2}], \\
\psi(t+3\cdot 2^{2k}) \in \mathcal{O}_{\varepsilon/8+\varepsilon/8+\varepsilon/16}(\psi(t)), \\
t \in [-2^{2k-1}-\dots-2^3-2, 2+2^2+\dots+2^{2k-2}], \\
\psi(t+2^{2k+2}) \in \mathcal{O}_{\varepsilon/16}(\psi(t)), \quad t \in [-2^{2k-1}-\dots-2^3-2, 2+2^2+\dots+2^{2k-2}],
\end{gathered}$$

$$\begin{aligned}&\psi(t+2^{2k}+2^{2k+2}) \in \mathcal{O}_{\varepsilon/8+\varepsilon/16}(\psi(t)),\\ &t\in[-2^{2k-1}-\dots-2^3-2, 2+2^2+\dots+2^{2k-2}],\end{aligned}$$
$$\dots$$

Since
$$\frac{\varepsilon}{8}+\frac{\varepsilon}{8}+\frac{\varepsilon}{16}+\frac{\varepsilon}{16}+\frac{\varepsilon}{32}+\dots=\frac{\varepsilon}{2},$$
we have
$$\begin{aligned}&\psi(t+l\cdot 2^{2k}) \in \mathcal{O}_{\varepsilon/2}(\psi(t)),\\ &t\in[-2^{2k-1}-\dots-2^3-2, 2+2^2+\dots+2^{2k-2}], \quad l\in\mathbb{Z}.\end{aligned} \tag{3.1}$$
We express any $t\in\mathbb{R}$ as the sum of numbers $p(t)$ and $q(t)$ for which
$$p(t)\in[-2^{2k-1}-\dots-2^3-2, 2+2^2+\dots+2^{2k-2}],$$
$$q(t)\in\mathbb{Z} \quad \text{and} \quad q(t)=j2^{2k} \quad \text{for some } j\in\mathbb{Z}.$$
Using (3.1), we obtain
$$\begin{aligned}&\varrho(\psi(t),\psi(t+l\cdot 2^{2k}))\\ &\le \varrho(\psi(p(t)+q(t)),\psi(p(t)))+\varrho(\psi(p(t)),\psi(p(t)+(j+l)\,2^{2k}))<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned} \tag{3.2}$$
for any $t\in\mathbb{R}$, $l\in\mathbb{Z}$, which completes the proof. $\square$

The process mentioned in the previous theorem is easily modifiable. We illustrate this fact by the following two theorems. We remark that Theorem 3.2 is used in [34].

Theorem 3.2. *Let $M>0$, $x_0\in\mathcal{X}$, and $j\in\mathbb{N}$ be given. Let $\varphi:[0,M]\to\mathcal{X}$ satisfy $\varphi(0)=\varphi(M)=x_0$. If $\{r_n\}_{n\in\mathbb{N}}\subset\mathbb{R}_0^+$ has the property that*
$$\sum_{n=1}^{\infty} r_n<\infty, \tag{3.3}$$
then an arbitrary continuous function $\psi:\mathbb{R}\to\mathcal{X}$, $\psi|_{[0,M]}\equiv\varphi$ for which
$$\begin{aligned}\psi(t)=x_0, \quad t\in{}&\{iM,\, 2\le i\le j+1\}\cup\{-i(j+1)M,\, 1\le i\le j\}\cup\\ &\cup_{n=1}^{\infty}\{((j+1)+\dots+j(j+1)^{2n-2}+i(j+1)^{2n})M;\, 1\le i\le j\}\cup\\ &\cup_{n=1}^{\infty}\{-((j+1)+\dots+j(j+1)^{2n-1}+i(j+1)^{2n+1})M;\, 1\le i\le j\}\end{aligned}$$
and, at the same time, for which it is valid
$$\psi(t)\in\mathcal{O}_{r_1}(\psi(t-M)), \quad t\in(M,2M),$$
$$\dots$$
$$\psi(t)\in\mathcal{O}_{r_1}(\psi(t-jM)), \quad t\in(jM,(j+1)M),$$
$$\psi(t)\in\mathcal{O}_{r_2}(\psi(t+(j+1)M)), \quad t\in(-(j+1)M,0),$$
$$\dots$$
$$\psi(t)\in\mathcal{O}_{r_2}(\psi(t+j(j+1)M)), \quad t\in(-j(j+1)M,-(j-1)(j+1)M),$$
$$\psi(t)\in\mathcal{O}_{r_3}(\psi(t-(j+1)^2M)), \quad t\in((j+1)M,((j+1)+(j+1)^2)M),$$
$$\dots$$
$$\begin{aligned}&\psi(t)\in\mathcal{O}_{r_3}(\psi(t-j(j+1)^2M)),\\ &t\in(((j+1)+(j-1)(j+1)^2)M,((j+1)+j(j+1)^2)M),\end{aligned}$$

$$
\begin{aligned}
&\dots\\
&\psi(t) \in \mathcal{O}_{r_{2n}}(\psi(t + (j+1)^{2n-1}M)),\\
&t \in \big(-((j+1)^{2n-1} + j(j+1)^{2n-3} + \dots + j(j+1)^3 + j(j+1))M,\\
&\quad -(j(j+1)^{2n-3} + \dots + j(j+1)^3 + j(j+1))M\big),\\
&\dots\\
&\psi(t) \in \mathcal{O}_{r_{2n}}(\psi(t + j(j+1)^{2n-1}M)),\\
&t \in \big(-(j(j+1)^{2n-1} + j(j+1)^{2n-3} + \dots + j(j+1)^3 + j(j+1))M,\\
&\quad -((j-1)(j+1)^{2n-1} + j(j+1)^{2n-3} + \dots + j(j+1)^3 + j(j+1))M\big),\\
&\psi(t) \in \mathcal{O}_{r_{2n+1}}(\psi(t - (j+1)^{2n}M)),\\
&t \in \big(((j+1) + j(j+1)^2 + \dots + j(j+1)^{2n-2})M,\\
&\quad ((j+1) + j(j+1)^2 + \dots + j(j+1)^{2n-2} + (j+1)^{2n})M\big),\\
&\dots\\
&\psi(t) \in \mathcal{O}_{r_{2n+1}}(\psi(t - j(j+1)^{2n}M)),\\
&t \in \big(((j+1) + j(j+1)^2 + \dots + j(j+1)^{2n-2} + (j-1)(j+1)^{2n})M,\\
&\quad ((j+1) + j(j+1)^2 + \dots + j(j+1)^{2n-2} + j(j+1)^{2n})M\big),\\
&\dots
\end{aligned}
$$

is almost periodic.

Proof. We can prove this theorem analogously as Theorem 3.1. Let ε be a positive number and let an odd integer $n(\varepsilon) \geq 2$ have the property (see (3.3)) that

$$\sum_{n=n(\varepsilon)}^{\infty} r_n < \frac{\varepsilon}{2}. \tag{3.4}$$

We will prove that $l(j+1)^{n(\varepsilon)-1}M$ is an ε-translation number of ψ for all $l \in \mathbb{Z}$. Arbitrarily choosing $l \in \mathbb{Z}$ and $t \in \mathbb{R}$, if we put

$$s := l(j+1)^{n(\varepsilon)-1}M, \tag{3.5}$$

then it suffices to show that the inequality

$$d(\psi(t), \psi(t+s)) < \varepsilon \tag{3.6}$$

holds; i.e., this inequality proves the theorem.

We can write t as the sum of numbers t_1 and t_2, where

$$
\begin{gathered}
t_1 \geq -\big(j(j+1)^{n(\varepsilon)-2} + \dots + j(j+1)^3 + j(j+1)\big)M,\\
t_1 \leq \big(j + 1 + j(j+1)^2 + \dots + j(j+1)^{n(\varepsilon)-3}\big)M
\end{gathered}
\tag{3.7}
$$

and

$$t_2 = i(j+1)^{n(\varepsilon)-1}M \quad \text{for some } i \in \mathbb{Z}. \tag{3.8}$$

Now we have (see (3.7) and the proof of Theorem 3.1)

$$
\begin{aligned}
\varrho(\psi(t), \psi(t+s)) &\leq \varrho(\psi(t_1 + t_2), \psi(t_1)) + \varrho(\psi(t_1), \psi(t_1 + t_2 + s))\\
&< \sum_{n=n(\varepsilon)}^{n(\varepsilon)+p-1} r_n + \sum_{n=n(\varepsilon)}^{n(\varepsilon)+q-1} r_n.
\end{aligned}
\tag{3.9}
$$

Indeed, we can express (consider (3.5) and (3.8))

$$
\begin{gathered}
t_2 = \big(i_1(j+1)^{n(\varepsilon)-1} + i_2(j+1)^{n(\varepsilon)} + \dots + i_p(j+1)^{n(\varepsilon)+p-1}\big)(m+1), \\
t_2 + s = \big(l_1(j+1)^{n(\varepsilon)-1} + l_2(j+1)^{n(\varepsilon)} + \dots + l_q(j+1)^{n(\varepsilon)+q-1}\big)(m+1),
\end{gathered}
$$

where $i_1, \dots, i_p, l_1, \dots, l_q \subseteq \{-j, \dots, 0, \dots, j\}$ satisfy

$$
i_1 \geq 0, i_2 \leq 0, \dots, (-1)^p \, i_p \leq 0, \quad l_1 \geq 0, l_2 \leq 0, \dots, (-1)^q \, l_q \leq 0,
$$

and use (iii). It is sure that (3.4) and (3.9) give (3.6). □

Theorem 3.3. *Let $\varphi : (-r, r] \to \mathcal{X}$, $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_0^+$, and $\{j_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ be arbitrary such that*

$$
\sum_{n=1}^{\infty} r_n j_n < \infty \tag{3.10}
$$

holds, and let a function $\psi : \mathbb{R} \to \mathcal{X}$ satisfy $\psi|_{(-r,r]} \equiv \varphi$ and

$$
\begin{gathered}
\psi(t) \in \mathcal{O}_{r_1}(\varphi(t-2r)), \quad t \in (r, r+2r], \\
\dots \\
\psi(t) \in \mathcal{O}_{r_1}(\varphi(t-2r)), \quad t \in (r + (j_1 - 1)2r, r + j_1 2r], \\
\psi(t) \in \mathcal{O}_{r_1}(\varphi(t+2r)), \quad t \in (-2r - r, -r], \\
\dots \\
\psi(t) \in \mathcal{O}_{r_1}(\varphi(t+2r)), \quad t \in (-j_1 2r - r, -(j_1 - 1)2r - r], \\
\dots \\
\psi(t) \in \mathcal{O}_{r_n}(\varphi(t-p_n)), \quad t \in (p_1 + \dots + p_{n-1}, p_1 + \dots + p_{n-1} + p_n], \\
\dots \\
\psi(t) \in \mathcal{O}_{r_n}(\varphi(t-p_n)), \\
t \in (p_1 + \dots + p_{n-1} + (j_n - 1)p_n, p_1 + \dots + p_{n-1} + j_n p_n], \\
\psi(t) \in \mathcal{O}_{r_n}(\varphi(t+p_n)), \quad t \in (-p_n - p_{n-1} - \dots - p_1, -p_{n-1} - \dots - p_1], \\
\dots \\
\psi(t) \in \mathcal{O}_{r_n}(\varphi(t+p_n)), \\
t \in (-j_n p_n - p_{n-1} - \dots - p_1, -(j_n - 1)p_n - p_{n-1} - \dots - p_1], \\
\dots
\end{gathered}
$$

where

$$
\begin{gathered}
p_1 := r + j_1 2r, \quad p_2 := 2(r + j_1 2r), \\
p_3 := (2j_2 + 1)p_2, \quad \dots, \quad p_n := (2j_{n-1} + 1)p_{n-1}, \dots
\end{gathered}
$$

If ψ is continuous on $\mathbb{R}$, then it is almost periodic.

Proof. It is not difficult to prove Theorem 3.3 analogously as Theorems 3.1 and 3.2. For given $\varepsilon > 0$, let an integer $n(\varepsilon) \geq 2$ satisfy

$$
\sum_{n=n(\varepsilon)}^{\infty} r_n j_n < \frac{\varepsilon}{4}.
$$

One can prove the inclusion

$$
\{l p_{n(\varepsilon)}; l \in \mathbb{Z}\} \subseteq \mathfrak{T}(\psi, \varepsilon) \tag{3.11}
$$

which yields the almost periodicity of ψ. $\square$

Remark 3.4. From the proofs of Theorems 3.1, 3.2, 3.3 (see (3.2), (3.5) and (3.6), (3.11)), we obtain a property of the set of all ε-translation numbers of the resulting function ψ. For any $\varepsilon > 0$, there exists nonzero $c \in \mathbb{R}$ for which

$$\{lc; l \in \mathbb{Z}\} \subseteq \mathfrak{T}(\psi, \varepsilon).$$

An important class of almost periodic functions is the class of *limit-periodic* functions. To this class belong the uniform limits of sequences of periodic continuous functions (in general, having different periods). It is seen that, applying the method from the above theorems, we obtain limit-periodic functions.

4. An application

Let $m \in \mathbb{N}$ be arbitrarily given. In this section, we will use the following notations: $\mathcal{I}m(\varphi)$ for the range of a function φ, $\mathcal{M}at(\mathbb{C}, m)$ for the set of all $m \times m$ matrices with complex elements, $U(m) \subset \mathcal{M}at(\mathbb{C}, m)$ for the group of all unitary matrices of dimension m, A^* for the conjugate transpose of $A \in \mathcal{M}at(\mathbb{C}, m)$, I for the identity matrix, 0 for the zero matrix, and i for the imaginary unit.

We will analyse systems of m homogeneous linear differential equations of the form

$$x'(t) = A(t)x(t), \quad t \in \mathbb{R}, \tag{4.1}$$

where A is an almost periodic function with $\mathcal{I}m(A) \subset \mathcal{M}at(\mathbb{C}, m)$ and with the property that $A(t) + A^*(t) = 0$ for any $t \in \mathbb{R}$; i.e., $A : \mathbb{R} \to \mathcal{M}at(\mathbb{C}, m)$ is an almost periodic function of skew-Hermitian (skew-adjoint) matrices. Let $\mathcal{S}$ be *the set of all systems* (4.1). We will identify the function A with the system (4.1) which is determined by A. Especially, we will write $A \in \mathcal{S}$.

In the vector space $\mathbb{C}^m$, we will consider the absolute norm $\|\cdot\|_1$ (one can also consider the Euclidean norm or the maximum norm). Let $\|\cdot\|$ be the corresponding matrix norm. Considering that every almost periodic function is bounded (see Lemma 2.3), the distance between two systems $A, B \in \mathcal{S}$ is defined by the norm of the matrix valued functions A, B, uniformly on $\mathbb{R}$; i.e., we introduce the metric

$$\sigma(A, B) := \sup_{t \in \mathbb{R}} \|A(t) - B(t)\|, \quad A, B \in \mathcal{S}.$$

For $\varepsilon > 0$, the symbol $\mathcal{O}^{\sigma}_{\varepsilon}(A)$ will denote the ε-*neighbourhood* of A in $\mathcal{S}$.

Now we recall the notion of the frequency module and its rational hull which can be introduced for all almost periodic function with values in a Banach space. *The frequency module* $\mathcal{F}$ of an almost periodic function $A : \mathbb{R} \to \mathcal{M}at(\mathbb{C}, m)$ is the $\mathbb{Z}$-module of the real numbers, generated by the λ such that

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T A(t) e^{2\pi i \lambda t}\, dt \neq 0.$$

The rational hull of $\mathcal{F}$ is the set

$$\{\lambda/l; \lambda \in \mathcal{F}, l \in \mathbb{Z}\}.$$

For the frequency modules of almost periodic linear differential systems and their solutions, we refer to [12, Chapters 4, 6], [25].

In [28], it is proved that, in any neighbourhood of a system (4.1) with frequency module $\mathcal{F}$, there exists a system with a frequency module contained in the rational hull of $\mathcal{F}$ possessing all almost periodic solutions with frequencies belonging to the rational hull of $\mathcal{F}$ as well. From [31, Theorem 1] it follows that there exists a system (4.1) which cannot be approximated by the so-called reducible systems with frequency module $\mathcal{F}$ (there exists an open set of irreducible systems with a fixed frequency module); i.e., a neighbourhood of a system (4.1) may not contain a system with almost periodic solutions and frequency module $\mathcal{F}$. In this case, see also [9] and [32] for reducible constant systems and systems reducing to diagonal form by Lyapunov transformation with frequency module $\mathcal{F}$, respectively.

In addition, it is proved in [30] that the systems with k-dimensional frequency basis of A, having solutions which are not almost periodic, form a subset of the second category of the space of all considered systems with k-dimensional frequency basis of A. Thus, it is known (see [28, Corollary 1]) that the systems with k-dimensional frequency basis of A and with an almost periodic fundamental matrix form a dense set of the first category in the space of all systems (4.1) with k-dimensional frequency basis.

In this context, we formulate the following result that the systems having no nontrivial almost periodic solution form a dense subset of $\mathcal{S}$.

Theorem 4.1. *For any $A \in \mathcal{S}$ and $\varepsilon > 0$, there exists $B \in \mathcal{O}^{\sigma}_{\varepsilon}(A)$ which does not have an almost periodic solution other than the trivial one.*

Proof. Let $A, C \in \mathcal{S}$ and $\varepsilon > 0$ be arbitrary. Since the sum of skew-Hermitian matrices is also skew-Hermitian and since the sum of two almost periodic functions is almost periodic (consider Theorem 2.4), we have that $A+C \in \mathcal{S}$. Let $X_A(t)$, $t \in \mathbb{R}$ and $X_C(t)$, $t \in \mathbb{R}$ be the principal (i.e., $X_A(0) = X_C(0) = I$) fundamental matrices of $A \in \mathcal{S}$ and $C \in \mathcal{S}$, respectively. If the matrices $C(t)$, $X_A(t)$ commute for all $t \in \mathbb{R}$, then the matrix valued function $X_A(t) \cdot X_C(t)$, $t \in \mathbb{R}$ is the principal fundamental matrix of $A + C \in \mathcal{S}$. Indeed, from $X'_A(t) = A(t) \cdot X_A(t)$, $X'_C(t) = C(t) \cdot X_C(t)$ for $t \in \mathbb{R}$, we obtain

$$\begin{aligned}
(X_A(t)X_C(t))' &= A(t)X_A(t)X_C(t) + X_A(t)C(t)X_C(t)\\
&= A(t)X_A(t)X_C(t) + C(t)X_A(t)X_C(t)\\
&= (A + C)(t)X_A(t)X_C(t), \quad t \in \mathbb{R}.
\end{aligned}$$

It gives that it suffices to find $C \in \mathcal{O}^{\sigma}_{\varepsilon}(0)$ for which all matrices $C(t)$, $t \in \mathbb{R}$ have the form $\operatorname{diag}[ia, \dots, ia]$, $a \in \mathbb{R}$ and for which the vector valued function $X_A(t) \cdot X_C(t) \cdot u$, $t \in \mathbb{R}$ is not almost periodic for any vector $u \in \mathbb{C}^m$, $\|u\|_1 = 1$.

We will construct such an almost periodic function C applying Theorem 3.1 for $a = \varepsilon/4$. First of all we put

$$C(t) \equiv 0, \quad t \in [0, 1].$$

Then, in the first step of our construction, we define C on $(1, 2]$ arbitrarily so that it is constant on $[1 + 1/4, 1 + 3/4]$ and $\|C(t)\| < \varepsilon/4$ for t from this interval, $C(2) := C(1) = 0$, and it is linear between values 0, $C(3/2)$ on $[1, 1 + 1/4]$ and $[1 + 3/4, 2]$.

In the second step, we define continuous C satisfying $\|C(t) - C(t + 2)\| < \varepsilon/4$ for $t \in [-2, 0)$ arbitrarily so that it is constant on

$$[-2 + 1/16, -2 + 1 - 1/16], \quad [-2 + 1 + 1/4 + 1/16, -2 + 1 + 3/4 - 1/16];$$

at the same time, we put

$$C(-2) := C(0) = 0, \quad C(-1 + 1/4) := C(1 + 1/4) = C(3/2),$$

$$
\begin{gathered}
C(-1) := C(1) = 0, \quad C(-1/4) := C(2-1/4) = C(3/2), \\
C(t) \equiv C(3/2)/2, \quad t \in [-1+1/16, -1+1/4-1/16] \cup [-1/4+1/16, -1/16]
\end{gathered}
$$

and define C so that it is linear on

$$
\begin{gathered}
[-2, -2+1/16], \quad [-1-1/16, -1], \quad [-1, -1+1/16], \\
[-1+1/4-1/16, -1+1/4], \quad [-1+1/4, -1+1/4+1/16], \\
[-1/4-1/16, -1/4], \quad [-1/4, -1/4+1/16], \quad [-1/16, 0].
\end{gathered}
$$

Analogously, in the third step, we obtain C on $(2, 6]$ for which we can choose constant values on

$$
\begin{gathered}
[4-2+1/16+8^{-1}/16, 4-2+1-1/16-8^{-1}/16], \\
[4-2+1+1/4+1/16+8^{-1}/16, 4-2+1+3/4-1/16-8^{-1}/16], \\
[4-1+1/16+8^{-1}/16, 4-1+1/4-1/16-8^{-1}/16], \\
[4-1/4+1/16+8^{-1}/16, 4-1/16-8^{-1}/16], \\
[4+8^{-1}/16, 4+1-8^{-1}/16], \quad [4+1+1/4+8^{-1}/16, 4+1+3/4-8^{-1}/16]
\end{gathered}
$$

arbitrarily so that $\|C(t) - C(t-4)\| < \varepsilon/8$, $t \in (2, 6]$; at the same time, we put

$$
\begin{gathered}
C(4-2+1/16) := C(-2+1/16) = C(-3/2), \\
C(4-2+1-1/16) := C(-1-1/16) = C(-3/2), \\
C(4-1) := C(-1) = 0, \\
C(4-1+1/16) := C(-1+1/16) = C(3/2)/2, \\
C(4-1+1/4-1/16) := C(-1+1/4-1/16) = C(3/2)/2, \\
C(4-1+1/4) := C(-1+1/4) = C(3/2), \\
C(4-2+1+1/4+1/16) := C(-2+1+1/4+1/16) = C(-1/2), \\
C(4-2+1+3/4-1/16) := C(-2+1+3/4-1/16) = C(-1/2), \\
C(4-1/4) := C(-1/4) = C(3/2), \\
C(4-1/4+1/16) := C(-1/4+1/16) = C(3/2)/2, \\
C(4-1/16) := C(-1/16) = C(3/2)/2, \\
C(4) := C(0), \quad C(4+1) := C(1), \\
C(4+1+1/4) := C(1+1/4) = C(3/2), \\
C(4+1+3/4) := C(1+3/4) = C(3/2), \\
C(4+2) := C(2) = C(0) = 0,
\end{gathered}
$$

$$
\begin{aligned}
C(t) \equiv C(-3/2)/2, \quad & t \in [4-2+8^{-1}/16, 4-2+1/16-8^{-1}/16] \\
& \cup [4-1-1/16+8^{-1}/16, 4-1-8^{-1}/16], \\
C(t) \equiv C(3/2)/4, \quad & t \in [4-1+8^{-1}/16, 4-1+1/16-8^{-1}/16] \\
& \cup [4-1/16+8^{-1}/16, 4-8^{-1}/16], \\
C(t) \equiv 3\,C(3/2)/4, \quad & t \in [4-1+1/4-1/16+8^{-1}/16, 4-1+1/4-8^{-1}/16] \\
& \cup [4-1/4+8^{-1}/16, 4-1/4+1/16-8^{-1}/16],
\end{aligned}
$$

$$
\begin{aligned}
C(t) \equiv &(C(3/2)+C(-1/2))/2,\\
&t \in [4-1+1/4+8^{-1}/16, 4-1+1/4+1/16-8^{-1}/16]\\
&\cup [4-1/4-1/16+8^{-1}/16, 4-1/4-8^{-1}/16],\\
C(t) \equiv &(8\,C(4+1)+1\,C(4+1+1/4))/9,\\
&t \in [4+1+8^{-1}/16, 4+1+8^{-1}/16\cdot 3],\\
C(t) \equiv &(7\,C(4+1)+2\,C(4+1+1/4))/9,\\
&t \in [4+1+8^{-1}/16\cdot 5, 4+1+8^{-1}/16\cdot 7],\\
&\dots\\
C(t) \equiv &(1\,C(4+1)+8\,C(4+1+1/4))/9,\\
&t \in [4+1+8^{-1}/16\cdot 29, 4+1+8^{-1}/16\cdot 31],\\
C(t) \equiv &(8\,C(4+1+3/4)+1\,C(4+2))/9,\\
&t \in [4+1+3/4+8^{-1}/16, 4+1+3/4+8^{-1}/16\cdot 3],\\
C(t) \equiv &(7\,C(4+1+3/4)+2\,C(4+2))/9,\\
&t \in [4+1+3/4+8^{-1}/16\cdot 5, 4+1+3/4+8^{-1}/16\cdot 7],\\
&\dots\\
C(t) \equiv &(1\,C(4+1+3/4)+8\,C(4+2))/9,\\
&t \in [4+1+3/4+8^{-1}/16\cdot 29, 4+1+3/4+8^{-1}/16\cdot 31]\,.
\end{aligned}
$$

Then we define continuous C so that it is linear on the rest of subintervals.

If we denote

$$
\begin{gathered}
a_1^1 := 0, \quad b_1^1 := 0, \quad c_1^1 := 1,\\
a_2^1 := 1, \quad b_2^1 := 1+1/4, \quad c_2^1 := 1+3/4, \quad a_3^1 := 2
\end{gathered}
$$

and (compare with the situation after the second step)

$$
\begin{gathered}
a_1^2 := -2, \quad b_1^2 := -2, \quad c_1^2 := -2,\\
a_2^2 := -2, \quad b_2^2 := -2+1/16, \quad c_2^2 := -1-1/16,\\
a_3^2 := -1, \quad b_3^2 := -1, \quad c_3^2 := -1,\\
a_4^2 := -1, \quad b_4^2 := -1+1/16, \quad c_4^2 := -1+1/4-1/16,\\
a_5^2 := -1+1/4, \quad b_5^2 := -1+1/4+1/16, \quad c_5^2 := -1+3/4-1/16,\\
a_6^2 := -1+3/4, \quad b_6^2 := -1+3/4+1/16, \quad c_6^2 := -1/16,
\end{gathered}
$$

we see that C does not need to be constant only on

$$
\begin{gathered}
[a_j^1-2, a_j^1-2+4^{-2}], \quad [b_2^1-2-4^{-2}, b_2^1-2], \quad [b_j^1-2, b_j^1-2+4^{-2}],\\
[c_j^1-2-4^{-2}, c_j^1-2], \quad [c_j^1-2, c_j^1-2+4^{-2}], \quad [a_{j+1}^1-2-4^{-2}, a_{j+1}^1-2]
\end{gathered}
$$

for $j \in \{1,2\}$; i.e., on

$$
[a_j^2, b_j^2],\ j \in \{1,\dots,6\}, \quad [c_j^2, a_{j+1}^2],\ j \in \{1,\dots,5\}, \quad [c_6^2, 0],
$$

and it has to be constant on each one of the intervals

$$
\begin{gathered}
[a_2^1-2+4^{-2}, b_2^1-2-4^{-2}], \quad [c_2^1-2+4^{-2}, a_3^1-2-4^{-2}],\\
[b_j^1-2+4^{-2}, c_j^1-2-4^{-2}], \quad j \in \{1,2\},
\end{gathered}
$$

i.e., on $[b_j^2, c_j^2]$, $j \in \{1, \dots, 6\}$. It is also seen that

$$a_1^2 = d_1^1, \quad b_1^2 = d_2^1, \quad c_1^2 = d_3^1, \quad a_2^2 = d_4^1, \quad \dots \quad c_6^2 = d_{18}^1,$$

where $d_1^1, d_2^1, \dots, d_{18}^1$ is the nondecreasing sequence of all numbers

$$\begin{gathered} a_j^1 - 2, \quad b_j^1 - 2, \quad c_j^1 - 2, \\ \min\{a_j^1 - 2 + 4^{-2}, b_j^1 - 2\}, \quad \max\{a_j^1 - 2, b_j^1 - 2 - 4^{-2}\}, \\ \min\{c_j^1 - 2, b_j^1 - 2 + 4^{-2}\}, \quad \max\{c_j^1 - 2 - 4^{-2}, b_j^1 - 2\}, \\ \min\{c_j^1 - 2 + 4^{-2}, a_{j+1}^1 - 2\}, \quad \max\{c_j^1 - 2, a_{j+1}^1 - 2 - 4^{-2}\} \end{gathered}$$

for $j \in \{1, 2\}$. We put $a_7^2 := 0$.

Let $d_1^2, d_2^2, \dots, d_{168}^2$ be the nondecreasing sequence of all numbers

$$\begin{gathered} b_1^1 + 4, \quad b_1^1 + 4, \quad b_1^1 + 4, \quad b_1^1 + 4, \quad b_1^1 + 4, \quad b_1^1 + 4, \\ b_1^1 + 4, \quad b_1^1 + 4, \quad b_1^1 + 4, \quad b_1^1 + 4, \quad b_1^1 + 4, \quad b_1^1 + 4, \\ c_1^1 + 4, \quad \min\{c_1^1 + 4, b_1^1 + 4 + 8^{-1}/16\}, \quad \max\{c_1^1 + 4 - 8^{-1}/16, b_1^1 + 4\}, \\ c_2^1 + 4, \quad \min\{c_2^1 + 4, b_2^1 + 4 + 8^{-1}/16\}, \quad \max\{c_2^1 + 4 - 8^{-1}/16, b_2^1 + 4\}, \\ a_1^1 + (4k+1)(b_1^1 - a_1^1)/32 + 4, \quad a_1^1 + (4k+3)(b_1^1 - a_1^1)/32 + 4, \\ a_1^1 + (4k+4)(b_1^1 - a_1^1)/32 + 4, \quad k \in \{0, 1, \dots, 7\}, \\ c_1^1 + (4k+1)(a_2^1 - c_1^1)/32 + 4, \quad c_1^1 + (4k+3)(a_2^1 - c_1^1)/32 + 4, \\ c_1^1 + (4k+4)(a_2^1 - c_1^1)/32 + 4, \quad k \in \{0, 1, \dots, 7\}, \\ a_2^1 + (4k+1)(b_2^1 - a_2^1)/32 + 4, \quad a_2^1 + (4k+3)(b_2^1 - a_2^1)/32 + 4, \\ a_2^1 + (4k+4)(b_2^1 - a_2^1)/32 + 4, \quad k \in \{0, 1, \dots, 7\}, \\ c_2^1 + (4k+1)(a_3^1 - c_2^1)/32 + 4, \quad c_2^1 + (4k+3)(a_3^1 - c_2^1)/32 + 4, \\ c_2^1 + (4k+4)(a_3^1 - c_2^1)/32 + 4, \quad k \in \{0, 1, \dots, 7\} \end{gathered}$$

and

$$\begin{gathered} a_{j+1}^2 + 4, \quad b_j^2 + 4, \quad c_j^2 + 4, \\ \min\{a_j^2 + 4 + 8^{-1}/16, b_j^2 + 4\}, \quad \max\{a_j^2 + 4, b_j^2 + 4 - 8^{-1}/16\}, \\ \min\{c_j^2 + 4, b_j^2 + 4 + 8^{-1}/16\}, \quad \max\{c_j^2 + 4 - 8^{-1}/16, b_j^2 + 4\}, \\ \min\{c_j^2 + 4 + 8^{-1}/16, a_{j+1}^2 + 4\}, \quad \max\{c_j^2 + 4, a_{j+1}^2 + 4 - 8^{-1}/16\} \end{gathered}$$

for $j \in \{1, \dots, 6\}$. We denote

$$a_1^3 := 2, \quad b_1^3 := d_1^2, \quad c_1^3 := d_2^2, \quad a_2^3 := d_3^2, \quad \dots \quad a_{57}^3 := d_{168}^2.$$

We remark that, in the sequences of d_j^l, $l \in \mathbb{N}$, values are a number of time.

In the fourth step, we define C so that

$$\|C(t) - C(t + 2^3)\| < \frac{\varepsilon}{2^3}, \quad t \in [-2^3 - 2, -2).$$

We consider the nondecreasing sequence $d_1^3, d_2^3, \dots, d_{21\cdot 8^2}^3$ of values

$$\begin{gathered} a_j^3 - 2^3, \quad b_j^3 - 2^3, \quad c_j^3 - 2^3, \\ \min\{a_j^3 - 2^3 + 8^{-2}/16, b_j^3 - 2^3\}, \quad \max\{a_j^3 - 2^3, b_j^3 - 2^3 - 8^{-2}/16\}, \\ \min\{c_j^3 - 2^3, b_j^3 - 2^3 + 8^{-2}/16\}, \quad \max\{c_j^3 - 2^3 - 8^{-2}/16, b_j^3 - 2^3\}, \end{gathered}$$

$$\min\{c_j^3 - 2^3 + 8^{-2}/16, a_{j+1}^3 - 2^3\}, \quad \max\{c_j^3 - 2^3, a_{j+1}^3 - 2^3 - 8^{-2}/16\}$$

for $j \in \{1, \dots, 7 \cdot 8\}$, 144 numbers $b_1^1 - 2^3$, and

$$\begin{gathered}
c_1^1 - 2^3, \quad \min\{c_1^1 - 2^3, b_1^1 - 2^3 + 8^{-2}/16\}, \quad \max\{c_1^1 - 2^3 - 8^{-2}/16, b_1^1 - 2^3\}, \\
c_2^1 - 2^3, \quad \min\{c_2^1 - 2^3, b_2^1 - 2^3 + 8^{-2}/16\}, \quad \max\{c_2^1 - 2^3 - 8^{-2}/16, b_2^1 - 2^3\}, \\
\min\{a_1^1 + (k-1)(b_1^1 - a_1^1)/(8 \cdot 4) - 2^3 + 8^{-2}/16, a_1^1 + k(b_1^1 - a_1^1)/(8 \cdot 4) - 2^3\}, \\
\max\{a_1^1 + (k-1)(b_1^1 - a_1^1)/(8 \cdot 4) - 2^3, a_1^1 + k(b_1^1 - a_1^1)/(8 \cdot 4) - 2^3 - 8^{-2}/16\}, \\
a_1^1 + k(b_1^1 - a_1^1)/(8 \cdot 4) - 2^3, \quad k \in \{1, \dots, 8 \cdot 4\}, \\
\min\{c_1^1 + (k-1)(a_2^1 - c_1^1)/(8 \cdot 4) - 2^3 + 8^{-2}/16, c_1^1 + k(a_2^1 - c_1^1)/(8 \cdot 4) - 2^3\}, \\
\max\{c_1^1 + (k-1)(a_2^1 - c_1^1)/(8 \cdot 4) - 2^3, c_1^1 + k(a_2^1 - c_1^1)/(8 \cdot 4) - 2^3 - 8^{-2}/16\}, \\
c_1^1 + k(a_2^1 - c_1^1)/(8 \cdot 4) - 2^3, \quad k \in \{1, \dots, 8 \cdot 4\}, \\
\min\{a_2^1 + (k-1)(b_2^1 - a_2^1)/(8 \cdot 4) - 2^3 + 8^{-2}/16, a_2^1 + k(b_2^1 - a_2^1)/(8 \cdot 4) - 2^3\}, \\
\max\{a_2^1 + (k-1)(b_2^1 - a_2^1)/(8 \cdot 4) - 2^3, a_2^1 + k(b_2^1 - a_2^1)/(8 \cdot 4) - 2^3 - 8^{-2}/16\}, \\
a_2^1 + k(b_2^1 - a_2^1)/(8 \cdot 4) - 2^3, \quad k \in \{1, \dots, 8 \cdot 4\}, \\
\min\{c_2^1 + (k-1)(a_3^1 - c_2^1)/(8 \cdot 4) - 2^3 + 8^{-2}/16, c_2^1 + k(a_3^1 - c_2^1)/(8 \cdot 4) - 2^3\}, \\
\max\{c_2^1 + (k-1)(a_3^1 - c_2^1)/(8 \cdot 4) - 2^3, c_2^1 + k(a_3^1 - c_2^1)/(8 \cdot 4) - 2^3 - 8^{-2}/16\}, \\
c_2^1 + k(a_3^1 - c_2^1)/(8 \cdot 4) - 2^3, \quad k \in \{1, \dots, 8 \cdot 4\}, \\
c_1^2 - 2^3, \quad \min\{c_1^2 - 2^3, b_1^2 - 2^3 + 8^{-2}/16\}, \quad \max\{c_1^2 - 2^3 - 8^{-2}/16, b_1^2 - 2^3\}, \\
c_2^2 - 2^3, \quad \min\{c_2^2 - 2^3, b_2^2 - 2^3 + 8^{-2}/16\}, \quad \max\{c_2^2 - 2^3 - 8^{-2}/16, b_2^2 - 2^3\}, \\
c_3^2 - 2^3, \quad \min\{c_3^2 - 2^3, b_3^2 - 2^3 + 8^{-2}/16\}, \quad \max\{c_3^2 - 2^3 - 8^{-2}/16, b_3^2 - 2^3\}, \\
c_4^2 - 2^3, \quad \min\{c_4^2 - 2^3, b_4^2 - 2^3 + 8^{-2}/16\}, \quad \max\{c_4^2 - 2^3 - 8^{-2}/16, b_4^2 - 2^3\}, \\
c_5^2 - 2^3, \quad \min\{c_5^2 - 2^3, b_5^2 - 2^3 + 8^{-2}/16\}, \quad \max\{c_5^2 - 2^3 - 8^{-2}/16, b_5^2 - 2^3\}, \\
c_6^2 - 2^3, \quad \min\{c_6^2 - 2^3, b_6^2 - 2^3 + 8^{-2}/16\}, \quad \max\{c_6^2 - 2^3 - 8^{-2}/16, b_6^2 - 2^3\}, \\
\min\{a_1^2 + (k-1)(b_1^2 - a_1^2)/8 - 2^3 + 8^{-2}/16, a_1^2 + k(b_1^2 - a_1^2)/8 - 2^3\}, \\
\max\{a_1^2 + (k-1)(b_1^2 - a_1^2)/8 - 2^3, a_1^2 + k(b_1^2 - a_1^2)/8 - 2^3 - 8^{-2}/16\}, \\
a_1^2 + k(b_1^2 - a_1^2)/8 - 2^3, \quad k \in \{1, \dots, 8\}, \\
\min\{c_1^2 + (k-1)(a_2^2 - c_1^2)/8 - 2^3 + 8^{-2}/16, c_1^2 + k(a_2^2 - c_1^2)/8 - 2^3\}, \\
\max\{c_1^2 + (k-1)(a_2^2 - c_1^2)/8 - 2^3, c_1^2 + k(a_2^2 - c_1^2)/8 - 2^3 - 8^{-2}/16\}, \\
c_1^2 + k(a_2^2 - c_1^2)/8 - 2^3, \quad k \in \{1, \dots, 8\}, \\
\dots \\
\min\{a_6^2 + (k-1)(b_6^2 - a_6^2)/8 - 2^3 + 8^{-2}/16, a_6^2 + k(b_6^2 - a_6^2)/8 - 2^3\}, \\
\max\{a_6^2 + (k-1)(b_6^2 - a_6^2)/8 - 2^3, a_6^2 + k(b_6^2 - a_6^2)/8 - 2^3 - 8^{-2}/16\}, \\
a_6^2 + k(b_6^2 - a_6^2)/8 - 2^3, \quad k \in \{1, \dots, 8\}, \\
\min\{c_6^2 + (k-1)(a_7^2 - c_6^2)/8 - 2^3 + 8^{-2}/16, c_6^2 + k(a_7^2 - c_6^2)/8 - 2^3\}, \\
\max\{c_6^2 + (k-1)(a_7^2 - c_6^2)/8 - 2^3, c_6^2 + k(a_7^2 - c_6^2)/8 - 2^3 - 8^{-2}/16\}, \\
c_6^2 + k(a_7^2 - c_6^2)/8 - 2^3, \quad k \in \{1, \dots, 8\}.
\end{gathered}$$

We put

$$a_1^4 := d_1^3, \quad b_1^4 := d_2^3, \quad c_1^4 := d_3^3, \quad \dots \quad c_{7\cdot 8^2}^4 := d_{21\cdot 8^2}^3, \quad a_{7\cdot 8^2+1}^4 := -2.$$

We recall that C can be increasing or decreasing only on

$$[a_j^4, b_j^4], \quad [c_j^4, a_{j+1}^4], \quad j \in \{1, \dots, 7\cdot 8^2\}.$$

We proceed further in the same way (as in the third and the fourth step). In the $2n$-th step, we define continuous C so that

$$\|C(t) - C(t + 2^{2n-1})\| < \frac{\varepsilon}{2^{n+1}}, \quad t \in [-2^{2n-1} - \dots - 2, -2^{2n-3} - \dots - 2).$$

We get the nondecreasing sequence $\{d_l^{2n-1}\}$ from

$$\begin{gathered}
a_j^{2n-1} - 2^{2n-1}, \quad b_j^{2n-1} - 2^{2n-1}, \quad c_j^{2n-1} - 2^{2n-1},\\
\min\{a_j^{2n-1} - 2^{2n-1} + 8^{2-2n}/16, b_j^{2n-1} - 2^{2n-1}\},\\
\max\{a_j^{2n-1} - 2^{2n-1}, b_j^{2n-1} - 2^{2n-1} - 8^{2-2n}/16\},\\
\min\{c_j^{2n-1} - 2^{2n-1}, b_j^{2n-1} - 2^{2n-1} + 8^{2-2n}/16\},\\
\max\{c_j^{2n-1} - 2^{2n-1} - 8^{2-2n}/16, b_j^{2n-1} - 2^{2n-1}\},\\
\min\{c_j^{2n-1} - 2^{2n-1} + 8^{2-2n}/16, a_{j+1}^{2n-1} - 2^{2n-1}\},\\
\max\{c_j^{2n-1} - 2^{2n-1}, a_{j+1}^{2n-1} - 2^{2n-1} - 8^{2-2n}/16\}
\end{gathered}$$

for $j \in \{1, \dots, 7\cdot 8^{2n-3}\}$, from

$$\begin{gathered}
c_1^1 - 2^{2n-1}, \quad \min\{c_1^1 - 2^{2n-1}, b_1^1 - 2^{2n-1} + 8^{2-2n}/16\},\\
\max\{c_1^1 - 2^{2n-1} - 8^{2-2n}/16, b_1^1 - 2^{2n-1}\},\\
c_2^1 - 2^{2n-1}, \quad \min\{c_2^1 - 2^{2n-1}, b_2^1 - 2^{2n-1} + 8^{2-2n}/16\},\\
\max\{c_2^1 - 2^{2n-1} - 8^{2-2n}/16, b_2^1 - 2^{2n-1}\},\\
\min\{a_1^1 + (k-1)(b_1^1 - a_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1} + 8^{2-2n}/16,\\
a_1^1 + k(b_1^1 - a_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1}\},\\
\max\{a_1^1 + (k-1)(b_1^1 - a_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1},\\
a_1^1 + k(b_1^1 - a_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1} - 8^{2-2n}/16\},\\
a_1^1 + k(b_1^1 - a_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1}, \quad k \in \{1, \dots, 8\cdot 4^{2n-3}\},\\
\min\{c_1^1 + (k-1)(a_2^1 - c_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1} + 8^{2-2n}/16,\\
c_1^1 + k(a_2^1 - c_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1}\},\\
\max\{c_1^1 + (k-1)(a_2^1 - c_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1},\\
c_1^1 + k(a_2^1 - c_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1} - 8^{2-2n}/16\},\\
c_1^1 + k(a_2^1 - c_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1}, \quad k \in \{1, \dots, 8\cdot 4^{2n-3}\},\\
\min\{a_2^1 + (k-1)(b_2^1 - a_2^1)/(8\cdot 4^{2n-3}) - 2^{2n-1} + 8^{2-2n}/16,\\
a_2^1 + k(b_2^1 - a_2^1)/(8\cdot 4^{2n-3}) - 2^{2n-1}\},\\
\max\{a_2^1 + (k-1)(b_2^1 - a_2^1)/(8\cdot 4^{2n-3}) - 2^{2n-1},\\
a_2^1 + k(b_2^1 - a_2^1)/(8\cdot 4^{2n-3}) - 2^{2n-1} - 8^{2-2n}/16\},\\
a_2^1 + k(b_2^1 - a_2^1)/(8\cdot 4^{2n-3}) - 2^{2n-1}, \quad k \in \{1, \dots, 8\cdot 4^{2n-3}\},
\end{gathered}$$

$$
\begin{gathered}
\min\{c_2^1+(k-1)(a_3^1-c_2^1)/(8\cdot 4^{2n-3})-2^{2n-1}+8^{2-2n}/16,\\
c_2^1+k(a_3^1-c_2^1)/(8\cdot 4^{2n-3})-2^{2n-1}\},\\
\max\{c_2^1+(k-1)(a_3^1-c_2^1)/(8\cdot 4^{2n-3})-2^{2n-1},\\
c_2^1+k(a_3^1-c_2^1)/(8\cdot 4^{2n-3})-2^{2n-1}-8^{2-2n}/16\},\\
c_2^1+k(a_3^1-c_2^1)/(8\cdot 4^{2n-3})-2^{2n-1},\quad k\in\{1,\dots,8\cdot 4^{2n-3}\},\\
c_1^2-2^{2n-1},\quad \min\{c_1^2-2^{2n-1},b_1^2-2^{2n-1}+8^{2-2n}/16\},\\
\max\{c_1^2-2^{2n-1}-8^{2-2n}/16,b_1^2-2^{2n-1}\},\\
\dots\\
c_6^2-2^{2n-1},\quad \min\{c_6^2-2^{2n-1},b_6^2-2^{2n-1}+8^{2-2n}/16\},\\
\max\{c_6^2-2^{2n-1}-8^{2-2n}/16,b_6^2-2^{2n-1}\},\\
\min\{a_1^2+(k-1)(b_1^2-a_1^2)/(8\cdot 4^{2n-4})-2^{2n-1}+8^{2-2n}/16,\\
a_1^2+k(b_1^2-a_1^2)/(8\cdot 4^{2n-4})-2^{2n-1}\},\\
\max\{a_1^2+(k-1)(b_1^2-a_1^2)/(8\cdot 4^{2n-4})-2^{2n-1},\\
a_1^2+k(b_1^2-a_1^2)/(8\cdot 4^{2n-4})-2^{2n-1}-8^{2-2n}/16\},\\
a_1^2+k(b_1^2-a_1^2)/(8\cdot 4^{2n-4})-2^{2n-1},\quad k\in\{1,\dots,8\cdot 4^{2n-4}\},\\
\min\{c_1^2+(k-1)(a_2^2-c_1^2)/(8\cdot 4^{2n-4})-2^{2n-1}+8^{2-2n}/16,\\
c_1^2+k(a_2^2-c_1^2)/(8\cdot 4^{2n-4})-2^{2n-1}\},\\
\max\{c_1^2+(k-1)(a_2^2-c_1^2)/(8\cdot 4^{2n-4})-2^{2n-1},\\
c_1^2+k(a_2^2-c_1^2)/(8\cdot 4^{2n-4})-2^{2n-1}-8^{2-2n}/16\},\\
c_1^2+k(a_2^2-c_1^2)/(8\cdot 4^{2n-4})-2^{2n-1},\quad k\in\{1,\dots,8\cdot 4^{2n-4}\},\\
\dots\\
\min\{a_6^2+(k-1)(b_6^2-a_6^2)/(8\cdot 4^{2n-4})-2^{2n-1}+8^{2-2n}/16,\\
a_6^2+k(b_6^2-a_6^2)/(8\cdot 4^{2n-4})-2^{2n-1}\},\\
\max\{a_6^2+(k-1)(b_6^2-a_6^2)/(8\cdot 4^{2n-4})-2^{2n-1},\\
a_6^2+k(b_6^2-a_6^2)/(8\cdot 4^{2n-4})-2^{2n-1}-8^{2-2n}/16\},\\
a_6^2+k(b_6^2-a_6^2)/(8\cdot 4^{2n-4})-2^{2n-1},\quad k\in\{1,\dots,8\cdot 4^{2n-4}\},\\
\min\{c_6^2+(k-1)(a_7^2-c_6^2)/(8\cdot 4^{2n-4})-2^{2n-1}+8^{2-2n}/16,\\
c_6^2+k(a_7^2-c_6^2)/(8\cdot 4^{2n-4})-2^{2n-1}\},\\
\max\{c_6^2+(k-1)(a_7^2-c_6^2)/(8\cdot 4^{2n-4})-2^{2n-1},\\
c_6^2+k(a_7^2-c_6^2)/(8\cdot 4^{2n-4})-2^{2n-1}-8^{2-2n}/16\},\\
c_6^2+k(a_7^2-c_6^2)/(8\cdot 4^{2n-4})-2^{2n-1},\quad k\in\{1,\dots,8\cdot 4^{2n-4}\},\\
\dots\\
c_1^{2n-2}-2^{2n-1},\quad \min\{c_1^{2n-2}-2^{2n-1},b_1^{2n-2}-2^{2n-1}+8^{2-2n}/16\},\\
\max\{c_1^{2n-2}-2^{2n-1}-8^{2-2n}/16,b_1^{2n-2}-2^{2n-1}\},\\
\dots
\end{gathered}
$$

$$
\begin{gathered}
c^{2n-2}_{7\cdot 8^{2n-4}} - 2^{2n-1}, \quad \min\{c^{2n-2}_{7\cdot 8^{2n-4}} - 2^{2n-1}, b^{2n-2}_{7\cdot 8^{2n-4}} - 2^{2n-1} + 8^{2-2n}\},\\
\max\{c^{2n-2}_{7\cdot 8^{2n-4}} - 2^{2n-1} - 8^{2-2n}/16, b^{2n-2}_{7\cdot 8^{2n-4}} - 2^{2n-1}\},\\
\min\{a^{2n-2}_1 + (k-1)(b^{2n-2}_1 - a^{2n-2}_1)/8 - 2^{2n-1} + 8^{2-2n}/16,\\
a^{2n-2}_1 + k(b^{2n-2}_1 - a^{2n-2}_1)/8 - 2^{2n-1}\},\\
\max\{a^{2n-2}_1 + (k-1)(b^{2n-2}_1 - a^{2n-2}_1)/8 - 2^{2n-1},\\
a^{2n-2}_1 + k(b^{2n-2}_1 - a^{2n-2}_1)/8 - 2^{2n-1} - 8^{2-2n}/16\},\\
a^{2n-2}_1 + k(b^{2n-2}_1 - a^{2n-2}_1)/8 - 2^{2n-1}, \quad k \in \{1, \dots, 8\},\\
\min\{c^{2n-2}_1 + (k-1)(a^{2n-2}_2 - c^{2n-2}_1)/8 - 2^{2n-1} + 8^{2-2n}/16,\\
c^{2n-2}_1 + k(a^{2n-2}_2 - c^{2n-2}_1)/8 - 2^{2n-1}\},\\
\max\{c^{2n-2}_1 + (k-1)(a^{2n-2}_2 - c^{2n-2}_1)/8 - 2^{2n-1},\\
c^{2n-2}_1 + k(a^{2n-2}_2 - c^{2n-2}_1)/8 - 2^{2n-1} - 8^{2-2n}/16\},\\
c^{2n-2}_1 + k(a^{2n-2}_2 - c^{2n-2}_1)/8 - 2^{2n-1}, \quad k \in \{1, \dots, 8\},\\
\dots\\
\min\{a^{2n-2}_{7\cdot 8^{2n-4}} + (k-1)(b^{2n-2}_{7\cdot 8^{2n-4}} - a^{2n-2}_{7\cdot 8^{2n-4}})/8 - 2^{2n-1} + 8^{2-2n}/16,\\
a^{2n-2}_{7\cdot 8^{2n-4}} + k(b^{2n-2}_{7\cdot 8^{2n-4}} - a^{2n-2}_{7\cdot 8^{2n-4}})/8 - 2^{2n-1}\},\\
\max\{a^{2n-2}_{7\cdot 8^{2n-4}} + (k-1)(b^{2n-2}_{7\cdot 8^{2n-4}} - a^{2n-2}_{7\cdot 8^{2n-4}})/8 - 2^{2n-1},\\
a^{2n-2}_{7\cdot 8^{2n-4}} + k(b^{2n-2}_{7\cdot 8^{2n-4}} - a^{2n-2}_{7\cdot 8^{2n-4}})/8 - 2^{2n-1} - 8^{2-2n}/16\},\\
a^{2n-2}_{7\cdot 8^{2n-4}} + k(b^{2n-2}_{7\cdot 8^{2n-4}} - a^{2n-2}_{7\cdot 8^{2n-4}})/8 - 2^{2n-1}, \quad k \in \{1, \dots, 8\},\\
\min\{c^{2n-2}_{7\cdot 8^{2n-4}} + (k-1)(a^{2n-2}_{7\cdot 8^{2n-4}+1} - c^{2n-2}_{7\cdot 8^{2n-4}})/8 - 2^{2n-1} + 8^{2-2n}/16,\\
c^{2n-2}_{7\cdot 8^{2n-4}} + k(a^{2n-2}_{7\cdot 8^{2n-4}+1} - c^{2n-2}_{7\cdot 8^{2n-4}})/8 - 2^{2n-1}\},\\
\max\{c^{2n-2}_{7\cdot 8^{2n-4}} + (k-1)(a^{2n-2}_{7\cdot 8^{2n-4}+1} - c^{2n-2}_{7\cdot 8^{2n-4}})/8 - 2^{2n-1},\\
c^{2n-2}_{7\cdot 8^{2n-4}} + k(a^{2n-2}_{7\cdot 8^{2n-4}+1} - c^{2n-2}_{7\cdot 8^{2n-4}})/8 - 2^{2n-1} - 8^{2-2n}/16\},\\
c^{2n-2}_{7\cdot 8^{2n-4}} + k(a^{2n-2}_{7\cdot 8^{2n-4}+1} - c^{2n-2}_{7\cdot 8^{2n-4}})/8 - 2^{2n-1}, \quad k \in \{1, \dots, 8\},
\end{gathered}
$$

and from a number of $b^1_1 - 2^{2n-1}$ such that the total number of d^{2n-1}_l is $21 \cdot 8^{2n-2}$. We denote

$$
\begin{gathered}
a^{2n}_1 := d^{2n-1}_1, \quad b^{2n}_1 := d^{2n-1}_2, \quad c^{2n}_1 := d^{2n-1}_3, \quad \dots\\
c^{2n-1}_{7\cdot 8^{2n-2}} := d^3_{21\cdot 8^{2n-2}}, \quad a^{2n-1}_{7\cdot 8^{2n-2}+1} := -2^{2n-3} - \dots - 2.
\end{gathered}
$$

In the $(2n+1)$-th step, we define continuous C so that

$$
\|C(t) - C(t - 2^{2n})\| < \frac{\varepsilon}{2^{n+2}}, \quad t \in (2 + \dots + 2^{2n-2}, 2 + \dots + 2^{2n}].
$$

Now C has constant values on $[b^{2n+1}_j, c^{2n+1}_j]$, $j \in \{1, \dots, 7 \cdot 8^{2n-1}\}$, where we put

$$
a^{2n+1}_1 := 2 + 2^2 + \dots + 2^{2n-2}
$$

and we obtain

$$
b^{2n+1}_1, \quad c^{2n+1}_1, \quad a^{2n+1}_2, \quad \dots \quad c^{2n+1}_{7\cdot 8^{2n-1}}, \quad a^{2n+1}_{7\cdot 8^{2n-1}+1}
$$

from the nondecreasing sequence of

$$
a^{2n}_{j+1} + 2^{2n}, \quad b^{2n}_j + 2^{2n}, \quad c^{2n}_j + 2^{2n},
$$

$$\min\{a_j^{2n}+2^{2n}+8^{1-2n}/16, b_j^{2n}+2^{2n}\},\quad \max\{a_j^{2n}+2^{2n}, b_j^{2n}+2^{2n}-8^{1-2n}/16\},$$
$$\min\{c_j^{2n}+2^{2n}, b_j^{2n}+2^{2n}+8^{1-2n}/16\},\quad \max\{c_j^{2n}+2^{2n}-8^{1-2n}/16, b_j^{2n}+2^{2n}\},$$
$$\min\{c_j^{2n}+2^{2n}+8^{1-2n}/16, a_{j+1}^{2n}+2^{2n}\},\quad \max\{c_j^{2n}+2^{2n}, a_{j+1}^{2n}+2^{2n}-8^{1-2n}/16\}$$

for $j \in \{1,\dots,7\cdot 8^{2n-2}\}$ and

$$\begin{gathered}
c_1^1+2^{2n},\quad \min\{c_1^1+2^{2n}, b_1^1+2^{2n}+8^{1-2n}/16\},\\
\max\{c_1^1+2^{2n}-8^{1-2n}/16, b_1^1+2^{2n}\},\\
c_2^1+2^{2n},\quad \min\{c_2^1+2^{2n}, b_2^1+2^{2n}+8^{1-2n}/16\},\\
\max\{c_2^1+2^{2n}-8^{1-2n}/16, b_2^1+2^{2n}\},\\
\min\{a_1^1+(k-1)(b_1^1-a_1^1)/(8\cdot 4^{2n-2})+2^{2n}+8^{1-2n}/16,\\
a_1^1+k(b_1^1-a_1^1)/(8\cdot 4^{2n-2})+2^{2n}\},\\
\max\{a_1^1+(k-1)(b_1^1-a_1^1)/(8\cdot 4^{2n-2})+2^{2n},\\
a_1^1+k(b_1^1-a_1^1)/(8\cdot 4^{2n-2})+2^{2n}-8^{1-2n}/16\},\\
a_1^1+k(b_1^1-a_1^1)/(8\cdot 4^{2n-2})+2^{2n},\quad k\in\{1,\dots,8\cdot 4^{2n-2}\},\\
\min\{c_1^1+(k-1)(a_2^1-c_1^1)/(8\cdot 4^{2n-2})+2^{2n}+8^{1-2n}/16,\\
c_1^1+k(a_2^1-c_1^1)/(8\cdot 4^{2n-2})+2^{2n}\},\\
\max\{c_1^1+(k-1)(a_2^1-c_1^1)/(8\cdot 4^{2n-2})+2^{2n},\\
c_1^1+k(a_2^1-c_1^1)/(8\cdot 4^{2n-2})+2^{2n}-8^{1-2n}/16\},\\
c_1^1+k(a_2^1-c_1^1)/(8\cdot 4^{2n-2})+2^{2n},\quad k\in\{1,\dots,8\cdot 4^{2n-2}\},\\
\min\{a_2^1+(k-1)(b_2^1-a_2^1)/(8\cdot 4^{2n-2})+2^{2n}+8^{1-2n}/16,\\
a_2^1+k(b_2^1-a_2^1)/(8\cdot 4^{2n-2})+2^{2n}\},\\
\max\{a_2^1+(k-1)(b_2^1-a_2^1)/(8\cdot 4^{2n-2})+2^{2n},\\
a_2^1+k(b_2^1-a_2^1)/(8\cdot 4^{2n-2})+2^{2n}-8^{1-2n}/16\},\\
a_2^1+k(b_2^1-a_2^1)/(8\cdot 4^{2n-2})+2^{2n},\quad k\in\{1,\dots,8\cdot 4^{2n-2}\},\\
\min\{c_2^1+(k-1)(a_3^1-c_2^1)/(8\cdot 4^{2n-2})+2^{2n}+8^{1-2n}/16,\\
c_2^1+k(a_3^1-c_2^1)/(8\cdot 4^{2n-2})+2^{2n}\},\\
\max\{c_2^1+(k-1)(a_3^1-c_2^1)/(8\cdot 4^{2n-2})+2^{2n},\\
c_2^1+k(a_3^1-c_2^1)/(8\cdot 4^{2n-2})+2^{2n}-8^{1-2n}/16\},\\
c_2^1+k(a_3^1-c_2^1)/(8\cdot 4^{2n-2})+2^{2n},\quad k\in\{1,\dots,8\cdot 4^{2n-2}\},\\
c_1^2+2^{2n},\quad \min\{c_1^2+2^{2n}, b_1^2+2^{2n}+8^{1-2n}/16\},\\
\max\{c_1^2+2^{2n}-8^{1-2n}/16, b_1^2+2^{2n}\},\\
\dots\\
c_6^2+2^{2n},\quad \min\{c_6^2+2^{2n}, b_6^2+2^{2n}+8^{1-2n}/16\},\\
\max\{c_6^2+2^{2n}-8^{1-2n}/16, b_6^2+2^{2n}\},\\
\min\{a_1^2+(k-1)(b_1^2-a_1^2)/(8\cdot 4^{2n-3})+2^{2n}+8^{1-2n}/16,\\
a_1^2+k(b_1^2-a_1^2)/(8\cdot 4^{2n-3})+2^{2n}\},
\end{gathered}$$

$$
\begin{gathered}
\max\{a_1^2+(k-1)(b_1^2-a_1^2)/(8\cdot 4^{2n-3})+2^{2n},\\
a_1^2+k(b_1^2-a_1^2)/(8\cdot 4^{2n-3})+2^{2n}-8^{1-2n}/16\},\\
a_1^2+k(b_1^2-a_1^2)/(8\cdot 4^{2n-3})+2^{2n},\quad k\in\{1,\dots,8\cdot 4^{2n-3}\},\\
\min\{c_1^2+(k-1)(a_2^2-c_1^2)/(8\cdot 4^{2n-3})+2^{2n}+8^{1-2n}/16,\\
c_1^2+k(a_2^2-c_1^2)/(8\cdot 4^{2n-3})+2^{2n}\},\\
\max\{c_1^2+(k-1)(a_2^2-c_1^2)/(8\cdot 4^{2n-3})+2^{2n},\\
c_1^2+k(a_2^2-c_1^2)/(8\cdot 4^{2n-3})+2^{2n}-8^{1-2n}/16\},\\
c_1^2+k(a_2^2-c_1^2)/(8\cdot 4^{2n-3})+2^{2n},\quad k\in\{1,\dots,8\cdot 4^{2n-3}\},\\
\dots\\
\min\{a_6^2+(k-1)(b_6^2-a_6^2)/(8\cdot 4^{2n-3})+2^{2n}+8^{1-2n}/16,\\
a_6^2+k(b_6^2-a_6^2)/(8\cdot 4^{2n-3})+2^{2n}\},\\
\max\{a_6^2+(k-1)(b_6^2-a_6^2)/(8\cdot 4^{2n-3})+2^{2n},\\
a_6^2+k(b_6^2-a_6^2)/(8\cdot 4^{2n-3})+2^{2n}-8^{1-2n}/16\},\\
a_6^2+k(b_6^2-a_6^2)/(8\cdot 4^{2n-3})+2^{2n},\quad k\in\{1,\dots,8\cdot 4^{2n-3}\},\\
\min\{c_6^2+(k-1)(a_7^2-c_6^2)/(8\cdot 4^{2n-3})+2^{2n}+8^{1-2n}/16,\\
c_6^2+k(a_7^2-c_6^2)/(8\cdot 4^{2n-3})+2^{2n}\},\\
\max\{c_6^2+(k-1)(a_7^2-c_6^2)/(8\cdot 4^{2n-3})+2^{2n},\\
c_6^2+k(a_7^2-c_6^2)/(8\cdot 4^{2n-3})+2^{2n}-8^{1-2n}/16\},\\
c_6^2+k(a_7^2-c_6^2)/(8\cdot 4^{2n-3})+2^{2n},\quad k\in\{1,\dots,8\cdot 4^{2n-3}\},\\
\dots\\
c_1^{2n-1}+2^{2n},\quad \min\{c_1^{2n-1}+2^{2n},b_1^{2n-1}+2^{2n}+8^{1-2n}/16\},\\
\max\{c_1^{2n-1}+2^{2n}-8^{1-2n}/16,b_1^{2n-1}+2^{2n}\},\\
\dots\\
c_{7\cdot 8^{2n-3}}^{2n-1}+2^{2n},\quad \min\{c_{7\cdot 8^{2n-3}}^{2n-1}+2^{2n},b_{7\cdot 8^{2n-3}}^{2n-1}+2^{2n}+8^{1-2n}/16\},\\
\max\{c_{7\cdot 8^{2n-3}}^{2n-1}+2^{2n}-8^{1-2n}/16,b_{7\cdot 8^{2n-3}}^{2n-1}+2^{2n}\},\\
\min\{a_1^{2n-1}+(k-1)(b_1^{2n-1}-a_1^{2n-1})/8+2^{2n}+8^{1-2n}/16,\\
a_1^{2n-1}+k(b_1^{2n-1}-a_1^{2n-1})/8+2^{2n}\},\\
\max\{a_1^{2n-1}+(k-1)(b_1^{2n-1}-a_1^{2n-1})/8+2^{2n},\\
a_1^{2n-1}+k(b_1^{2n-1}-a_1^{2n-1})/8+2^{2n}-8^{1-2n}/16\},\\
a_1^{2n-1}+k(b_1^{2n-1}-a_1^{2n-1})/8+2^{2n},\quad k\in\{1,\dots,8\},\\
\min\{c_1^{2n-1}+(k-1)(a_2^{2n-1}-c_1^{2n-1})/8+2^{2n}+8^{1-2n}/16,\\
c_1^{2n-1}+k(a_2^{2n-1}-c_1^{2n-1})/8+2^{2n}\},\\
\max\{c_1^{2n-1}+(k-1)(a_2^{2n-1}-c_1^{2n-1})/8+2^{2n},\\
c_1^{2n-1}+k(a_2^{2n-1}-c_1^{2n-1})/8+2^{2n}-8^{1-2n}/16\},\\
c_1^{2n-1}+k(a_2^{2n-1}-c_1^{2n-1})/8+2^{2n},\quad k\in\{1,\dots,8\},
\end{gathered}
$$

$$
\begin{gathered}
\dots \\
\min\{a^{2n-1}_{7\cdot 8^{2n-3}} + (k-1)(b^{2n-1}_{7\cdot 8^{2n-3}} - a^{2n-1}_{7\cdot 8^{2n-3}})/8 + 2^{2n} + 8^{1-2n}/16, \\
a^{2n-1}_{7\cdot 8^{2n-3}} + k(b^{2n-1}_{7\cdot 8^{2n-3}} - a^{2n-1}_{7\cdot 8^{2n-3}})/8 + 2^{2n}\}, \\
\max\{a^{2n-1}_{7\cdot 8^{2n-3}} + (k-1)(b^{2n-1}_{7\cdot 8^{2n-3}} - a^{2n-1}_{7\cdot 8^{2n-3}})/8 + 2^{2n}, \\
a^{2n-1}_{7\cdot 8^{2n-3}} + k(b^{2n-1}_{7\cdot 8^{2n-3}} - a^{2n-1}_{7\cdot 8^{2n-3}})/8 + 2^{2n} - 8^{1-2n}/16\}, \\
a^{2n-1}_{7\cdot 8^{2n-3}} + k(b^{2n-1}_{7\cdot 8^{2n-3}} - a^{2n-1}_{7\cdot 8^{2n-3}})/8 + 2^{2n}, \quad k \in \{1,\dots,8\}, \\
\min\{c^{2n-1}_{7\cdot 8^{2n-3}} + (k-1)(a^{2n-1}_{7\cdot 8^{2n-3}+1} - c^{2n-1}_{7\cdot 8^{2n-3}})/8 + 2^{2n} + 8^{1-2n}/16, \\
c^{2n-1}_{7\cdot 8^{2n-3}} + k(a^{2n-1}_{7\cdot 8^{2n-3}+1} - c^{2n-1}_{7\cdot 8^{2n-3}})/8 + 2^{2n}\}, \\
\max\{c^{2n-1}_{7\cdot 8^{2n-3}} + (k-1)(a^{2n-1}_{7\cdot 8^{2n-3}+1} - c^{2n-1}_{7\cdot 8^{2n-3}})/8 + 2^{2n}, \\
c^{2n-1}_{7\cdot 8^{2n-3}} + k(a^{2n-1}_{7\cdot 8^{2n-3}+1} - c^{2n-1}_{7\cdot 8^{2n-3}})/8 + 2^{2n} - 8^{1-2n}/16\}, \\
c^{2n-1}_{7\cdot 8^{2n-3}} + k(a^{2n-1}_{7\cdot 8^{2n-3}+1} - c^{2n-1}_{7\cdot 8^{2n-3}})/8 + 2^{2n}, \quad k \in \{1,\dots,8\},
\end{gathered}
$$

and the corresponding number of $b^1_1 + 2^{2n}$.

Using this construction, we obtain a continuous function C on $\mathbb{R}$. From Theorem 3.1 it follows that C is almost periodic. Since

$$
\begin{gathered}
\|C(t)\| = 0, \quad t \in [0,1], \quad \|C(t)\| < \varepsilon/4, \quad t \in (1,2], \\
\|C(t) - C(t+2)\| < \varepsilon/4, \quad t \in [-2,0), \quad \|C(t) - C(t-4)\| < \varepsilon/8, \quad t \in (2,6], \\
\dots \\
\|C(t) - C(t+2^{2n-1})\| < \varepsilon/2^{n+1}, \\
t \in [-2^{2n-1} - \dots - 2^3 - 2, -2^{2n-3} - \dots - 2^3 - 2), \\
\|C(t) - C(t-2^{2n})\| < \varepsilon/2^{n+2}, \quad t \in (2 + 2^2 + \dots + 2^{2n-2}, 2 + 2^2 + \dots + 2^{2n}],
\end{gathered}
$$

we see that

$$\|C(t)\| < \sum_{j=1}^{\infty} \frac{2\varepsilon}{2^{j+1}} = \varepsilon, \quad t \in \mathbb{R}. \tag{4.2}$$

We denote

$$I_n := [2 + 2^2 + \dots + 2^{2n-2}, 2 + 2^2 + \dots + 2^{2n}].$$

We will prove that we can choose constant values of $C(t)$, $t \in I_n$ on subintervals with the total length denoted by r_{2n+1} which is grater than 2^{2n-1} for all $n \in \mathbb{N}$. We can choose C on

$$
\begin{gathered}
[4 - 2 + 1/16 + 8^{-1}/16, 4 - 2 + 1 - 1/16 - 8^{-1}/16] \subset [2,6], \\
[4 - 2 + 1 + 1/4 + 1/16 + 8^{-1}/16, 4 - 2 + 1 + 3/4 - 1/16 - 8^{-1}/16] \subset [2,6], \\
[4 + 8^{-1}/16, 4 + 1 - 8^{-1}/16], [4 + 1 + 1/4 + 8^{-1}/16, 4 + 1 + 3/4 - 8^{-1}/16] \subset [2,6].
\end{gathered}
$$

Hence,

$$r_3 \ge 55/64 + 23/64 + 63/64 + 31/64 = 43/16; \tag{4.3}$$

i.e., the statement is valid for $n = 1$. We use the induction principle with respect to n. Assume that the statement is true for $1, 2, \dots, n-1$ and prove it for n. Without loss of the generality (consider the below given process), we can also assume that the estimation $r_{2j} > 2^{2(j-1)}$ is valid for $j \in \{1,\dots,n\}$ (note that $r_2 = 5/4 > 2^0$) if we use analogous notation.

In view of the construction, we see that we can choose C on any interval

$$[s + 2^{2n} + 8^{1-2n}/16, t + 2^{2n} - 8^{1-2n}/16]$$

if we can choose C on $[s,t]$, where $s = b_j^l < c_j^l = t$, $l < 2n+1$. Especially, we can choose C on

$$\begin{gathered}
[2^{2n} + 8^{1-2n}/16, 1 + 2^{2n} - 8^{1-2n}/16],\\
[1 + 1/4 + 2^{2n} + 8^{1-2n}/16, 1 + 3/4 + 2^{2n} - 8^{1-2n}/16],\\
[-2 + 1/16 + 2^{2n} + 8^{1-2n}/16, -2 + 1 - 1/16 + 2^{2n} - 8^{1-2n}/16],\\
[-2 + 1 + 1/4 + 1/16 + 2^{2n} + 8^{1-2n}/16, -2 + 1 + 3/4 - 1/16 + 2^{2n} - 8^{1-2n}/16]
\end{gathered}$$

and on less than $7 \cdot 8^{2n-1} - 4$ subintervals of I_n. Expressing

$$\begin{aligned}
I_n =& [0 + 2^{2n}, 1 + 2^{2n}] \cup [1 + 2^{2n}, 2 + 2^{2n}] \cup [-2 + 2^{2n}, 0 + 2^{2n}] \cup \dots\\
&\cup [2 + 2^2 + \dots + 2^{2n-4} + 2^{2n}, 2 + 2^2 + \dots + 2^{2n-2} + 2^{2n}]\\
&\cup [-2^{2n-1} - \dots - 2^3 - 2 + 2^{2n}, -2^{2n-3} - \dots - 2^3 - 2 + 2^{2n}]
\end{aligned} \tag{4.4}$$

and using the induction hypothesis, the construction, and (4.3), we obtain that we can choose C on intervals of the lengths grater than or equal to

$$\begin{gathered}
1 - 2 \cdot 8^{1-2n}/16, \quad 1/2 - 2 \cdot 8^{1-2n}/16,\\
1 - 1/8 - 2 \cdot 8^{1-2n}/16, \quad 1/2 - 1/8 - 2 \cdot 8^{1-2n}/16,\\
43/16 + 2^2 + 2^3 + \dots + 2^{2n-3} + 2^{2n-2} - 2 \cdot 8^{1-2n}/16 \cdot (7 \cdot 8^{2n-1} - 4).
\end{gathered}$$

Summing, we obtain

$$r_{2n+1} \geq 1 + \frac{1}{2} + \frac{7}{8} + \frac{3}{8} + \frac{11}{16} + 2^{2n-1} - 2 - \frac{7}{8} > 2^{2n-1}, \tag{4.5}$$

which is the above statement. Analogously, we can prove

$$r_{2n} > 2^{2n-2}, \quad n \in \mathbb{N}. \tag{4.6}$$

Now we describe the principal fundamental matrix X_C on I_n for arbitrary $n \in \mathbb{N}$. Since C is constant a has the form $\operatorname{diag}[ia, ia, \dots, ia]$ for some $a \in \mathbb{R}$ on each interval $[b_j^{2n+1}, c_j^{2n+1}]$, $j \in \{1, \dots, 6 \cdot 4^{2n-1}\}$, from

$$X_C(t_2) - X_C(t_1) = \int_{t_1}^{t_2} C(\tau) X_C(\tau)\, d\tau, \quad t_1, t_2 \in \mathbb{R},$$

we obtain

$$\begin{aligned}
&\|X_C(t) - X_C^{2n+1}(t)\| \leq\\
&\sum_{j=1}^{k} \Big(\int_{a_j^{2n+1}}^{b_j^{2n+1}} \|C(\tau) X_C(\tau)\| d\tau + \int_{c_j^{2n+1}}^{a_{j+1}^{2n+1}} \|C(\tau) X_C(\tau)\| d\tau \Big)
\end{aligned} \tag{4.7}$$

if $t \leq a_{k+1}^{2n+1}$, $t \in I_n$, where

$$\begin{gathered}
X_C^{2n+1}(t) := X_C(2 + 2^2 + \dots + 2^{2n-2}), \quad t \in [2 + 2^2 + \dots + 2^{2n-2}, b_1^{2n+1}],\\
X_C^{2n+1}(t) := \exp(C(b_1^{2n+1})(t - b_1^{2n+1})) \cdot X_C^{2n+1}(b_1^{2n+1}), \ t \in (b_1^{2n+1}, c_1^{2n+1}],\\
X_C^{2n+1}(t) := {X_C}^{2n+1}(c_1^{2n+1}), \quad t \in (c_1^{2n+1}, b_2^{2n+1}],\\
\dots
\end{gathered}$$

$$\begin{aligned} X_C^{2n+1}(t) &:= \exp(C(b^{2n+1}_{7\cdot 8^{2n-1}})(t - b^{2n+1}_{7\cdot 8^{2n-1}})) \cdot X_C^{2n+1}(b^{2n+1}_{7\cdot 8^{2n-1}}), \\ &\quad t \in (b^{2n+1}_{7\cdot 8^{2n-1}}, c^{2n+1}_{7\cdot 8^{2n-1}}], \\ X_C^{2n+1}(t) &:= X_C^{2n+1}(c^{2n+1}_{7\cdot 8^{2n-1}}), \quad t \in (c^{2n+1}_{7\cdot 8^{2n-1}}, 2 + 2^2 + \dots + 2^{2n}]. \end{aligned}$$

It is seen that X_C is bounded (see also the below given, where it is shown that $X_C(t) \in U(m)$ for all t) as almost periodic C. Any interval

$$[2 + \dots + 2^{2n-2} + l - 1, 2 + \dots + 2^{2n-2} + l], \quad l \in \{1, \dots, 2^{2n}\}, \, n \in \mathbb{N}$$

contains at most 4^{2n+1} subintervals where C can be linear. Indeed, it suffices to consider the construction. We repeat that the length of each one of the considered subintervals is $8^{1-2n}/16$ which implies that the total length of them on

$$J_n^l := [2 + 2^2 + \dots + 2^{2n-2}, 2 + 2^2 + \dots + 2^{2n-2} + 2^{2n-l}], \quad l \in \{1, \dots, n\}$$

is less than 2^{1-l}. Thus (consider also (4.7)), there exists $K \in \mathbb{R}$ such that

$$\|X_C(t) - X_C^{2n+1}(t)\| \le \frac{K}{2^l}, \quad t \in J_n^l, \, l \in \{1, \dots, n\}, \, n \in \mathbb{N}. \tag{4.8}$$

From the form $\operatorname{diag}[ia(t), \dots, ia(t)]$ of all matrices $C(t)$, we see that

$$\|C(t)\| = |a(t)|, \quad t \in \mathbb{R}.$$

For simplicity, let $a(t) \ge 0$, $t \in \mathbb{R}$. Let $a_j^n \in \mathbb{R}$, $j \in \{1, \dots, n\}$ be arbitrarily chosen. Considering the construction and combining (4.5) and (4.6), we obtain that we can choose constant values of $C(t)$, $t \in [2 + \dots + 2^{2n-2} + (l-1)2^n, 2 + \dots + 2^{2n-2} + l\, 2^n]$ on subintervals with the total length grater than 2^{n-2} for each $l \in \{1, \dots, 2^n\}$ and all sufficiently large $n \in \mathbb{N}$. Since we choose C only so that

$$\|C(t) - C(t - 2^{2n})\| < \varepsilon / 2^{n+2}, \quad t \in I_n,$$

we see that we can obtain

$$X_C^{2n+1}(t_j^n) = \operatorname{diag}[\exp(ia_j^n), \dots, \exp(ia_j^n)]$$

for arbitrary t_j^n such that

$$\begin{aligned} &t_1^n \ge 2 + 2^2 + 2^4 + \dots + 2^{2n-2} + 3^n - 3^0, \quad t_2^n \ge t_1^n + 3^n - 3^1, \\ &\dots \quad t_n^n \ge t_{n-1}^n + 3^n - 3^{n-1}, \quad 2 + 2^2 + 2^4 + \dots + 2^{2n} \ge t_n^n \end{aligned} \tag{4.9}$$

because we have

$$4^n > n(3^n - 3^0) > 3^n - 3^0 > \dots > 3^n - 3^{n-1} > 2^{2n-k+1}$$

for sufficiently large $n \in \mathbb{N}$ and some $k = k(n) \in \{1, \dots, n\}$ satisfying

$$2^{2n-k-2} \cdot \varepsilon \cdot 2^{-n-2} > 2\pi.$$

We recall that we need to prove the existence of such C, given by the above construction, for which the vector valued function $X_A(t) \cdot X_C(t) \cdot u$, $t \in \mathbb{R}$ is not almost periodic for any $u \in \mathbb{C}^m$, $\|u\|_1 = 1$. Since

$$(X_A(t) X_A^*(t))' = A(t)\, X_A(t)\, X_A^*(t) - X_A(t)\, X_A^*(t)\, A(t), \quad t \in \mathbb{R}$$

and since the constant function given by I is a solution of $X' = A \cdot X - X \cdot A$, $X(0) = I$, we have $X_A(t) \in U(m)$ for all t. Thus, $X_C(t) \in U(m)$, $t \in \mathbb{R}$ as well. We add that $X_A(t) \cdot X_A^*(t) = I$, $t \in \mathbb{R}$ implies $A^*(t) + A(t) = 0$, $t \in \mathbb{R}$.

Let $c \in \mathbb{C}$, $|c| = 1$, and $N \in U(m)$ be arbitrarily given. Obviously, for any $M \in U(m)$, we can choose a number $a(M, c) \in [0, 2\pi)$ in order that all eigenvalues of matrix $P := M \cdot \operatorname{diag}[\exp(ia(M,c)), \dots, \exp(ia(M,c))]$ are not in the neighbourhood of c with a given radius which depends only on dimension m. Indeed, if M has eigenvalues $\lambda_1, \dots, \lambda_m$, then the eigenvalues of P are $\lambda_1 \exp(ia(M,c))$, ..., $\lambda_m \exp(ia(M,c))$. Considering

$$M \operatorname{diag}[\exp(ia(M,c)), \dots, \exp(ia(M,c))]u - Nu$$

and expressing vectors $u \in \mathbb{C}^m$, $|u_1| + \dots + |u_m| = 1$, as linear combinations of the eigenvectors of P, we see that

$$M \operatorname{diag}[\exp(ia(M,c)), \dots, \exp(ia(M,c))]u$$

cannot be in a neighbourhood of $N \cdot u$ for some $c \in \mathbb{C}$, $|c| = 1$. Thus (the considered multiplication of matrices and vectors is uniformly continuous), there exist $\vartheta > 0$ and $\xi > 0$ such that, for any matrices $M, N \in U(m)$, one can find $a(M,N) \in [0, 2\pi)$ satisfying

$$\begin{gathered} \|M \operatorname{diag}[\exp(i\tilde{a}), \dots, \exp(i\tilde{a})]u - Nu\|_1 > \vartheta, \\ u \in \mathbb{C}^m, \|u\|_1 = 1, \quad \tilde{a} \in (a(M,N) - \xi, a(M,N) + \xi). \end{gathered} \tag{4.10}$$

We showed that we can construct C so that we obtain

$$X_C^{2n+1}(t_j^n) = \operatorname{diag}[\exp(ia_j^n), \dots, \exp(ia_j^n)]$$

for arbitrarily given $a_j^n \in [0, 2\pi)$ and any t_j^n satisfying (4.9) if $n \in \mathbb{N}$ is sufficiently large and $j \in \{1, \dots, n\}$. Especially, for sufficiently large $n \in \mathbb{N}$ and for

$$\begin{gathered} t_1^n := 2 + 2^2 + 2^4 + \dots + 2^{2n-2} + 3^n - 3^0, \\ t_2^n := t_1^n + 3^n - 3^1, \quad \dots \quad t_n^n := t_{n-1}^n + 3^n - 3^{n-1}, \end{gathered} \tag{4.11}$$

we can choose all $X_C^{2n+1}(t_j^n)$ in the form without any conditions. Hence, we obtain diagonal matrices $X_C^{2n+1}(t_j^n)$, $j \in \{1, \dots, n\}$, given by numbers

$$\exp\Big(ia(X_A(t_j^n), X_A(t_j^n - 3^n + 3^{j-1})X_C(t_j^n - 3^n + 3^{j-1}))\Big)$$

on their diagonals.

It is seen from (4.11) that each

$$t_j^n \in [2 + 2^2 + \dots + 2^{2n-2}, 2 + 2^2 + \dots + 2^{2n-2} + n3^n].$$

Thus (see (4.8)), for any $\eta > 0$, we have

$$\|X_C(t_j^n) - X_C^{2n+1}(t_j^n)\| < \eta \tag{4.12}$$

for sufficiently large $n = n(\eta) \in \mathbb{N}$ and $j \in \{1, \dots, n\}$. From (4.10) and (4.12) it follows that

$$\|X_A(t_j^n)\, X_C(t_j^n)u - X_A(t_j^n - 3^n + 3^{j-1})X_C(t_j^n - 3^n + 3^{j-1})u\|_1 > \vartheta \tag{4.13}$$

for $u \in \mathbb{C}^m$, $\|u\|_1 = 1$, sufficiently large $n \in \mathbb{N}$, and $j \in \{1, \dots, n\}$.

By contradiction, suppose that there exists $u \in \mathbb{C}^m$, $\|u\|_1 = 1$, with the property that $X_A(t) \cdot X_C(t) \cdot u$, $t \in \mathbb{R}$ is almost periodic. Applying Theorem 2.4 for

$$\psi(t) = X_A(t)X_C(t)u, \quad t \in \mathbb{R},\ s_n = 3^n,\ n \in \mathbb{N},\ \varepsilon = \vartheta,$$

we obtain

$$\|X_A(t + 3^{n_1})X_C(t + 3^{n_1})u - X_A(t + 3^{n_2})X_C(t + 3^{n_2})u\|_1 < \vartheta, \quad t \in \mathbb{R} \tag{4.14}$$

for all n_1, n_2 from an infinite set $N(\vartheta) \subseteq \mathbb{N}$. If we rewrite (4.14) as

$$\|X_A(t)X_C(t)u - X_A(t + 3^{n_2} - 3^{n_1})X_C(t + 3^{n_2} - 3^{n_1})u\|_1 < \vartheta, \quad t \in \mathbb{R},$$

then it is easy to see that (4.13) is not valid for infinitely many $n \in \mathbb{N}$. This contradiction proves the theorem. $\square$

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