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GLOBAL DYNAMICS OF A REACTION-DIFFUSION SYSTEM

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ABSTRACT. In this work the existence of a global attractor for the semiflow of weak solutions of a two-cell Brusselator system is proved. The method of grouping estimation is exploited to deal with the challenge in proving the absorbing property and the asymptotic compactness of this type of coupled reaction-diffusion systems with cubic autocatalytic nonlinearity and linear coupling. It is proved that the Hausdorff dimension and the fractal dimension of the global attractor are finite. Moreover, the existence of an exponential attractor for this solution semiflow is shown.

1. INTRODUCTION

Consider a reaction-diffusion systems consisting of four coupled two-cell Brusselator equations associated with cubic autocatalytic kinetics [12, 17, 19, 30],

$$\frac{\partial u}{\partial t} = d_1 \Delta u + a - (b+1)u + u^2 v + D_1(w-u), \tag{1.1}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v + bu - u^2 v + D_2(z-v), \tag{1.2}$$

$$\frac{\partial w}{\partial t} = d_1 \Delta w + a - (b+1)w + w^2 z + D_1(u-w), \tag{1.3}$$

$$\frac{\partial z}{\partial t} = d_2 \Delta z + bw - w^2 z + D_2(v-z), \tag{1.4}$$

for $t > 0$, on a bounded domain $\Omega \subset \Re^n, n \le 3$, that has a locally Lipschitz continuous boundary, with the homogeneous Dirichlet boundary condition

$$u(t,x) = v(t,x) = w(t,x) = z(t,x) = 0, \quad t > 0,\ x \in \partial\Omega, \tag{1.5}$$

and an initial condition

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad w(0,x) = w_0(x), \quad z(0,x) = z_0(x), \quad x \in \Omega, \tag{1.6}$$

where d_1, d_2, a, b, D_1, and D_2 are positive constants. In this work, we shall study the asymptotic dynamics of the solution semiflow generated by this problem.

The Brusselator model is originally a system of two ordinary differential equations describing kinetics of cubic autocatalytic chemical or biochemical reactions, proposed by the scientists in the Brussels school led by the renowned Nobel Prize

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laureate (1977), Ilya Prigogine, cf. [26, 2]. Brusselator kinetics describes the following scheme of chemical reactions

$$\begin{gathered} \mathrm{A} \longrightarrow \mathrm{U}, \\ \mathrm{B} + \mathrm{U} \longrightarrow \mathrm{V} + \mathrm{D}, \\ 2\mathrm{U} + \mathrm{V} \longrightarrow 3\mathrm{U}, \\ \mathrm{U} \longrightarrow \mathrm{E}, \end{gathered}$$

where A, B, D, E, U, and V are chemical reactants or products. Let $u(t, x)$ and $v(t, x)$ be the concentrations of U and V, and assume that the concentrations of the input compounds A and B are held constant during the reaction process, denoted by a and b respectively. Then by the law of mass action and the Fick's law one obtains a system of two nonlinear reaction-diffusion equations called (diffusive) *Brusselator equations*,

$$\frac{\partial u}{\partial t} = d_1 \Delta u + u^2 v - (b+1)u + a, \tag{1.7}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v - u^2 v + bu, \tag{1.8}$$

Several known examples of autocatalysis which can be modelled by the Brusselator equations, such as ferrocyanide-iodate-sulphite reaction, chlorite-iodide-malonic acid reaction, arsenite-iodate reaction, and some enzyme catalytic reactions, cf. [1, 2, 5].

Numerous studies by numerical simulations or by mathematical analysis, especially after the seminal publications [21, 24] in 1993, have shown that the autocatalytic reaction-diffusion systems such as the Brusselator equations and the Gray-Scott equations [13, 14] exhibit rich spatial patterns (including but not restricted to Turing patterns) and complex bifurcations [1, 4, 5, 8, 27, 25, 36] as well as interesting dynamics [6, 11, 16, 20, 28, 29, 37] on 1D or 2D domains.

For Brusselator equations and the other cubic autocatalytic model equations of space dimension $n \le 3$, however, we have not seen substantial research results in the front of global dynamics until recently [38, 39, 40, 41].

In this paper, we shall prove the existence of a global attractor in the product L^2 phase space for the solution semiflow of the coupled two-cell Brusselator system (1.1)–(1.4) with homogeneous Dirichlet boundary conditions (1.5).

This study of global dynamics of such a reaction-diffusion system of two cells or two compartments consisting of four coupled components is a substantial advance from the one-cell model of two-component reaction-diffusion systems toward the biological network dynamics [12, 18]. Multi-cell or multi-compartment models generically mean the coupled ODEs or PDEs with large number of unknowns (interpreted as components in chemical kinetics or species in ecology), which appear widely in the literature of systems biology as well as cell biology. Here understandably "cell" is a generic term that may not be narrowly or directly interpreted as a biological cell. Coupled cells with diffusive reaction and mutual mass exchange are often adopted as model systems for description of processes in living cells and tissues, or in distributed chemical reactions and transport for compartmental reactors [35, 30].

In this regard, unfortunately, the problems with high dimensionality can occur and puzzle the research, when the number of molecular species in the system turns out to be very large, which makes the behavior simulation extremely difficult or computationally too inefficient. Thus theoretical research results on multi-cell dynamics can give insights to deeper exploration of various signal transductions and spatio-temporal pattern formations or chaos.

For most reaction-diffusion systems consisting of two or more equations arising from the scenarios of autocatalytic chemical reactions or biochemical activator-inhibitor reactions, such as the Brusselator equations and the coupled two-cell Brusselator systems here, the asymptotically dissipative sign condition in vector version

$$\lim_{|s|\to\infty} F(s)\cdot s \le C,$$

where $C \ge 0$ is a constant, is inherently not satisfied by the opposite-signed and coupled nonlinear terms, see (1.11) later. Besides serious challenge arises in dealing with the coupling of the two groups of variables u, v and w, z. The novel mathematical feature in this paper is to overcome this coupling obstacle and make the *a priori* estimates by a method of *grouping estimation* combined with the other techniques to show the globally dissipative and attractive dynamics.

We start with the formulation of an evolutionary equation associated with the two-cell Brusselator equations. Define the product Hilbert spaces as follows,

$$H = [L^2(\Omega)]^4, \quad E = [H_0^1(\Omega)]^4, \quad \text{and} \quad \Pi = [(H_0^1(\Omega)\cap H^2(\Omega))]^4.$$

The norm and inner-product of H or the component space $L^2(\Omega)$ will be denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. The norm of $L^p(\Omega)$ will be denoted by $\|\cdot\|_{L^p}$ if $p \ne 2$. By the Poincaré inequality and the homogeneous Dirichlet boundary condition (1.5), there is a constant $\gamma > 0$ such that

$$\|\nabla\varphi\|^2 \ge \gamma\|\varphi\|^2, \quad \text{for } \varphi \in H_0^1(\Omega) \text{ or } E, \tag{1.9}$$

and we shall take $\|\nabla\varphi\|$ to be the equivalent norm $\|\varphi\|_E$ of the space E and of the component space $H_0^1(\Omega)$. We use $|\cdot|$ to denote an absolute value or a vector norm in a Euclidean space.

It is easy to check that, by the Lumer-Phillips theorem and the analytic semigroup generation theorem [33], the linear operator

$$A = \begin{pmatrix} d_1\Delta & 0 & 0 & 0 \\ 0 & d_2\Delta & 0 & 0 \\ 0 & 0 & d_1\Delta & 0 \\ 0 & 0 & 0 & d_2\Delta \end{pmatrix} : D(A)(=\Pi) \longrightarrow H \tag{1.10}$$

is the generator of an analytic C_0-semigroup on the Hilbert space H, which will be denoted by $\{e^{At}, t \ge 0\}$. It is known [23, 33, 34] that A in (1.10) is extended to be a bounded linear operator from E to E^*. By the fact that $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ is a continuous embedding for $n \le 3$ and using the generalized Hölder inequality,

$$\|u^2 v\| \le \|u\|_{L^6}^2 \|v\|_{L^6}, \quad \|w^2 z\| \le \|w\|_{L^6}^2 \|z\|_{L^6}, \quad \text{for } u, v, w, z \in L^6(\Omega),$$

one can verify that the nonlinear mapping

$$F(g) = \begin{pmatrix} a - (b+1)u + u^2 v + D_1(w-u) \\ bu - u^2 v + D_2(z-v) \\ a - (b+1)w + w^2 z + D_1(u-w) \\ bw - w^2 z + D_2(v-z) \end{pmatrix} : E \longrightarrow H, \tag{1.11}$$

where $g = (u, v, w, z)$, is well defined on E and is locally Lipschitz continuous. Thus the initial-boundary value problem (1.1)–(1.6) is formulated into the following initial value problem,

$$\begin{gathered}\frac{dg}{dt} = Ag + F(g), \quad t > 0, \\ g(0) = g_0 = \operatorname{col}(u_0, v_0, w_0, z_0).\end{gathered} \tag{1.12}$$

where $g(t) = \operatorname{col}(u(t,\cdot), v(t,\cdot), w(t,\cdot), z(t,\cdot))$, which is simply written as $(u(t,\cdot), v(t,\cdot), w(t,\cdot), z(t,\cdot))$. We shall also simply write $g_0 = (u_0, v_0, w_0, z_0)$.

The local existence of solution to a multi-component reaction-diffusion system such as (1.12) with certain regularity requirement is not a trivial issue. There are two different approaches to get a solution. One is the mild solution provided by the "variation-of-constant formula" in terms of the associated linear semigroup $\{e^{At}\}_{t\ge 0}$ but the the parabolic theory of mild solution requires that $g_0 \in E$ instead of $g_0 \in H$ assumed here. The other is the weak solution obtained through the Galerkin approximation (the spectral approximation) and the Lions-Magenes type of compactness approach, cf. [7, 22, 33].

Definition 1.1. A function $g(t, x), (t, x) \in [0, \tau] \times \Omega$, is called a weak solution to the initial value problem of the parabolic evolutionary equation (1.12), if the following two conditions are satisfied:

(i) $\frac{d}{dt}(g, \zeta) = (Ag, \zeta) + (F(g), \zeta)$ is satisfied for a.e. $t \in [0, \tau]$ and for any $\zeta \in E$;
(ii) $g(t,\cdot) \in L^2(0, \tau; E) \cap C_w([0, \tau]; H)$ such that $g(0) = g_0$.

Here $(\cdot,\cdot)$ stands for the dual product of E^* (the dual space of E) and E, C_w stands for the weakly continuous functions valued in H, and (1.12) is satisfied in the space E^*.

Proposition 1.2. *For any given initial data $g_0 \in H$, there exists a unique, local weak solution $g(t) = (u(t), v(t), w(t), z(t)), t \in [0, \tau]$ for some $\tau > 0$, of the Brusselator evolutionary equation* (1.12)*, which becomes a strong solution on* $(0, \tau]$*, namely, it satisfies*

$$g \in C([0, \tau]; H) \cap C^1((0, \tau); H) \cap L^2(0, \tau; E) \tag{1.13}$$

and (1.12) *is satisfied in the space H for $t \in (0, \tau]$.*

The proof of Proposition 1.2 is made by conducting *a priori* estimates on the Galerkin approximate solutions of the initial value problem (1.12) (these estimates are similar to what we shall present in Section 2) and by the weak/weak* convergence argument, as well as the use of the properties of the function space, cf. [7, 22],

$$\Phi(0, \tau) = \{\varphi(\cdot) : \varphi \in L^2(0, \tau; E), \text{(distributional)}\, \partial_t\varphi \in L^2(0, \tau; E^*)\},$$

with the norm

$$\|\varphi\|_\Phi = \|\varphi\|_{L^2(0,\tau;E)} + \|\partial_t\varphi\|_{L^2(0,\tau;E^*)}.$$

The detail is omitted here.

We refer to [15, 33, 34] and many references therein for the concepts and basic facts in the theory of infinite dimensional dynamical systems, including few given below for clarity.

Definition 1.3. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space X. A bounded subset B_0 of X is called an *absorbing set* in X if, for any bounded subset $B \subset X$, there is some finite time $t_0 \geq 0$ depending on B such that $S(t)B \subset B_0$ for all $t > t_0$.

Definition 1.4. A semiflow $\{S(t)\}_{t\geq 0}$ on a Banach space X is called *asymptotically compact* if for any bounded sequences $\{x_n\}$ in X and $\{t_n\} \subset (0, \infty)$ with $t_n \to \infty$, there exist subsequences $\{x_{n_k}\}$ of $\{x_n\}$ and $\{t_{n_k}\}$ of $\{t_n\}$, such that $\lim_{k\to\infty} S(t_{n_k})x_{n_k}$ exists in X.

Definition 1.5. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space X. A subset $\mathscr{A}$ of X is called a *global attractor* for this semiflow, if the following conditions are satisfied:

(i) $\mathscr{A}$ is a nonempty, compact, and invariant set in the sense that
$$S(t)\mathscr{A} = \mathscr{A} \quad \text{for any } t \geq 0.$$

(ii) $\mathscr{A}$ attracts any bounded set B of X in terms of the Hausdorff distance, i.e.
$$\operatorname{dist}(S(t)B, \mathscr{A}) = \sup_{x\in B} \inf_{y\in\mathscr{A}} \|S(t)x - y\|_X \to 0, \quad \text{as } t \to \infty.$$

Now we state the main result of this paper. We emphasize that this result is established unconditionally, neither assuming initial data or solutions are nonnegative, nor imposing any restriction on any positive parameters involved in the equations (1.1)–(1.4).

Theorem 1.6 (Main Theorem)**.** *For any positive parameters d_1, d_2, a, b, D_1, D_2, there exists a global attractor $\mathscr{A}$ in the phase space H for the solution semiflow $\{S(t)\}_{t\geq 0}$ generated by the Brusselator evolutionary equation* (1.12).

The following proposition states concisely the basic result on the existence of a global attractor for a semiflow, cf. [15, 33, 34].

Proposition 1.7. *Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space X. If the following conditions are satisfied:*

(i) *$\{S(t)\}_{t\geq 0}$ has a bounded absorbing set B_0 in X, and*
(ii) *$\{S(t)\}_{t\geq 0}$ is asymptotically compact,*
then there exists a global attractor $\mathscr{A}$ in X for this semiflow, which is given by
$$\mathscr{A} = \omega(B_0) \overset{\text{def}}{=} \cap_{\tau\geq 0} \operatorname{Cl}_X \cup_{t\geq\tau} (S(t)B_0).$$

In Section 2 we shall prove the global existence of the weak solutions of the Brusselator evolutionary equation (1.12) and the absorbing property of this solution semiflow. In Section 3 we shall prove the asymptotic compactness of this solutions semiflow. In Section 4 we show the existence of a global attractor in space H for this Busselator semiflow and its properties as being the (H, E) global attractor and the $\mathbb{L}^\infty$ regularity. We also prove that the global attractor has a finite Hausdorff dimension and a finite fractal dimension. In Section 5, the existence of an exponential attractor for this semiflow is shown.

As a remark, with some adjustment in proof, these results are also valid for the homogeneous Neumann boundary condition. Furthermore, corresponding results can be shown for the coupled two-cell Gray-Scott equations, Selkov equations, and Schnackenberg equations.

2. Global solutions and absorbing property

In this article, we shall write $u(t,x)$, $v(t,x)$, $w(t,x)$, and $z(t,x)$ simply as $u(t)$, $v(t)$, $w(t)$, and $z(t)$, or even as u, v, w, and z, and similarly for other functions of (t,x).

Lemma 2.1. *For any initial data $g_0 = (u_0, v_0, w_0, z_0) \in H$, there exists a unique, global weak solution $g(t) = (u(t), v(t), w(t), z(t))$, $t \in [0, \infty)$, of the Brusselator evolutionary equation* (1.12) *and it becomes a strong solution on the time interval* $(0, \infty)$.

Proof. By Proposition 1.2, the local weak solution $g(t) = (u(t), v(t), w(t), z(t))$ exists uniquely on $[0, T_{\max})$, the maximal interval of existence. Taking the inner products $\langle (1.2), v(t)\rangle$ and $\langle (1.4), z(t)\rangle$ and summing up, we obtain

$$\begin{aligned}
&\frac{1}{2}\Big(\frac{d}{dt}\|v\|^2 + \frac{d}{dt}\|z\|^2\Big) + d_2\left(\|\nabla v\|^2 + \|\nabla z\|^2\right) \\
&= \int_\Omega \left(-u^2v^2 + buv - w^2z^2 + bwz - D_2[v^2 - 2vz + z^2]\right) dx \\
&= \int_\Omega -\Big[\big(uv - \frac{b}{2}\big)^2 + \big(wz - \frac{b}{2}\big)^2 + D_2(v-z)^2\Big]\, dx + \frac{1}{2}b^2|\Omega| \\
&\le \frac{1}{2}b^2|\Omega|.
\end{aligned} \tag{2.1}$$

It follows that

$$\frac{d}{dt}\left(\|v\|^2 + \|z\|^2\right) + 2\gamma d_2\left(\|v\|^2 + \|z\|^2\right) \le b^2|\Omega|,$$

which yields

$$\|v(t)\|^2 + \|z(t)\|^2 \le e^{-2\gamma d_2 t}\left(\|v_0\|^2 + \|z_0\|^2\right) + \frac{b^2|\Omega|}{2\gamma d_2}, \quad \text{for } t \in [0, T_{\max}). \tag{2.2}$$

Let $y(t,x) = u(t,x) + v(t,x) + w(t,x) + z(t,x)$. In order to treat the u-component and the w-component, first we add up (1.1), (1.2), (1.3) and (1.4) altogether to get the following equation satisfied by $y(t) = y(t,x)$,

$$\frac{\partial y}{\partial t} = d_1 \Delta y - y + \left[(d_2 - d_1)\Delta(v+z) + (v+z) + 2a\right]. \tag{2.3}$$

Taking the inner-product $\langle (2.3), y(t)\rangle$ we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|y\|^2 + d_1\|\nabla y\|^2 + \|y\|^2 \\
&= \int_\Omega \left[(d_2 - d_1)\Delta(v+z) + (v+z) + 2a\right] y\, dx \\
&\le |d_1 - d_2|\|\nabla(v+z)\|\|\nabla y\| + \|v+z\|\|y\| + 2a|\Omega|^{1/2}\|y\| \\
&\le \frac{d_1}{2}\|\nabla y\|^2 + \frac{|d_1 - d_2|^2}{2d_1}\|\nabla(v+z)\|^2 + \frac{1}{2}\|y\|^2 + \|v+z\|^2 + 4a^2|\Omega|,
\end{aligned}$$

so that

$$\frac{d}{dt}\|y\|^2 + d_1\|\nabla y\|^2 + \|y\|^2 \le \frac{|d_1 - d_2|^2}{d_1}\|\nabla(v+z)\|^2 + 4\left(\|v\|^2 + \|z\|^2\right) + 8a^2|\Omega|. \tag{2.4}$$

By substituting (2.2) for $\|v\|^2 + \|z\|^2$ in the above inequality, we obtain

$$\frac{d}{dt}\|y\|^2 + d_1\|\nabla y\|^2 + \|y\|^2 \le \frac{|d_1-d_2|^2}{d_1}\|\nabla(v+z)\|^2 + C_1(v_0,z_0,t), \tag{2.5}$$

where

$$C_1(v_0,z_0,t) = 4e^{-2\gamma d_2 t}\left(\|v_0\|^2 + \|z_0\|^2\right) + \big(\frac{4b^2}{\gamma d_2} + 8a^2\big)|\Omega|.$$

Integrate the inequality (2.5). Then the weak solution $y(t)$ of (2.3) satisfies the estimate

$$\begin{aligned} \|y(t)\|^2 &\le \|u_0+v_0+w_0+z_0\|^2 + \frac{|d_1-d_2|^2}{d_1}\int_0^t \|\nabla(v(s)+z(s))\|^2\,ds \\ &\quad + \frac{2}{\gamma d_2}\left(\|v_0\|^2+\|v_0\|^2\right) + \big(\frac{4b^2}{\gamma d_2}+8a^2\big)|\Omega|t, \quad t\in[0,T_{\max}). \end{aligned} \tag{2.6}$$

From (2.1) we also have

$$\begin{aligned} d_2\int_0^t \|\nabla(v(s)+z(s))\|^2\,ds &\le 2d_2\int_0^t \Big(\|\nabla v(s)\|^2 + \|\nabla z(s)\|^2\Big)\,ds \\ &\le \left(\|v_0\|^2+\|z_0\|^2\right) + b^2|\Omega|t. \end{aligned}$$

Substitute this into (2.6) to obtain

$$\begin{aligned} \|y(t)\|^2 &\le \|u_0+v_0+w_0+z_0\|^2 + \Big(\frac{|d_1-d_2|^2}{d_1\,d_2} + \frac{2}{\gamma d_2}\Big)\left(\|v_0\|^2+\|z_0\|^2\right) \\ &\quad + \Big[\Big(\frac{|d_1-d_2|^2}{d_1\,d_2} + \frac{4}{\gamma d_2}\Big)b^2 + 8a^2\Big]|\Omega|\,t, \quad t\in[0,T_{\max}). \end{aligned} \tag{2.7}$$

Let $p(t) = u(t)+w(t)$. Then by (2.2) and (2.7) we have shown that

$$\begin{aligned} &\|p(t)\|^2 = \|u(t)+w(t)\|^2 = \|y(t)-(v(t)+z(t))\|^2 \\ &\le 2\Big(\|u_0+v_0+w_0+z_0\|^2 + \Big(1+\frac{|d_1-d_2|^2}{d_1\,d_2}+\frac{2}{\gamma d_2}\Big)(\|v_0\|^2+\|z_0\|^2)\Big) + C_2\,t, \end{aligned} \tag{2.8}$$

for $t\in[0,T_{\max})$, where C_2 is a constant independent of the initial data g_0.

On the other hand, let $\psi(t,x) = u(t,x)+v(t,x)-w(t,x)-z(t,x)$, which satisfies the equation

$$\frac{\partial\psi}{\partial t} = d_1\Delta\psi - (1+2D_1)\psi + \left[(d_2-d_1)\Delta(v-z) + (1+2(D_1-D_2))(v-z)\right]. \tag{2.9}$$

Taking the inner-product $\langle(2.9),\psi(t)\rangle$ we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\psi\|^2 + d_1\|\nabla\psi\|^2 + \|\psi\|^2 \le \frac{1}{2}\frac{d}{dt}\|\psi\|^2 + d_1\|\nabla\psi\|^2 + (1+2D_1)\|\psi\|^2 \\ &\le (d_1-d_2)\|\nabla(v-z)\|\|\nabla\psi\| + |1+2(D_1-D_2)|\|v-z\|\|\psi\| \\ &\le \frac{d_1}{2}\|\nabla\psi\|^2 + \frac{|d_1-d_2|^2}{2d_1}\|\nabla(v-z)\|^2 + \frac{1}{2}\|\psi\|^2 + \frac{1}{2}|1+2(D_1-D_2)|^2\|v-z\|^2, \end{aligned}$$

so that

$$\frac{d}{dt}\|\psi\|^2 + d_1\|\nabla\psi\|^2 + \|\psi\|^2 \le \frac{|d_1-d_2|^2}{d_1}\|\nabla(v-z)\|^2 + C_3(v_0,z_0,t), \tag{2.10}$$

where

$$C_3(v_0, z_0, t) = 2|1 + 2(D_1 - D_2)|^2 \Big(e^{-2\gamma d_2 t} (\|v_0\|^2 + \|z_0\|^2) + \frac{b^2}{2\gamma d_2} |\Omega| \Big).$$

Integration of (2.10) yields

$$\begin{aligned} \|\psi\|^2 \le \|u_0 + v_0 - w_0 - z_0\|^2 + \frac{|d_1 - d_2|^2}{d_1} \int_0^t \|\nabla(v(s) - z(s))\|^2 \, ds \\ + |1 + 2(D_1 - D_2)|^2 \Big(\frac{1}{\gamma d_2} (\|v_0\|^2 + \|z_0\|^2) + \frac{b^2 |\Omega|}{\gamma d_2} \, t \Big), \quad t \in [0, T_{\max}). \end{aligned} \tag{2.11}$$

Note that

$$\begin{aligned} d_2 \int_0^t \|\nabla(v(s) - z(s))\|^2 \, ds &\le 2 d_2 \int_0^t \left(\|\nabla v(s)\|^2 + \|\nabla z(s)\|^2 \right) \, ds \\ &\le \left(\|v_0\|^2 + \|z_0\|^2 \right) + b^2 |\Omega| t. \end{aligned}$$

From (2.11) it follows that

$$\begin{aligned} \|\psi\|^2 \le \|u_0 + v_0 - w_0 - z_0\|^2 + \frac{|d_1 - d_2|^2}{d_1 \, d_2} \left(\|v_0\|^2 + \|z_0\|^2 + b^2 |\Omega| \, t \right) \\ + |1 + 2(D_1 - D_2)|^2 \Big(\frac{1}{\gamma d_2} (\|v_0\|^2 + \|z_0\|^2) + \frac{b^2 |\Omega|}{\gamma d_2} \, t \Big), \quad t \in [0, T_{\max}). \end{aligned} \tag{2.12}$$

Let $q(t) = u(t) - w(t)$. Then by (2.2) and (2.12) we find that

$$\begin{aligned} \|q(t)\|^2 = \|u(t) - w(t)\|^2 = \|\psi(t) - (v(t) - z(t))\|^2 \le 2\|u_0 + v_0 - w_0 - z_0\|^2 \\ + 2 \Big(1 + \frac{|d_1 - d_2|^2}{d_1 d_2} + \frac{|1 + 2(D_1 - D_2)|^2}{\gamma d_2} \Big) (\|v_0\|^2 + \|z_0\|^2) + C_4 t, \end{aligned} \tag{2.13}$$

for $t \in [0, T_{\max})$, where C_4 is a constant independent of the initial data g_0.

Finally combining (2.8) and (2.13) we can conclude that for each initial data $g_0 \in H$, the components $u(t) = (1/2)(p(t) + q(t))$ and $w(t) = (1/2)(p(t) - q(t))$ are bounded if $T_{\max}$ of the maximal interval of existence of the solution is finite. Together with (2.2), it shows that, for each $g_0 \in H$, the weak solution $g(t) = (u(t), v(t), w(t), z(t))$ of the Brusselator evolutionary equation (1.12) will never blow up in H at any finite time and it exists globally. □

By the global existence and uniqueness of the weak solutions and their continuous dependence on initial data shown in Proposition 1.2 and Lemma 2.1, the family of all the global weak solutions $\{g(t; g_0) : t \ge 0, g_0 \in H\}$ defines a semiflow on H,

$$S(t) : g_0 \mapsto g(t; g_0), \quad g_0 \in H, \ t \ge 0,$$

which is called the two-cell Brusselator semiflow, or simply the *Brusselator semiflow*, generated by the Brusselator evolutionary equation (1.12).

Lemma 2.2. *There exists a constant $K_1 > 0$, such that the set*

$$B_0 = \left\{ \|g\| \in H : \|g\|^2 \le K_1 \right\} \tag{2.14}$$

is an absorbing set in H for the Brusselator semiflow $\{S(t)\}_{t \ge 0}$.

Proof. For this two-cell Brusselator semiflow, from (2.2) we obtain

$$\limsup_{t\to\infty} (\|v(t)\|^2 + \|z(t)\|^2) < R_0 = \frac{b^2|\Omega|}{\gamma d_2} \tag{2.15}$$

and that for any given bounded set $B \subset H$ and $g_0 \in B$ there is a finite time $t_1(B) \geq 0$ such that

$$\|v(t;g_0)\|^2 + \|z(t;g_0)\|^2 < R_0, \quad \text{for any } t > t_1(B). \tag{2.16}$$

Moreover, for any $t \geq 0$, (2.1) also implies that

$$\begin{aligned} &\int_t^{t+1} (\|\nabla v(s)\|^2 + \|\nabla z(s)\|^2)\, ds \\ &\leq \frac{1}{d_2}(\|v(t)\|^2 + \|z(t)\|^2 + b^2|\Omega|) \\ &\leq \frac{1}{d_2}\Big(e^{-2\gamma d_2 t}(\|v_0\|^2 + \|z_0\|^2) + \frac{b^2|\Omega|}{2\gamma d_2}\Big) + \frac{b^2|\Omega|}{d_2}. \end{aligned} \tag{2.17}$$

which is for later use.

From (2.5) we can deduce that

$$\frac{d}{dt}\left(e^t\|y(t)\|^2\right) \leq \frac{|d_1-d_2|^2}{d_1}\, e^t\|\nabla(v(t)+z(t))\|^2 + e^t C_1(v_0,z_0,t).$$

Integrate this differential inequality to obtain

$$\begin{aligned} \|y(t)\|^2 &\leq e^{-t}\|u_0+v_0+w_0+z_0\|^2 \\ &\quad + \frac{|d_1-d_2|^2}{d_1}\int_0^t e^{-(t-\tau)}\|\nabla(v(\tau)+z(\tau))\|^2\, d\tau + C_5(v_0,z_0,t), \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} C_5(v_0,z_0,t) &= e^{-t}\int_0^t 4e^{(1-2\gamma d_2)\tau}\, d\tau\, (\|v_0\|^2 + \|z_0\|^2) + \big(\frac{4b^2}{\gamma d_2} + 8a^2\big)|\Omega| \\ &\leq 4\alpha(t)(\|v_0\|^2 + \|z_0\|^2) + \big(\frac{4b^2}{\gamma d_2} + 8a^2\big)|\Omega|, \end{aligned}$$

in which

$$\alpha(t) = e^{-t}\int_0^t e^{(1-2\gamma d_2)\tau}\, d\tau = \begin{cases} \frac{1}{|1-2\gamma d_2|}|e^{-2\gamma d_2 t} - e^{-t}|, & \text{if } 1-2\gamma d_2 \neq 0; \\ te^{-t} \leq 2e^{-1}e^{-t/2}, & \text{if } 1-2\gamma d_2 = 0. \end{cases} \tag{2.19}$$

On the other hand, multiplying (2.1) by e^t and then integrating each term of the resulting inequality, we obtain

$$\frac{1}{2}\int_0^t e^\tau \frac{d}{d\tau}\left(\|v(\tau)\|^2 + \|z(\tau)\|^2\right)\, d\tau + d_2\int_0^t e^\tau(\|\nabla v(\tau)\|^2 + \|z(\tau)\|^2)\, d\tau \leq \frac{1}{2}b^2|\Omega|e^t,$$

so that, by integration by parts and using (2.2), we obtain

$$
\begin{aligned}
& d_2 \int_0^t e^{\tau}(\|\nabla v(\tau)\|^2 + \|\nabla z(\tau)\|^2)\, d\tau \\
&\le \frac{1}{2} b^2 |\Omega| e^t - \frac{1}{2}\int_0^t e^{\tau} \frac{d}{d\tau}\left(\|v(\tau)\|^2 + \|\nabla z(\tau)\|^2\right) d\tau \\
&= \frac{1}{2} b^2 |\Omega| e^t - \frac{1}{2}\Big[e^t(\|v(t)\|^2 + \|z(t)\|^2) - (\|v_0\|^2 + \|z_0\|^2) \\
&\quad - \int_0^t e^{\tau}(\|v(\tau)\|^2 + \|z(\tau)\|^2)\, d\tau\Big] \\
&\le b^2|\Omega| e^t + (\|v_0\|^2 + \|z_0\|^2) + \int_0^t e^{(1-2\gamma d_2)\tau}(\|v_0\|^2 + \|z_0\|^2)\, d\tau + \frac{b^2|\Omega|}{2\gamma d_2} e^t \\
&\le \big(1 + \frac{1}{2\gamma d_2}\big) b^2 |\Omega| e^t + \big(1 + \alpha(t) e^t\big)\left(\|v_0\|^2 + \|z_0\|^2\right), \quad \text{for } t \ge 0.
\end{aligned}
\tag{2.20}
$$

Substituting (2.20) into (2.18), we obtain that for $t \ge 0$,

$$
\begin{aligned}
& \|y(t)\|^2 \\
&\le e^{-t}\|u_0 + v_0 + w_0 + z_0\|^2 + C_5(v_0, z_0, t) \\
&\quad + \frac{2|d_1 - d_2|^2}{d_1\, d_2} e^{-t}\Big[\big(1 + \frac{1}{2\gamma d_2}\big) b^2 |\Omega| e^t + \big(1 + e^t \alpha(t)\big)(\|v_0\|^2 + \|z_0\|^2)\Big] \\
&\le e^{-t}\|u_0 + v_0 + w_0 + z_0\|^2 + 4\alpha(t)(\|v_0\|^2 + \|z_0\|^2) + \big(\frac{4b^2}{\gamma d_2} + 8a^2\big)|\Omega| \\
&\quad + \frac{2|d_1 - d_2|^2}{d_1\, d_2} e^{-t}\Big[\big(1 + \frac{1}{2\gamma d_2}\big) b^2 |\Omega| e^t + \big(1 + e^t \alpha(t)\big)\left(\|v_0\|^2 + \|z_0\|^2\right)\Big].
\end{aligned}
\tag{2.21}
$$

Note that (2.19) shows $\alpha(t) \to 0$, as $t \to 0$. From (2.21) we find that

$$
\limsup_{t\to\infty} \|y(t)\|^2 < R_1 = 1 + \big(\frac{4b^2}{\gamma d_2} + 8a^2\big)|\Omega| + \frac{2|d_1 - d_2|^2}{d_1\, d_2}\big(1 + \frac{1}{2\gamma d_2}\big) b^2 |\Omega|. \tag{2.22}
$$

The combination of (2.15) and (2.22) gives us

$$
\limsup_{t\to\infty} \|u(t) + w(t)\|^2 = \limsup_{t\to\infty} \|y(t) - (v(t) + z(t))\|^2 < 4R_0 + 2R_1. \tag{2.23}
$$

Similarly, from the inequality (2.10) satisfied by $\psi(t) = u(t) + v(t) - w(t) - z(t)$, we obtain

$$
\frac{d}{dt}\left(e^t \|\psi(t)\|^2\right) \le \frac{|d_1 - d_2|^2}{d_1} e^t \|\nabla(v(t) - z(t))\|^2 + e^t C_3(v_0, z_0, t).
$$

Integrate this differential inequality to obtain

$$
\begin{aligned}
\|\psi(t)\|^2 &\le e^{-t}\|u_0 + v_0 - w_0 - z_0\|^2 \\
&\quad + \frac{|d_1 - d_2|^2}{d_1} \int_0^t e^{-(t-\tau)} \|\nabla(v(\tau) - z(\tau))\|^2\, d\tau + C_6(v_0, z_0, t),
\end{aligned}
\tag{2.24}
$$

where

$$
\begin{aligned}
C_6(v_0, z_0, t) &= 2|1 + 2(D_1 - D_2)|^2\Big(e^{-t}\int_0^t e^{(1-2\gamma d_2)\tau}\, d\tau\, (\|v_0\|^2 + \|z_0\|^2) + \frac{b^2}{\gamma d_2}|\Omega|\Big) \\
&\le 2|1 + 2(D_1 - D_2)|^2\Big(\alpha(t)(\|v_0\|^2 + \|z_0\|^2) + \frac{b^2}{\gamma d_2}|\Omega|\Big).
\end{aligned}
$$

Using (2.20) to treat the integral term in (2.24), we obtain that

$$
\begin{aligned}
&\|\psi(t)\|^2 \\
&\le e^{-t}\|u_0+v_0-w_0-z_0\|^2 + C_6(v_0,z_0,t) \\
&\quad + \frac{2|d_1-d_2|^2}{d_1\,d_2}e^{-t}\int_0^t e^{\tau}(\|\nabla v(\tau)\|^2+\|\nabla z(\tau)\|^2)\,d\tau \\
&\le e^{-t}\|u_0+v_0-w_0-z_0\|^2 + 2|1+2(D_1-D_2)|^2 \\
&\quad \times \Big(\alpha(t)(\|v_0\|^2+\|z_0\|^2)+\frac{b^2}{\gamma d_2}|\Omega|\Big) \\
&\quad + \frac{2|d_1-d_2|^2}{d_1\,d_2}e^{-t}\Big[\big(1+\frac{1}{2\gamma d_2}\big)b^2|\Omega|e^t + \big(1+e^t\alpha(t)\big)\,(\|v_0\|^2+\|z_0\|^2)\Big],
\end{aligned}
\tag{2.25}
$$

for $t \ge 0$. Therefore, since $\alpha(t) \to 0$ as $t \to 0$, from (2.25) we obtain

$$
\limsup_{t\to\infty}\|\psi(t)\|^2 < R_2 = 1 + 2b^2|\Omega|\Big[\frac{|1+2(D_1-D_2)|^2}{\gamma d_2} + \frac{|d_1-d_2|^2}{d_1\,d_2}\big(1+\frac{1}{2\gamma d_2}\big)\Big].
\tag{2.26}
$$

The combination of (2.15) and (2.26) gives us

$$
\limsup_{t\to\infty}\|u(t)-w(t)\|^2 = \limsup_{t\to\infty}\|\psi(t)-(v(t)-z(t))\|^2 < 4R_0 + 2R_2.
\tag{2.27}
$$

Finally, putting together (2.23) and (2.27), we assert that

$$
\limsup_{t\to\infty}(\|u(t)\|^2+\|w(t)\|^2) < 8R_0 + 2(R_1+R_2).
\tag{2.28}
$$

Moreover, from (2.2), (2.21) and (2.25) we see that for any given bounded set $B \subset H$ and $g_0 \in B$ there is a finite time $t_2(B) \ge 0$ such that

$$
\|u(t;g_0)\|^2 + \|w(t;g_0)\|^2 < 8R_0 + 2(R_1+R_2), \quad \text{for any } t > t_2(B).
\tag{2.29}
$$

Then assembling (2.15) and (2.28), we end up with

$$
\limsup_{t\to\infty}\|g(t)\|^2 = \limsup_{t\to\infty}(\|u(t)\|^2+\|v(t)\|^2+\|w(t)\|^2+\|z(t)\|^2) < 9R_0+2(R_1+R_2).
$$

Moreover, (2.16) and (2.29) show that for any given bounded set $B \subset H$ and $g_0 \in B$ the solution $g(t;g_0)$ satisfies

$$
\|g(t;g_0)\|^2 < 9R_0 + 2(R_1+R_2), \quad \text{for any } t > \max\{t_1(B), t_2(B)\}.
$$

Thus this lemma is proved with $K_1 = 9R_0 + 2(R_1+R_2)$ in (2.14). And K_1 is a universal positive constant independent of initial data. $\square$

Next we show the absorbing properties of the (v,z) components of this Brusselator semiflow in the product Banach spaces $[L^{2p}(\Omega)]^2$, for any integer $1 \le p \le 3$.

Lemma 2.3. *For any given integer $1 \le p \le 3$, there exists a positive constant K_p such that the absorbing inequality*

$$
\limsup_{t\to\infty}\ \|(v(t),z(t))\|_{L^{2p}}^{2p} < K_p
\tag{2.30}
$$

is satisfied by the (v,z) components of the Brusselator semiflow $\{S(t)\}_{t\ge 0}$ for any initial data $g_0 \in H$.

Proof. The case $p = 1$ has been shown in Lemma 2.2. According to the solution property (1.13) satisfied by all the global weak solutions on $[0, \infty)$, we know that for any given initial status $g_0 \in H$ there exists a time $t_0 \in (0, 1)$ such that

$$S(t_0)g_0 \in E = [H_0^1(\Omega)]^6 \hookrightarrow \mathbb{L}^6(\Omega) \hookrightarrow \mathbb{L}^4(\Omega). \tag{2.31}$$

Then the weak solution $g(t) = S(t)g_0$ becomes a strong solution on $[t_0, \infty)$ and satisfies

$$S(\cdot)g_0 \in C([t_0, \infty); E) \cap L^2(t_0, \infty; \Pi) \subset C([t_0, \infty); \mathbb{L}^6(\Omega)) \subset C([t_0, \infty); \mathbb{L}^4(\Omega)), \tag{2.32}$$

for $n \le 3$. Based on this observation, without loss of generality, we can simply *assume* that $g_0 \in \mathbb{L}^6(\Omega)$ for the purpose of studying the long-time dynamics. Thus parabolic regularity (2.32) of strong solutions implies the $S(t)g_0 \in E \subset \mathbb{L}^6(\Omega), t \ge 0$. Then by the bootstrap argument, again without loss of generality, one can *assume* that $g_0 \in \Pi \subset \mathbb{L}^8(\Omega)$ so that $S(t)g_0 \in \Pi \subset \mathbb{L}^8(\Omega), t \ge 0$.

Take the L^2 inner-product $\langle (1.2), v^5 \rangle$ and $\langle (1.4), z^5 \rangle$ and sum up to obtain

$$\begin{aligned}
&\frac{1}{6}\frac{d}{dt}\left(\|v(t)\|_{L^6}^6 + \|z(t)\|_{L^6}^6\right) + 5d_2\left(\|v(t)^2\nabla v(t)\|^2 + \|z(t)^2\nabla z(t)\|^2\right) \\
&= \int_\Omega \left(bu(t,x)v^5(t,x) - u^2(t,x)v^6(t,x) + bw(t,x)z^5(t,x) - w^2(t,x)z^6(t,x)\right) dx \\
&+ D_2 \int_\Omega \left[(z(t,x) - v(t,x))v^5(t,x) + (v(t,x) - z(t,x))z^5(t,x)\right] dx.
\end{aligned} \tag{2.33}$$

By Young's inequality, we have

$$\begin{aligned}
&\int_\Omega \left[\left(buv^5 - u^2v^6\right) + \left(bwz^5 - w^2z^6\right)\right] dx \\
&\le \frac{1}{2}\Big(\int_\Omega b^2(v^4 + z^4)\, dx - \int_\Omega (u^2v^6 + w^2z^6)\, dx\Big),
\end{aligned}$$

and

$$\int_\Omega \left[(z - v)v^5 + (v - z)z^5\right] dx \le \int_\Omega \Big[-v^6 + \big(\frac{1}{6}z^6 + \frac{5}{6}v^6\big) + \big(\frac{1}{6}v^6 + \frac{5}{6}z^6\big) - z^6 \Big] dx = 0.$$

Substitute the above two inequalities into (2.33) and use Poincaré inequality, we obtain the following inequality relating $\|(v, z)\|_{L^6}^6$ to $\|(v, z)\|_{L^4}^4$,

$$\begin{aligned}
&\frac{d}{dt}\left(\|v(t)\|_{L^6}^6 + \|z(t)\|_{L^6}^6\right) + 10\gamma d_2\left(\|v(t)\|_{L^6}^6 + \|z(t)\|_{L^6}^6\right) \\
&\le \frac{d}{dt}\left(\|v(t)\|_{L^6}^6 + \|z(t)\|_{L^6}^6\right) + 10d_2\left(\|\nabla v^3(t)\|^2 + \|\nabla z^3(t)\|^2\right) \\
&\le 3b^2(\|(v(t))\|_{L^4}^4 + \|(z(t))\|_{L^4}^4).
\end{aligned}$$

Similarly we can get the corresponding inequality relating $\|(v, z)\|_{L^4}^4$ to $\|(v, z)\|^2$,

$$\begin{aligned}
&\frac{d}{dt}\left(\|(v(t))\|_{L^4}^4 + \|(z(t))\|_{L^4}^4\right) + 6\gamma d_2\left(\|(v(t))\|_{L^4}^4 + \|(z(t))\|_{L^4}^4\right) \\
&\le \frac{d}{dt}\left(\|(v(t))\|_{L^4}^4 + \|(z(t))\|_{L^4}^4\right) + 6d_2\left(\|\nabla v^2(t)\|^2 + \|\nabla z^2(t)\|^2\right) \\
&\le 2b^2(\|v(t)\|^2 + \|z(t)\|^2).
\end{aligned}$$

Applying Gronwall inequality to the above two inequalities and using (2.2), we obtain

$$
\begin{aligned}
&\|(v(t))\|_{L^4}^4 + \|(z(t))\|_{L^4}^4 \\
&\le e^{-6\gamma d_2 t}\left(\|v_0\|_{L^4}^4 + \|z_0\|_{L^4}^4\right) \\
&\quad + \int_0^t e^{-6\gamma d_2 (t-\tau)} 2b^2 (\|v(\tau)\|^2 + \|z(\tau)\|^2) d\tau \\
&\le e^{-6\gamma d_2 t}\left(\|v_0\|_{L^4}^4 + \|z_0\|_{L^4}^4\right) + \int_\Omega e^{-6\gamma d_2 (t-\tau) - 2\gamma d_2 \tau} 2b^2 (\|v_0\|^2 + \|z_0\|^2)\, d\tau + \frac{b^4|\Omega|}{6\gamma^2 d_2^2} \\
&\le e^{-2\gamma d_2 t} C_7 \left(\|v_0\|_{L^6}^6 + \|z_0\|_{L^6}^6\right) + \frac{b^4|\Omega|}{6\gamma^2 d_2^2}, \quad t \ge 0,
\end{aligned}
$$

where C_7 is a uniform positive constant, and then

$$
\begin{aligned}
&\|v(t)\|_{L^6}^6 + \|z(t)\|_{L^6}^6 \\
&\le e^{-10\gamma d_2 t}\left(\|v_0\|_{L^6}^6 + \|z_0\|_{L^6}^6\right) + \int_0^t e^{-10\gamma d_2 (t-\tau)} 3b^2 (\|v(\tau)\|_{L^4}^4 + \|z(\tau)\|_{L^4}^4) d\tau \\
&\le e^{-10\gamma d_2 t}\left(\|v_0\|_{L^6}^6 + \|z_0\|_{L^6}^6\right) + \int_\Omega e^{-10\gamma d_2 (t-\tau) - 2\gamma d_2 \tau} 3b^2 C_7 (\|v_0\|_{L^6}^6 + \|z_0\|_{L^6}^6)\, d\tau \\
&\quad + \frac{b^6|\Omega|}{20\gamma^3 d_2^3} \\
&\le e^{-2\gamma d_2 t}\Big(1 + \frac{3b^2 C_7}{8\gamma d_2}\Big)\left(\|v_0\|_{L^6}^6 + \|z_0\|_{L^6}^6\right) + \frac{b^6|\Omega|}{20\gamma^3 d_2^3}, \quad t \ge 0.
\end{aligned}
$$

It follows that

$$
\limsup_{t\to\infty} \left(\|v(t)\|_{L^4}^4 + \|z(t)\|_{L^4}^4\right) < K_2 = 1 + \frac{b^4|\Omega|}{6\gamma^2 d_2^2}, \tag{2.34}
$$

$$
\limsup_{t\to\infty} \left(\|v(t)\|_{L^6}^6 + \|z(t)\|_{L^6}^6\right) < K_3 = 1 + \frac{b^6|\Omega|}{20\gamma^3 d_2^3}. \tag{2.35}
$$

Thus (2.30) is proved. □

3. Asymptotic compactness

The lack of inherent dissipation and the appearance of cross-cell coupling make the attempt of showing the asymptotic compactness of the two-cell Brusselator semiflow also challenging. In this section we shall prove this asymptotic compactness through the following two lemmas.

Since $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ are continuous embeddings, there are constants $\delta > 0$ and $\eta > 0$ such that $\|\cdot\|_{L^4}^2 \le \delta \|\nabla(\cdot)\|^2$ and $\|\cdot\|_{L^6}^2 \le \eta \|\nabla(\cdot)\|^2$. We shall use the notation $\|(y_1, y_2)\|^2 = \|y_1\|^2 + \|y_2\|^2$ and $\|\nabla(y_1, y_2)\|^2 = \|\nabla y_1\|^2 + \|\nabla y_2\|^2$ for conciseness. The following proposition is about the uniform Gronwall inequality, which is an instrumental tool in the analysis of asymptotic compactness, cf. [23, 33, 34].

Proposition 3.1. *Let β, ζ, and h be nonnegative functions in $L^1_{loc}([0,\infty); \mathbb{R})$. Assume that β is absolutely continuous on $(0,\infty)$ and the following differential inequality is satisfied,*

$$
\frac{d\beta}{dt} \le \zeta\beta + h, \quad \text{for } t > 0.
$$

If there is a finite time $t_1 > 0$ and some $r > 0$ such that

$$\int_t^{t+r} \zeta(\tau)\,d\tau \le A, \quad \int_t^{t+r} \beta(\tau)\,d\tau \le B, \quad \int_t^{t+r} h(\tau)\,d\tau \le C,$$

for any $t > t_1$, where $A, B,$ and C are some positive constants, then

$$\beta(t) \le \big(\frac{B}{r} + C\big)e^A, \quad \text{for any } t > t_1 + r.$$

Lemma 3.2. *For any given initial data $g_0 \in B_0$, the (u, w) components of the solution trajectories $g(t) = S(t)g_0$ of the IVP (1.12) satisfy*

$$\|\nabla(u(t), w(t))\|^2 \le M_1, \quad \text{for } t > T_1, \tag{3.1}$$

where $M_1 > 0$ is a uniform constant depending on K_1 and $|\Omega|$ but independent of initial data, and $T_1 > 0$ is finite and only depends on the absorbing ball B_0.

Proof. Take the inner-products $\langle (1.1), -\Delta u(t)\rangle$ and $\langle (1.3), -\Delta w(t)\rangle$ and then sum up the two equalities to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla(u,w)\|^2 + d_1\|\Delta(u,w)\|^2 + (b+1)\|\nabla(u,w)\|^2 \\
&= -\int_\Omega a(\Delta u + \Delta w)\,dx - \int_\Omega (u^2 v\Delta u + w^2 z\Delta w)\,dx \\
&\quad - D_1\int_\Omega (|\nabla u|^2 - 2\nabla u\cdot\nabla w + |\nabla w|^2)\,dx \\
&\le \big(\frac{d_1}{4} + \frac{d_1}{4} + \frac{d_1}{2}\big)\|\Delta(u,w)\|^2 + \frac{a^2}{d_1}|\Omega| + \frac{1}{2d_1}\int_\Omega \big(u^4v^2 + w^4v^2\big)\,dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{d}{dt}\|\nabla(u,w)\|^2 + 2(b+1)\|\nabla(u,w)\|^2 \\
&\le \frac{2a^2}{d_1}|\Omega| + \frac{1}{d_1}\left(\|u^2\|^2\|v\|^2 + \|w^2\|^2\|z\|^2\right) \\
&\le \frac{2a^2}{d_1}|\Omega| + \frac{\delta^2}{d_1}\left(\|v\|^2\|\nabla u\|^4 + \|z\|^2\|\nabla w\|^4\right).
\end{aligned} \tag{3.2}$$

By the absorbing property shown in Lemma 2.2, there is a finite time $T_0 = T_0(B_0) \ge 0$ such that $S(t)B_0 \subset B_0$ for all $t > T_0$. Therefore, for any $g_0 \in B_0$, by (2.14) we have

$$\|(u(t), w(t))\|^2 + \|(v(t), z(t))\|^2 \le K_1, \quad \text{for } t > T_0. \tag{3.3}$$

Substitute (3.3) into (3.2) to obtain

$$\begin{aligned}
\frac{d}{dt}\|\nabla(u,w)\|^2 &\le \frac{d}{dt}\|\nabla(u,w)\|^2 + 2(b+1)\|\nabla(u,w)\|^2 \\
&\le \frac{\delta^2 K_1}{d_1}\|\nabla(u,w)\|^4 + \frac{2a^2}{d_1}|\Omega|,
\end{aligned} \tag{3.4}$$

which can be written as the inequality

$$\frac{d\rho}{dt} \le \beta\rho + \frac{2a^2}{d_1}|\Omega|, \tag{3.5}$$

where

$$\rho(t) = \|\nabla(u(t), w(t))\|^2 \quad \text{and} \quad \beta(t) = \frac{\delta^2 K_1}{d_1}\rho(t).$$

In view of the inequality (2.4), (2.17) and (3.3), we have

$$\begin{aligned}
&\int_t^{t+1} \|\nabla y(\tau)\|^2 \, d\tau \\
&\le \frac{2|d_1 - d_2|^2}{d_1^2} \int_t^{t+1} \|\nabla (v+z)\|^2 d\tau \\
&\quad + \frac{1}{d_1}\Big(\|y(t)\|^2 + \int_t^{t+1} \frac{8}{\gamma}(\|v(\tau)\|^2 + \|z(\tau)\|^2 + 2a^2|\Omega|)\, d\tau \Big) \le C_8,
\end{aligned} \tag{3.6}$$

for $t > T_0$, where

$$C_8 = \frac{4|d_1 - d_2|^2}{d_1^2 d_2}\big[K_1 + \big(1 + \frac{1}{2\gamma d_2}\big) b^2|\Omega|\big] + \frac{1}{d_1}\Big(K_1 + \frac{8}{\gamma}(K_1 + 2a^2|\Omega|)\Big).$$

From the inequality (2.10), (2.17) and (3.3) and with a similar estimation, there exists a uniform constant $C_9 > 0$ such that

$$\int_t^{t+1} \|\nabla \psi(\tau)\|^2 \, d\tau \le C_9, \quad \text{for } t > T_0. \tag{3.7}$$

Then we can put together (2.17), (3.6) and (3.7) to get

$$\begin{aligned}
&\int_t^{t+1} \rho(\tau)\, d\tau \\
&= \int_t^{t+1} (\|\nabla u(\tau)\|^2 + \|\nabla w(\tau)\|^2)\, d\tau \\
&\le \frac{1}{2} \int_t^{t+1} \big(\|\nabla (y(\tau) - (v(\tau) + z(\tau))\|^2 + \|\nabla(\psi(\tau) - (v(\tau) - z(\tau))\|^2\big)\, d\tau \\
&\le \int_t^{t+1} \big(\|\nabla y(\tau)\|^2 + \|\nabla \psi(\tau)\|^2 + \|\nabla(v+z)\|^2 + \|\nabla(v-z)\|^2\big)\, d\tau \\
&\le C_8 + C_9 + \frac{4}{d_2}\big[K_1 + \big(1 + \frac{1}{2\gamma d_2}\big) b^2|\Omega|\big] \overset{\text{def}}{=} C_{10}, \quad \text{for } t > T_0.
\end{aligned} \tag{3.8}$$

Now we can apply the uniform Gronwall inequality in Proposition 3.1 where $r = 1$ to (3.5) and use (3.8) to reach the conclusion (3.1) with

$$M_1 = \Big(C_{10} + \frac{2a^2}{d_1}|\Omega|\Big) e^{\delta^2 K_1 C_{10}/d_1}$$

and $T_1 = T_0(B_0) + 1$. The proof is completed. □

Lemma 3.3. *For any given initial data $g_0 \in B_0$, the (v, z) components of the trajectory $g(t) = S(t)g_0$ of the IVP (1.12) satisfy*

$$\|\nabla(v(t), z(t))\|^2 \le M_2, \quad \textit{for } t > T_2, \tag{3.9}$$

where $M_2 > 0$ is a uniform constants depending on K_1 and $|\Omega|$ but independent of initial data, and $T_2\,(> T_1 > 0)$ is finite and only depends on the absorbing ball B_0.

Proof. Take the inner-products $\langle (1.2), -\Delta v(t)\rangle$ and $\langle (1.4), -\Delta z(t)\rangle$ and sum up the two equalities to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla(v,z)\|^2 + d_2\|\Delta(v,z)\|^2 \\
&= -\int_\Omega b(u\Delta v + w\Delta z)\, dx
\end{aligned}$$

$$
\begin{aligned}
&+ \int_\Omega (u^2 v\Delta v + w^2 z\Delta z)\,dx - D_2 \int_\Omega [(z-v)\Delta v + (v-z)\Delta z]\,dx \\
&\le \frac{d_2}{2}\|\Delta(v,z)\|^2 + \frac{b^2}{d_2}\|(u,w)\|^2 + \frac{1}{d_2}\int_\Omega (u^4 v^2 + w^4 z^2)\,dx \\
&\quad - D_2 \int_\Omega (|\nabla v|^2 - 2\nabla v\cdot\nabla z + |\nabla z|^2)\,dx \\
&\le \frac{d_2}{2}\|\Delta(v,z)\|^2 + \frac{b^2}{d_2}\|(u,w)\|^2 + \frac{1}{d_2}\int_\Omega (u^4 v^2 + w^4 z^2)\,dx, \quad t > T_0.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\|\nabla(v(t),z(t))\|^2 \\
&= -(\langle v,\Delta v\rangle + \langle z,\Delta z\rangle) \le \frac{1}{2}\Big(\|v(t)\|^2 + \|z(t)|^2\| + \|\Delta v(t)\|^2 + \|\Delta z(t)\|^2\Big),
\end{aligned}
$$

by using Hölder inequality and the embedding inequality mentioned in the beginning of this section and by Lemma 3.2, from the above inequality we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla(v,z)\|^2 + d_2\|\nabla(v,z)\|^2 \\
&\le d_2\|(v,z)\|^2 + \frac{2b^2}{d_2}\|(u,w)\|^2 + \frac{2}{d_2}(\|u\|_{L^6}^4\|v\|_{L^6}^2 + \|w\|_{L^6}^4\|z\|_{L^6}^2) \\
&\le \big(d_2 + \frac{2b^2}{d_2}\big)K_1 + \frac{2\eta^6}{d_2}(\|\nabla u\|^4 + \|\nabla w\|^4)\|\nabla(v,z)\|^2 \\
&\le K_1\big(d_2 + \frac{2b^2}{d_2}\big) + \frac{2\eta^6 M_1^2}{d_2}\|\nabla(v,z)\|^2,\ t > T_1.
\end{aligned} \tag{3.10}
$$

Applying the uniform Gronwall inequality in Proposition 3.1 to (3.10) and using (2.17), we can assert that

$$
\|\nabla(v(t),z(t))\|^2 \le M_2, \quad \text{for } t > T_1 + 1, \tag{3.11}
$$

where

$$
M_2 = \Big(\frac{1}{d_2}\big[K_1 + \big(1 + \frac{1}{2\gamma d_2}\big)b^2|\Omega|\big] + K_1\big[d_2 + \frac{2b^2}{d_2}\big]\Big)e^{2\eta^6 M_1^2/d_2}.
$$

Thus (3.9) is proved with this M_2 and $T_2 = T_1 + 1$. □

4. The existence of a global attractor and its properties

In this section we finally prove Theorem 1.6 on the existence of a global attractor, which will be denoted by $\mathscr{A}$, for the Brusselator semiflow $\{S(t)\}_{t\ge 0}$ and we shall investigate the properties of $\mathscr{A}$, including its finite fractal dimensionality.

Proof of Theorem 1.6. In Lemma 2.2, we have shown that the Brusselator semiflow $\{S(t)\}_{t\ge 0}$ has a bounded absorbing set B_0 in H. Combining Lemma 3.2 and Lemma 3.3 we proved that

$$
\|S(t)g_0\|_E^2 \le M_1 + M_2, \quad \text{for } t > T_2 \text{ and for } g_0 \in B_0,
$$

which implies that $\{S(t)B_0 : t > T_2\}$ is a bounded set in space E and consequently a precompact set in space H. Therefore, the Brusselator semiflow $\{S(t)\}_{t\ge 0}$ is asymptotically compact in H. Finally we apply Proposition 1.7 to reach the conclusion that there exists a global attractor $\mathscr{A}$ in H for this Brusselator semiflow $\{S(t)\}_{t\ge 0}$. □

Now we show that the global attractor $\mathscr{A}$ of the Brusselator semiflow is an (H, E) global attractor with the regularity $\mathscr{A} \subset \mathbb{L}^\infty(\Omega)$. The concept of (H, E) global attractor was introduced in [3].

Definition 4.1. Let $\{\Sigma(t)\}_{t\geq 0}$ be a semiflow on a Banach space X and let Y be a compactly imbedded subspace of X. A subset $\mathcal{A}$ of Y is called an (X, Y) global attractor for this semiflow if $\mathcal{A}$ has the following properties,

(i) $\mathcal{A}$ is a nonempty, compact, and invariant set in Y.
(ii) $\mathcal{A}$ attracts any bounded set $B \subset X$ with respect to the Y-norm; namely, there is a $\tau = \tau(B)$ such that $\Sigma(t)B \subset Y$ for $t > \tau$ and $\operatorname{dist}_Y(\Sigma(t)B, \mathcal{A}) \to 0$, as $t \to \infty$.

Lemma 4.2. *Let $\{g_m\}$ be a sequence in E such that $\{g_m\}$ converges to $g_0 \in E$ weakly in E and $\{g_m\}$ converges to g_0 strongly in H, as $m \to \infty$. Then*

$$\lim_{m\to\infty} S(t)g_m = S(t)g_0 \quad \textit{strongly in } E,$$

where the convergence is uniform with respect to t in any given compact interval $[t_0, t_1] \subset (0, \infty)$.

The proof of this lemma is found in [41, Lemma 10].

Theorem 4.3. *The global attractor $\mathscr{A}$ in H for the Brusselator semiflow $\{S(t)\}_{t\geq 0}$ is indeed an (H, E) global attractor and $\mathscr{A}$ is a bounded subset in $\mathbb{L}^\infty(\Omega)$.*

Proof. By Lemmas 2.2, 3.2 and 3.3, we can assert that for the Brusselator semiflow $\{S(t)\}_{t\geq 0}$ defined on H there exists a bounded absorbing set $B_1 \subset E$ and this absorbing is in the E-norm. Indeed,

$$B_1 = \{g \in E : \|g\|_E^2 = \|\nabla g\|^2 \leq M_1 + M_2\}.$$

Now we show that the Brusselator semiflow $\{S(t)\}_{t\geq 0}$ is asymptotically compact with respect to the strong topology in E. For any time sequence $\{t_n\}, t_n \to \infty$, and any bounded sequence $\{g_n\} \subset E$, there exists a finite time $t_0 \geq 0$ such that $S(t)\{g_n\} \subset B_0$, for any $t > t_0$. Then for an arbitrarily given $T > t_0 + T_2$, where T_2 is the time specified in Lemma 3.3, there is an integer $n_0 \geq 1$ such that $t_n > 2T$ for all $n > n_0$. By Lemma 3.2 and Lemma 3.3,

$$\{S(t_n - T)g_n\}_{n>n_0} \text{ is a bounded set in } E.$$

Since E is a Hilbert space, there is an increasing sequence of integers $\{n_j\}_{j=1}^\infty$ with $n_1 > n_0$, such that

$$\lim_{j\to\infty} S(t_{n_j} - T)g_{n_j} = g^* \quad \text{weakly in } E.$$

By the compact imbedding $E \hookrightarrow H$, there is a further subsequence of $\{n_j\}$, but relabeled as the same as $\{n_j\}$, such that

$$\lim_{j\to\infty} S(t_{n_j} - T)g_{n_j} = g^* \quad \text{strongly in } H.$$

Then by Lemma 4.2, we have the following convergence with respect to the E-norm,

$$\lim_{j\to\infty} S(t_{n_j})g_{n_j} = \lim_{j\to\infty} S(T)S(t_{n_j} - T)g_{n_j} = S(T)g^* \quad \text{strongly in } E.$$

This proves that $\{S(t)\}_{t\geq 0}$ is asymptotically compact in E.

Therefore, by Proposition 1.7, there exists a global attractor $\mathscr{A}_E$ for the extended Brusselator semiflow $\{S(t)\}_{t\geq 0}$ in the space E. According to Definition 4.1 and the fact that B_1 attracts B_0 in the E-norm due to the combination of Lemma 3.2 and Lemma 3.3, we see that this global attractor $\mathscr{A}_E$ is an (H, E) global attractor. Moreover, the invariance and the boundedness of $\mathscr{A}$ in H and the invariance and boundedness of $\mathscr{A}_E$ in E imply that

$\mathscr{A}_E$ attracts $\mathscr{A}$ in E, so that $\mathscr{A} \subset \mathscr{A}_E$, and
$\mathscr{A}$ attracts $\mathscr{A}_E$ in H, so that $\mathscr{A}_E \subset \mathscr{A}$.

Therefore, $\mathscr{A} = \mathscr{A}_E$ and we proved that the global attractor $\mathscr{A}$ in H is itself an (H, E) global attractor for this Brusselator semiflow.

Next we show that $\mathscr{A}$ is a bounded subset in $\mathbb{L}^\infty(\Omega)$. By the (L^p, L^∞) regularity of the analytic C_0-semigroup $\{e^{At}\}_{t\ge 0}$ stated in [33, Theorem 38.10], one has $e^{At}: \mathbb{L}^p(\Omega) \to \mathbb{L}^\infty(\Omega)$ for $t > 0$, and there is a constant $C(p) > 0$ such that

$$\|e^{At}\|_{\mathcal{L}(\mathbb{L}^p, \mathbb{L}^\infty)} \le C(p)\, t^{-\frac{n}{2p}}, \quad t > 0, \quad \text{where } n = \dim \Omega. \tag{4.1}$$

By the variation-of-constant formula satisfied by the mild solutions (of course by strong solutions), for any $g \in \mathscr{A}$ $(\subset E)$, we have

$$\begin{aligned} \|S(t)g\|_{L^\infty} &\le \|e^{At}\|_{\mathcal{L}(L^2, L^\infty)} \|g\| + \int_0^t \|e^{A(t-\sigma)}\|_{\mathcal{L}(L^2, L^\infty)} \|f(S(\sigma)g)\|\, d\sigma \\ &\le C(2) t^{-3/4} \|g\| + \int_0^t C(2)(t-\sigma)^{-3/4} L(M_1, M_2) \|S(\sigma)g\|_E\, d\sigma, \end{aligned} \tag{4.2}$$

$t \ge 0$, where $C(2)$ is in the sense of (4.1) with $p = 2$, and $L(M_1, M_2)$ is the Lipschitz constant of the nonlinear map f on the closed, bounded ball with radius $M_1 + M_2$ in E. By the invariance of the global attractor $\mathscr{A}$, surely we have

$$\{S(t)\mathscr{A} : t \ge 0\} \subset B_0\, (\subset H) \quad \text{and} \quad \{S(t)\mathscr{A} : t \ge 0\} \subset B_1\, (\subset E).$$

Then from (4.2) we obtain

$$\begin{aligned} \|S(t)g\|_{L^\infty} &\le C(2) K_1 t^{-3/4} + \int_0^t C(2) L(M_1, M_2)(M_1 + M_2)(t-\sigma)^{-3/4}\, d\sigma \\ &= C(2)[K_1 t^{-3/4} + 4L(M_1, M_2)(M_1 + M_2) t^{1/4}], \quad \text{for } t > 0. \end{aligned} \tag{4.3}$$

Specifically one can take $t = 1$ in (4.3) and use the invariance of $\mathscr{A}$ to obtain

$$\|g\|_{L^\infty} \le C(2)(K_1 + 4L(M_1, M_2)(M_1 + M_2)), \quad \text{for any } g \in \mathscr{A}.$$

Thus the global attractor $\mathscr{A}$ is a bounded subset in $\mathbb{L}^\infty(\Omega)$. $\square$

Now consider the Hausdorff dimension and fractal dimension of the global attractor of the Brusselator semiflow $\{S(t)\}_{t\ge 0}$ in H. Let $q_m = \limsup_{t\to\infty} q_m(t)$, where

$$q_m(t) = \sup_{g_0 \in \mathscr{A}} \sup_{\substack{g_i \in H, \|g_i\| = 1 \\ i = 1, \dots, m}} \Big(\frac{1}{t} \int_0^t \operatorname{Tr}\,(A + F'(S(\tau)g_0)) \circ Q_m(\tau)\, d\tau \Big), \tag{4.4}$$

in which $\operatorname{Tr}\,(A + F'(S(\tau)g_0))$ is the trace of the linear operator $A + F'(S(\tau)g_0)$, with $F(g)$ being the nonlinear map in (1.12), and $Q_m(t)$ stands for the orthogonal projection of space H on the subspace spanned by $G_1(t), \dots, G_m(t)$, with

$$G_i(t) = L(S(t), g_0) g_i, \quad i = 1, \dots, m. \tag{4.5}$$

Here $F'(S(\tau)g_0)$ is the Fréchet derivative of the map F at $S(\tau)g_0$, and $L(S(t), g_0)$ is the Fréchet derivative of the map $S(t)$ at g_0, with t being fixed.

Next we study Hausdorff and fractal dimensions of the global attractor $\mathscr{A}$. The following proposition is seen in [34, Chapter 5].

Proposition 4.4. *If there is an integer m such that $q_m < 0$, then the Hausdorff dimension $d_H(\mathscr{A})$ and the fractal dimension $d_F(\mathscr{A})$ of $\mathscr{A}$ satisfy*

$$d_H(\mathscr{A}) \le m, \quad \text{and} \quad d_F(\mathscr{A}) \le m \max_{1\le j\le m-1}\Big(1 + \frac{(q_j)_+}{|q_m|}\Big) \le 2m. \tag{4.6}$$

It is standard to show that for any given $t > 0$, $S(t)$ on H is Fréchet differentiable and its Fréchet derivative at g_0 is the bounded linear operator $L(S(t), g_0)$ given by

$$L(S(t), g_0)G_0 \stackrel{\text{def}}{=} G(t) = (U(t), V(t), W(t), Z(t)),$$

for any $G_0 = (U_0, V_0, W_0, Z_0) \in H$, where $(U(t), V(t), W(t), Z(t))$ is the strong solution of the following initial-boundary value problem of the variational equations

$$\begin{gathered}
\frac{\partial U}{\partial t} = d_1\Delta U + 2u(t)v(t)U + u^2(t)V - (b+1)U + D_1(W-U),\\
\frac{\partial V}{\partial t} = d_2\Delta V - 2u(t)v(t)U - u^2(t)V + bU + D_2(Z-V),\\
\frac{\partial W}{\partial t} = d_1\Delta W + 2w(t)z(t)W + w^2(t)Z - (b+1)W + D_1(U-W),\\
\frac{\partial Z}{\partial t} = d_2\Delta Z - 2w(t)z(t)W - w^2(t)Z + bW + D_2(V-Z),\\
U\mid_{\partial\Omega} = V\mid_{\partial\Omega} = W\mid_{\partial\Omega} = Z\mid_{\partial\Omega} = 0, \quad t > 0,\\
U(0) = U_0, \quad V(0) = V_0, \quad W(0) = W_0, \quad Z(0) = Z_0.
\end{gathered} \tag{4.7}$$

Here $g(t) = (u(t), v(t), w(t), z(t)) = S(t)g_0$ is the solution of (1.12) with the initial condition $g(0) = g_0$. The initial-boundary value problem (4.7) can be written as

$$\frac{dG}{dt} = (A + F'(S(t)g_0))G, \quad G(0) = G_0. \tag{4.8}$$

From Lemma 3.2, Lemma 3.3 and the invariance of $\mathscr{A}$ it follows that

$$\sup_{g_0\in\mathscr{A}} \|S(t)g_0\|_E^2 \le M_1 + M_2. \tag{4.9}$$

Theorem 4.5. *The global attractors $\mathscr{A}$ for the Brusselator semiflow $\{S(t)\}_{t\ge 0}$ has a finite Hausdorff dimension and a finite fractal dimension.*

Proof. By Proposition 4.4, we shall estimate $\mathrm{Tr}(A + F'(S(\tau)g_0)) \circ Q_m(\tau)$. At any given time $\tau > 0$, let $\{\varphi_j(\tau) : j = 1, \dots, m\}$ be an H-orthonormal basis for the subspace

$$Q_m(\tau)H = \mathrm{span}\{G_1(\tau), \dots, G_,(\tau)\},$$

where $G_1(t), \dots, G_m(t)$ satisfy (4.8) with the respective initial values $G_{1,0}, \dots, G_{m,0}$ and, without loss of generality, assuming that $G_{1,0}, \dots, G_{m,0}$ are linearly independent in H. By Gram-Schmidt orthogonalization scheme,

$$\varphi_j(\tau) = (\varphi_j^1(\tau), \varphi_j^2(\tau), \varphi_j^3(\tau), \varphi_j^4(\tau)) \in E, \quad j = 1, \dots, m,$$

and $\varphi_j(\tau)$ are strongly measurable in τ. Let $d_0 = \min\{d_1, d_2\}$. Then

$$
\begin{aligned}
&\operatorname{Tr}(A + F'(S(\tau)g_0) \circ Q_m(\tau) \\
&= \sum_{j=1}^{m} \left(\langle A\varphi_j(\tau), \varphi_j(\tau)\rangle + \langle F'(S(\tau)g_0)\varphi_j(\tau), \varphi_j(\tau)\rangle \right) \\
&\le -d_0 \sum_{j=1}^{m} \|\nabla \varphi_j(\tau)\|^2 + J_1 + J_2 + J_3,
\end{aligned}
\tag{4.10}
$$

where

$$
\begin{aligned}
J_1 = &\sum_{j=1}^{m} \int_\Omega 2u(\tau)v(\tau) \left(|\varphi_j^1(\tau)|^2 - \varphi_j^1(\tau)\varphi_j^2(\tau) \right) dx \\
&+ \sum_{j=1}^{m} \int_\Omega 2w(\tau)z(\tau) \left(|\varphi_j^3(\tau)|^2 - \varphi_j^3(\tau)\varphi_j^4(\tau) \right) dx,
\end{aligned}
$$

$$
\begin{aligned}
J_2 &= \sum_{j=1}^{m} \int_\Omega \left(u^2(\tau) \left(\varphi_j^1(\tau)\varphi_j^2(\tau) - |\varphi_j^2(\tau)|^2 \right) + w^2(\tau) \left(\varphi_j^3(\tau)\varphi_j^4(\tau) - |\varphi_j^4(\tau)|^2 \right) \right) dx \\
&\le \sum_{j=1}^{m} \int_\Omega \left(u^2(\tau)|\varphi_j^1(\tau)||\varphi_j^2(\tau)| + w^2(\tau)|\varphi_j^3(\tau)||\varphi_j^4(\tau)| \right) dx,
\end{aligned}
$$

and

$$
\begin{aligned}
J_3 = &\sum_{j=1}^{m} \int_\Omega \left(-(b+1)(|\varphi_j^1(\tau)|^2 + |\varphi_j^3(\tau)|^2) + b(\varphi_j^1(\tau)\varphi_j^2(\tau) + \varphi_j^3(\tau)\varphi_j^4(\tau)) \right) dx \\
&- \sum_{j=1}^{m} \int_\Omega \left(D_1 \left(\varphi_j^1(\tau) - \varphi_j^3(\tau) \right)^2 + D_2 \left(\varphi_j^3(\tau) - \varphi_j^4(\tau) \right)^2 \right) dx \\
\le &\sum_{j=1}^{m} \int_\Omega b \left(\varphi_j^1(\tau)\varphi_j^2(\tau) + \varphi_j^3(\tau)\varphi_j^4(\tau) \right) dx.
\end{aligned}
$$

By the generalized Hölder inequality and the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ for $n \le 3$, and using (4.9), we obtain

$$
\begin{aligned}
J_1 \le &\, 2 \sum_{j=1}^{m} \|u(\tau)\|_{L^4} \|v(\tau)\|_{L^4} \left(\|\varphi_j^1(\tau)\|_{L^4}^2 + \|\varphi_j^1(\tau)\|_{L^4} \|\varphi_j^2(\tau)\|_{L^4} \right) \\
&+ 2 \sum_{j=1}^{m} \|w(\tau)\|_{L^4} \|z(\tau)\|_{L^4} \left(\|\varphi_j^3(\tau)\|_{L^4}^2 + \|\varphi_j^3(\tau)\|_{L^4} \|\varphi_j^4(\tau)\|_{L^4} \right) \\
\le &\, 4 \sum_{j=1}^{m} \|S(\tau)g_0\|_{L^4}^2 \|\varphi_j(\tau)\|_{L^4}^2 \le 4\delta(M_1 + M_2) \sum_{j=1}^{m} \|\varphi_j(\tau)\|_{L^4}^2,
\end{aligned}
\tag{4.11}
$$

where δ is the Sobolev embedding coefficient given at the beginning of Section 3. Now we apply the Garliardo-Nirenberg interpolation inequality, cf. [33, Theorem B.3],

$$
\|\varphi\|_{W^{k,p}} \le C \|\varphi\|_{W^{m,q}}^{\theta} \|\varphi\|_{L^r}^{1-\theta}, \quad \text{for } \varphi \in W^{m,q}(\Omega), \tag{4.12}
$$

provided that $p, q, r \geq 1, 0 < \theta \leq 1$, and

$$k - \frac{n}{p} \leq \theta\big(m - \frac{n}{q}\big) - (1-\theta)\frac{n}{r}, \quad \text{where } n = \dim \Omega.$$

Here let $W^{k,p}(\Omega) = L^4(\Omega), W^{m,q}(\Omega) = H_0^1(\Omega), L^r(\Omega) = L^2(\Omega)$, and $\theta = n/4 \leq 3/4$. It follows from (4.12) that

$$\|\varphi_j(\tau)\|_{L^4} \leq C\|\nabla\varphi_j(\tau)\|^{n/4}\|\varphi_j(\tau)\|^{1-\frac{n}{4}} = C\|\nabla\varphi_j(\tau)\|^{n/4}, \quad j = 1, \dots, m, \tag{4.13}$$

since $\|\varphi_j(\tau)\| = 1$, where C is a uniform constant. Substitute (4.13) into (4.11) to obtain

$$J_1 \leq 4\delta(M_1 + M_2)C^2 \sum_{j=1}^m \|\nabla\varphi_j(\tau)\|^{n/2}. \tag{4.14}$$

Similarly, by the generalized Hölder inequality, we can get

$$J_2 \leq \delta(M_1 + M_2)\sum_{j=1}^m \|\varphi_j(\tau)\|_{L^4}^2 \leq \delta(M_1 + M_2)C^2 \sum_{j=1}^m \|\nabla\varphi_j(\tau)\|^{n/2}. \tag{4.15}$$

Moreover, we have

$$J_3 \leq \sum_{j=1}^m b\|\varphi_j(\tau)\|^2 = bm. \tag{4.16}$$

Substituting (4.14), (4.15) and (4.16) into (4.10), we obtain

$$\begin{aligned} &\operatorname{Tr}(A + F'(S(\tau)g_0) \circ Q_m(\tau) \\ &\leq -d_0 \sum_{j=1}^m \|\nabla\varphi_j(\tau)\|^2 + 5\delta(M_1 + M_2)C^2 \sum_{j=1}^m \|\nabla\varphi_j(\tau)\|^{n/2} + bm. \end{aligned} \tag{4.17}$$

By Young's inequality, for $n \leq 3$, we have

$$5\delta(M_1 + M_2)C^2 \sum_{j=1}^m \|\nabla\varphi_j(\tau)\|^{n/2} \leq \frac{d_0}{2}\sum_{j=1}^m \|\nabla\varphi_j(\tau)\|^2 + \Gamma(n)m,$$

where $\Gamma(n)$ is a uniform constant depending only on $n = \dim(\Omega)$ and the involved constants δ, C, d_0, M_1 and M_2. Hence,

$$\operatorname{Tr}(A + F'(S(\tau)g_0) \circ Q_m(\tau) \leq -\frac{d_0}{2}\sum_{j=1}^m \|\nabla\varphi_j(\tau)\|^2 + (\Gamma(n) + b)\, m, \quad \tau > 0,\ g_0 \in \mathscr{A}.$$

According to the generalized Sobolev-Lieb-Thirring inequality [34, Appendix, Cor. 4.1], since $\{\varphi_1(\tau), \dots, \varphi_m(\tau)\}$ is an orthonormal set in H, so there exists a constant $\Psi > 0$ only depending on the shape and dimension of Ω, such that

$$\sum_{j=1}^m \|\nabla\varphi_j(\tau)\|^2 \geq \frac{\Psi\, m^{1+\frac{2}{n}}}{|\Omega|^{2/n}}. \tag{4.18}$$

Therefore, we end up with

$$\operatorname{Tr}(A + F'(S(\tau)g_0) \circ Q_m(\tau) \leq -\frac{d_0\Psi}{2|\Omega|^{2/n}} m^{1+\frac{2}{n}} + (\Gamma(n) + b)\, m, \tag{4.19}$$

for $\tau > 0$ and $g_0 \in \mathscr{A}$. Then we can conclude that

$$\begin{aligned} q_m(t) &= \sup_{g_0 \in \mathscr{A}} \sup_{\substack{g_i \in H, \|g_i\|=1 \\ i=1,\dots,m}} \Big(\frac{1}{t} \int_0^t \operatorname{Tr}\left(A + F'(S(\tau)g_0)\right) \circ Q_m(\tau)\, d\tau \Big) \\ &\le -\frac{d_0 \Psi}{2|\Omega|^{2/n}} m^{1+\frac{2}{n}} + (\Gamma(n) + b)\, m, \quad \text{for any } t > 0, \end{aligned} \tag{4.20}$$

so that

$$q_m = \limsup_{t \to \infty} q_m(t) \le -\frac{d_0 \Psi}{2|\Omega|^{2/n}} m^{1+\frac{2}{n}} + (\Gamma(n) + b)\, m < 0, \tag{4.21}$$

if the integer m satisfies the condition

$$m - 1 \le \Big(\frac{2(\Gamma(n) + b)}{d_0 \Psi} \Big)^{n/2} |\Omega| < m. \tag{4.22}$$

According to Proposition 4.4, we have shown that the Hausdorff dimension and the fractal dimension of the global attractor $\mathscr{A}$ are finite and their upper bounds are given by

$$d_H(\mathscr{A}) \le m \quad \text{and} \quad d_F(\mathscr{A}) \le 2m,$$

respectively, where m satisfies (4.22). □

5. Existence of an exponential attractor

In this final section, we shall prove the existence of an exponential attractor for the Brusselator semiflow $\{S(t)\}_{t \ge 0}$.

Definition 5.1. Let X be a real Banach space and $\{\Sigma(t)\}_{t\ge 0}$ be a semiflow on X. A set $\mathscr{E} \subset X$ is an exponential attractor for the semiflow $\{\Sigma(t)\}_{t \ge 0}$ in X, if the following conditions are satisfied:

(i) $\mathscr{E}$ is a nonempty, compact, positively invariant set in X,
(ii) $\mathscr{E}$ has a finite fractal dimension, and
(iii) $\mathscr{E}$ attracts every bounded set $B \subset X$ exponentially: there exist positive constants μ and $C(B)$ which depends on B, such that

$$\operatorname{dist}_X(\Sigma(t)B, \mathscr{E}) \le C(B) e^{-\mu t}, \quad \text{for } t \ge 0.$$

The basic theory and construction of exponential attractors were established in [10] for discrete and continuous semiflows on Hilbert spaces. The existence theory was generalized to semiflows on Banach spaces in [9] and extended to some nonlinear reaction-diffusion equations on unbounded domains and other equations including chemotaxis equations and some quasilinear parabolic equations.

Global attractors, exponential attractors, and inertial manifolds are the three major research topics in the area of infinite dimensional dynamical systems. For a continuous semiflow on a Hilbert space, if all the three objects (a global attractor $\mathscr{A}$, an exponential attractor $\mathscr{E}$, and an inertial manifold $\mathscr{M}$ of the same exponential attraction rate) exist, then the following inclusion relationship holds,

$$\mathscr{A} \subset \mathscr{E} \subset \mathscr{M}.$$

Here we shall tackle the existence of exponential attractor for the Brusselator semiflow by the argument of squeezing property [10, 23].

Definition 5.2. For a spectral (orthogonal) projection P_N relative to a nonnegative, self-adjoint, linear operator $\Lambda : D(\Lambda) \to \mathcal{H}$ with a compact resolvent, which maps the Hilbert space $\mathcal{H}$ onto the N-dimensional subspace $\mathcal{H}_N$ spanned by a set of the first N eigenvectors of the operator Λ, we defined a cone

$$\mathscr{C}_{P_N} = \{y \in X : \|(I - P_N)(y)\|_{\mathcal{H}} \le \|P_N(y)\|_{\mathcal{H}}\}.$$

A continuous mapping S_* satisfies the *discrete squeezing property* relative to a set $B \subset \mathcal{H}$ if there exist a constant $\kappa \in (0, 1/2)$ and a spectral projection P_N on $\mathcal{H}$ such that for any pair of points $y_0, z_0 \in B$, if

$$S_*(y_0) - S_*(z_0) \notin \mathscr{C}_{P_N},$$

then

$$\|S_*(y_0) - S_*(z_0)\|_{\mathcal{H}} \le \kappa \|y_0 - z_0\|_{\mathcal{H}}.$$

We first present the following lemma, which is a modified version of the basic result [23, Theorem 4.5] on the sufficient conditions for the existence of an exponential attractor of a semiflow on a Hilbert space. In some sense, this lemma provides a more accessible way to check these sufficient conditions if we are sure there exists an (X, Y) global attractor, such as the (H, E) global attractor for the Brusselator semiflow here. The following lemma was proved in [42, Lemma 6.1].

Lemma 5.3. *Let X be a real Banach space and Y be a compactly embedded subspace of X. Consider a semilinear evolutionary equation*

$$\frac{d\varphi}{dt} + \Lambda\varphi = g(\varphi), \quad t > 0, \tag{5.1}$$

where $\Lambda : D(\Lambda) \to X$ is a nonnegative, self-adjoint, linear operator with compact resolvent, and $g : Y = D(\Lambda^{1/2}) \to W$ is a locally Lipschitz continuous mapping. Suppose that the weak solution of (5.1) for each initial point $w(0) = w_0 \in W$ uniquely exists for all $t \ge 0$, which turn out to be a strong solution for $t > 0$ and altogether form a semiflow denoted by $\{\Sigma(t)\}_{t \ge 0}$. Assume that the following four conditions are satisfied:

(i) *There exist a compact, positively invariant, absorbing set $\mathcal{B}_c$ in X.*
(ii) *There is a positive integer N such that the norm quotient $Q(t)$ defined by*

$$Q(t) = \frac{\|\Lambda^{1/2}\left(\varphi_1(t) - \varphi_2(t)\right)\|_X^2}{\|\varphi_1(t) - \varphi_2(t)\|_X^2} \tag{5.2}$$

for any two trajectories $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ of (5.1) starting from the set $\mathcal{B}_c \backslash \mathscr{C}_{P_N}$ satisfies a differential inequality

$$\frac{dQ}{dt} \le \rho\left(\mathcal{B}_c\right) Q(t), \quad t > 0,$$

where $\rho\left(\mathcal{B}_c\right)$ is a positive constant only depending on $\mathcal{B}_c$.
(iii) *For any given finite $T > 0$ and any given $\varphi \in \mathcal{B}_c$, $\Sigma(\cdot)\varphi : [0, T] \to \mathcal{B}_c$ is Hölder continuous with exponent $\theta = 1/2$ and the coefficient of Hölder continuity, $K(\varphi) : \mathcal{B}_c \to (0, \infty)$, is a bounded function.*
(iv) *For any $t \in [0, T]$, where $T > 0$ is arbitrarily given, $\Sigma(t)(\cdot) : \mathcal{B}_c \to \mathcal{B}_c$ is a Lipschitz continuous mapping and the Lipschitz constant $L(t) : [0, T] \to (0, \infty)$ is a bounded function.*

Then there exists an exponential attractor $\mathscr{E}$ in X for this semiflow $\{\Sigma(t)\}_{t \ge 0}$.

The next theorem is another main result in this paper and it shows the existence of an exponential attractor for the solution semiflow $\{S(t)\}_{t\geq 0}$ in H by the approach through the existence of an (H,E) global attractor and Lemma 5.3.

Theorem 5.4. *For any positive parameters d_1, d_2, a, b, D_1 and D_2, there exists an exponential attractor $\mathscr{E}$ in H for the solution semiflow $\{S(t)\}_{t\geq 0}$ generated by the Brusselator evolutionary equation* (1.12).

Proof. By Theorem 4.3, there exists an (H,E) global attractor $\mathscr{A}$, which is exactly the global attractor of the Brusselator semiflow $\{S(t)\}_{t\geq 0}$ in H. Consequently, by [42, Corollary 5.7], there exists a compact, positively invariant, absorbing set $\mathcal{B}_E$ in H, which is a bounded set in E, for this semiflow.

Next we prove that the second condition in Lemma 5.3 is satisfied by this Brusselator semiflow. Consider any two points $g_1(0), g_2(0) \in \mathcal{B}_E$ and let $g_i(t) = (u_i(t), v_i(t), w_i(t), z_i(t))$, $i = 1, 2$, be the corresponding solutions, respectively. Let $y(t) = g_1(t) - g_2(t)$, $t \geq 0$. The associated norm quotient of the difference $g_1 - g_2$ of two trajectories, where $g_1(0) \neq g_2(0)$, is given by

$$Q(t) = \frac{\|(-A)^{1/2} y(t)\|^2}{\|y(t)\|^2}, \quad t \geq 0.$$

Directly we can calculate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}Q(t) \\
&= \frac{1}{\|y(t)\|^4}\Big[\langle (-A)^{1/2}y(t), (-A)^{1/2}y_t\rangle \|y(t)\|^2 - \|(-A)^{1/2}y(t)\|^2\langle y(t), y_t\rangle\Big] \\
&= \frac{1}{\|y(t)\|^2}\left[\langle (-A)y(t), y_t\rangle - Q(t)\langle y(t), y_t\rangle\right] \\
&= \frac{1}{\|y(t)\|^2}\langle (-A)y(t) - Q(t)y(t), Ay(t) + F(g_1(t)) - F(g_2(t))\rangle \\
&= \frac{1}{\|y(t)\|^2}\langle (-A)y(t) - Q(t)y(t), Ay(t) + Q(t)y(t) + F(g_1(t)) - F(g_2(t))\rangle \\
&= \frac{1}{\|y(t)\|^2}\left[-\|Ay(t) + Q(t)y(t)\|^2 - \langle Ay(t) + Q(t)y(t), F(g_1(t)) - F(g_2(t))\rangle\right] \\
&\leq \frac{1}{\|y(t)\|^2}\Big(-\frac{1}{2}\|Ay(t) + Q(t)y(t)\|^2 + \frac{1}{2}\|F(g_1(t)) - F(g_2(t))\|\Big)
\end{aligned} \tag{5.3}$$

where we used the identity $-\langle Ay(t) + Q(t)y(t), Q(t)y(t)\rangle = 0$. Note that $\mathcal{B}_E$ is a bounded set in E and that $E \hookrightarrow [L^6(\Omega)]^4$ is a continuous imbedding so that there is a uniform constant $R > 0$ only depending on $\mathcal{B}_E$ such that

$$\|(u,v,w,z)\|^2_{L^6(\Omega)} \leq R, \quad \text{for any } (u,v,w,z) \in \mathcal{B}_E. \tag{5.4}$$

It is seen that

$$\begin{aligned}
&\|F(g_1(t)) - F(g_2(t))\| \\
&\leq \|-(b+1)(u_1 - u_2) + \big(u_1^2 v_1 - u_2^2 v_2\big) - D_1((u_1 - u_2) - (w_1 - w_2))\| \\
&\quad + \|b(u_1 - u_2) - \big(u_1^2 v_1 - u_2^2 v_2\big) - D_2((v_1 - v_2) - (z_1 - z_2))\| \\
&\quad + \|-(b+1)(w_1 - w_2) + \big(w_1^2 z_1 - w_2^2 z_2\big) + D_1((u_1 - u_2) - (w_1 - w_2))\| \\
&\quad + \|b(w_1 - w_2) - \big(w_1^2 z_1 - w_2^2 z_2\big) + D_2((v_1 - v_2) - (z_1 - z_2))\|.
\end{aligned} \tag{5.5}$$

Using the Hölder inequality, the imbedding inequality $\|\cdot\|_{L^6}^2 \le \eta\|\cdot\|_E^2$, and Poincaré inequality orderly, we have

$$\begin{aligned}\|u_1 - u_2\|^2 &\le |\Omega|^{2/3}\|u_1 - u_2\|_{L^6(\Omega)}^2 \\ &\le |\Omega|^{2/3}\eta\|\nabla(u_1 - u_2)\|^2 \\ &= c_1\|(-A)^{1/2}\,(g_1 - g_2)\,\|^2,\end{aligned}$$

where $c_1 = |\Omega|^{2/3}\eta\, d_1$. Similarly, we have

$$\|w_1 - w_2\|^2 \le c_1\|(-A)^{1/2}\,(g_1 - g_2)\,\|^2,$$

and

$$\|v_1 - v_2\|^2 \le c_2\|(-A)^{1/2}\,(g_1 - g_2)\,\|^2, \quad \|z_1 - z_2\|^2 \le c_2\|(-A)^{1/2}\,(g_1 - g_2)\,\|^2,$$

where $c_2 = |\Omega|^{2/3}\eta\, d_2$. By the generalized Hölder inequality and (5.4), we have

$$\begin{aligned}&\|u_1^2 v_1 - u_2^2 v_2\|^2 \\ &\le 2\|u_1 - u_2\|_{L^6(\Omega)}^2\|u_1 + u_2\|_{L^6(\Omega)}^2\|v_2\|_{L^6(\Omega)}^2 + 2\|v_1 - v_2\|_{L^6(\Omega)}^2\|u_1\|_{L^6(\Omega)}^4 \\ &\le 8R^2\|u_1 - u_2\|_{L^6(\Omega)}^2 + 2R^2\|v_1 - v_2\|_{L^6(\Omega)}^2 \\ &\le c_3(R)\|(-A)^{1/2}\,(g_1 - g_2)\,\|^2,\end{aligned}$$

and similarly,

$$\|w_1^2 z_1 - w_2^2 z_2\|^2 \le c_3(R)\|(-A)^{1/2}\,(g_1 - g_2)\,\|^2,$$

where $c_3(R) = 2\eta\,(4d_1 + d_2)\,R^2$.

Substituting these inequalities into (5.5), we obtain

$$\begin{aligned}&\|F\,(g_1(t)) - F\,(g_2(t))\,\| \\ &\le 2\left(\sqrt{c_1(b + 1 + D_1)} + \sqrt{c_2\,D_2} + \sqrt{c_3(R)}\right)\|(-A)^{1/2}y(t)\|.\end{aligned} \tag{5.6}$$

Then substitution of (5.6) into (5.3) yields

$$\frac{d}{dt}\,Q(t) \le \frac{1}{\|y(t)\|^2}\|F\,(g_1(t)) - F\,(g_2(t))\,\| \le \rho(\mathcal{B}_E)Q(t), \quad t > 0, \tag{5.7}$$

where

$$\rho(\mathcal{B}_E) = 2\left(\sqrt{c_1(b + 1 + D_1)} + \sqrt{c_2\,D_2} + \sqrt{c_3(R)}\right)$$

is a positive constant only depending on R which depends on $\mathcal{B}_E$. Thus the second condition in Lemma 5.3 is satisfied.

Now check the Hölder continuity of $S(\cdot)g : [0, T] \to \mathcal{B}_E$ for any given $g \in \mathcal{B}_E$ and any given compact interval $[0, T]$. By the mild solution formula, for any $0 \le t_1 < t_2 \le T$ we obtain

$$\begin{aligned}\|S(t_2)g - S\,(t_1)\,g\| \le \|\big(e^{A(t_2 - t_1)} - I\big)e^{At_1}g\| &+ \int_{t_1}^{t_2}\|e^{A(t_2-\sigma)}F(S(\sigma)g)\|d\sigma \\ &+ \int_0^{t_1}\|\big(e^{A(t_2-t_1)} - I\big)e^{A(t_1-\sigma)}F(S(\sigma)g)\|d\sigma.\end{aligned} \tag{5.8}$$

Since $\mathcal{B}_E$ is positively invariant with respect to the Brusselator semiflow $\{S(t)\}_{t\ge 0}$ and $\mathcal{B}_E$ is bounded in E, there exists a constant $K_{\mathcal{B}_E} > 0$ such that for any $g \in \mathcal{B}_E$, we have

$$\|S(t)g\|_E \le K_{\mathcal{B}_E}, \quad t \ge 0.$$

Since $F : E \to H$ is locally Lipschitz continuous, there is a Lipschitz constant $L_{\mathcal{B}_E} > 0$ of F relative to this positively invariant set $\mathcal{B}_E$. Moreover, by [33, Theorem 37.5], for the analytic, contracting, linear semigroup $\{e^{At}\}_{t\geq 0}$, there exist positive constants N_0 and N_1 such that

$$\|e^{At}g - g\|_H \leq N_0\, t^{1/2}\|g\|_E, \quad \text{for } t \geq 0,\ w \in E,$$

and

$$\|e^{At}\|_{\mathcal{L}(H,E)} \leq N_1 t^{-1/2}, \quad \text{for } t > 0.$$

It follows that

$$\|\big(e^{A(t_2-t_1)} - I\big)e^{At_1}g\| \leq N_0(t_2-t_1)^{1/2}K_{\mathcal{B}_E}$$

and

$$\int_{t_1}^{t_2} \|e^{A(t_2-\sigma)}F(S(\sigma)g)\|d\sigma \leq \int_{t_1}^{t_2} \frac{N_1 L_{\mathcal{B}_E} K_{\mathcal{B}_E}}{\sqrt{t_2-\sigma}}\, d\sigma = 2K_{\mathcal{B}_E}L_{\mathcal{B}_E}N_1(t_2-t_1)^{1/2}.$$

Moreover,

$$\begin{aligned}\int_0^{t_1} \|\big(e^{A(t_2-t_1)} - I\big)e^{A(t_1-\sigma)}F(S(\sigma)g)\|d\sigma &\leq N_0(t_2-t_1)^{1/2}\int_0^{t_1} \frac{N_1 L_{\mathcal{B}_E}K_{\mathcal{B}_E}}{\sqrt{t_1-\sigma}}\, d\sigma \\ &= 2K_{\mathcal{B}_E}L_{\mathcal{B}_E}N_0N_1\sqrt{T}(t_2-t_1)^{1/2}.\end{aligned}$$

Substituting the above three inequalities into (5.8), we obtain

$$\|S(t_2)g - S(t_1)g\| \leq K_{\mathcal{B}_E}\Big(N_0 + 2L_{\mathcal{B}_E}N_1(1+N_0\sqrt{T})\Big)(t_2-t_1)^{1/2}, \tag{5.9}$$

for $0 \leq t_1 < t_2 \leq T$. Thus the third condition in Lemma 5.3 is satisfied. Namely, for any given $T > 0$, the mapping $S(\cdot)g : [0,T] \to \mathcal{B}_E$ is Hölder continuous with the exponent $1/2$ and with a uniformly bounded coefficient independent of $g \in \mathcal{B}_E$.

We can use Theorem 47.8 (specifically (47.20) therein) in [33] to confirm the Lipschitz continuity of the mapping $S(t)(\cdot) : \mathcal{B}_E \to \mathcal{B}_E$ for any $t \in [0,T]$ where $T > 0$ is arbitrarily given. Thus the fourth condition in Lemma 5.3 is also satisfied. Finally, we apply Lemma 5.3 to reach the conclusion of this theorem. □

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