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POSITIVE SOLUTIONS FOR SECOND-ORDER MULTI-POINT BOUNDARY-VALUE PROBLEMS AT RESONANCE IN BANACH SPACES

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Abstract. In this article, we study the existence and multiplicity of positive solutions for a nonlinear second-order multi-point boundary-value problem at resonance in Banach spaces. The arguments are based upon a specially constructed equivalent equation and the fixed point theory in a cone for strict set contraction operators.

1. Introduction

The theory of ordinary differential equations in Banach spaces has become a new important branch (see, for example, [2, 6, 7, 12] and references cited therein). In 1988, Guo and Lakshmikantham [8] discussed the existence of multiple solutions for two-point boundary value problem of ordinary differential equations in Banach spaces. Since then, nonlinear second-order multi-point boundary value problems at non-resonance in Banach spaces have been studied by several authors (see, for example, [5, 13, 14, 18] and references cited therein). Recently, the existence of solutions for boundary value problems at resonance have been studied by many papers, (see, for example [3, 4, 9, 10, 11, 15, 16, 19]). Using the Krasnolsel'skii-Guo fixed point theorem, Han [9] studied a second order three-point BVP at resonance by rewriting the original BVP as an equivalent one. Motivated by their results, in this paper, we will discuss the existence of positive solutions for the second-order m-point boundary value problem at resonance

$$y''(t) = f(t, y), \quad 0 < t < 1, \tag{1.1}$$

$$y'(0) = \theta, \quad y(1) = \sum_{i=1}^{m-2} k_i y(\xi_i) \tag{1.2}$$

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in a real Banach space E, where θ is the zero element of E, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $k_i > 0$, $i = 1, 2, \dots, m-2$, $\sum_{i=1}^{m-2} k_i = 1$.

The boundary value problem (1.1)-(1.2) is at resonance when $\sum_{i=1}^{m-2} k_i = 1$; that is, the corresponding homogeneous boundary value problem

$$y''(t) = 0, \quad t \in [0, 1],$$
$$y'(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} k_i y(\xi_i)$$

has nontrivial solutions.

To the best of our knowledge, no paper has considered the existence of positive solutions for the boundary value problems at resonance in Banach spaces. We shall fill this gap in the literature. The organization of this paper is as follows. We shall introduce a theorem and some notations in the rest of this section. In Section 2, we provide some necessary background. In particular, we state some properties of Green's function associated with the equivalent problem of (1.1)-(1.2). In Section 3, the main results will be stated and proved.

Theorem 1.1 ([1, 17]). *Let K be a cone of the real Banach space X and $K_{r,R} = \{x \in K | r \le \|x\| \le R\}$ with $R > r > 0$. Assume that $A: K_{r,R} \to K$ is a strict set contraction such that one of the following two conditions is satisfied*

(i) *$Ax \not\le x$ for all $x \in K$, $\|x\| = r$ and $Ax \not\ge x$ for all $x \in K$, $\|x\| = R$.*
(ii) *$Ax \not\ge x$ for all $x \in K$, $\|x\| = r$ and $Ax \not\le x$ for all $x \in K$, $\|x\| = R$.*

Then A has at least one fixed point $x \in K$ satisfying $r < \|x\| < R$.

Let the real Banach space E with norm $\|\cdot\|$ be partially ordered by a normal cone P of E; i.e., $x \le y$ if and only if $y - x \in P$, and P^* denotes the dual cone of P; i.e., $P^* = \{\varphi \in E^* : \varphi(x) \ge 0, \ x \in P\}$. Denote the normal constant of P by N; i.e., $\theta \le x \le y$ implies $\|x\| \le N\|y\|$. Take $I = [0, 1]$. For any $x \in C[I, E]$, evidently, $(C[I, E], \|\cdot\|_c)$ is a Banach space with $\|x\|_c = \max_{t \in I} \|x(t)\|$, and $Q = \{x \in C[I, E] : x(t) \ge \theta \text{ for } t \in I\}$ is a cone of the Banach space $C[I, E]$. A function $y \in C^2[I, E]$ is called a positive solution of the boundary value problem (1.1)-(1.2) if it satisfies (1.1)-(1.2) and $y \in Q$, $y(t) \not\equiv \theta$.

In this paper, we denote $\alpha(\cdot)$ the Kuratowski measure of non-compactness of a bounded set in E and $C[I, E]$. The closed balls in spaces E and $C[I, E]$ are denoted by $T_r = \{x \in E : \|x\| \le r\} (r > 0)$ and $B_r = \{y \in C[I, E] : \|y\|_c \le r\} (r > 0)$, respectively.

Define

$$F(t, y) := f(t, y) + \beta^2 y,$$

where $\beta \in (0, \frac{\pi}{2})$. Obviously, $y(t)$ is a solution of the problem (1.1)-(1.2) if and only if it is a solution of the problem

$$y''(t) + \beta^2 y(t) = F(t, y(t)), \quad 0 < t < 1, \tag{1.3}$$
$$y'(0) = \theta, \quad y(1) = \sum_{i=1}^{m-2} k_i y(\xi_i), \tag{1.4}$$

and the problem (1.3)-(1.4) is at non-resonance.

For convenience, we set

$$a_0 = \sum_{i=1}^{m-2} k_i \cos\beta\xi_i - \cos\beta, \quad a_1 = (a_0+1)\sin\beta + \sum_{i=1}^{m-2} k_i \sin\beta\xi_i,$$

$$a_2 = 1 - \sum_{i=1}^{m-2} k_i \cos\beta(1-\xi_i).$$

In this paper, we assume the following conditions hold.

- (H1) $k_i > 0$, $i = 1, 2, \dots, m-2$, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $\sum_{i=1}^{m-2} k_i = 1$.
- (H2) P is a normal cone of E and N is the normal constant; $F : I \times P \to P$, $F(t,\theta) = \theta$ for all $t \in I$; for any $r > 0$, $F(t,x)$ is uniformly continuous and bounded on $I \times (P \cap T_r)$ and there exists a constant L_r with $0 \le L_r < (\beta a_0)/(2a_1)$ such that
$$\alpha(F(I \times D)) \le L_r \alpha(D), \quad \forall D \subset P \cap T_r.$$

2. Preliminary lemmas

Lemma 2.1. *Assume $\sum_{i=1}^{m-2} k_i = 1$, then for $h(t) \in C[I, E]$, the problem*

$$y''(t) + \beta^2 y(t) = h(t), \quad 0 < t < 1, \tag{2.1}$$

$$y'(0) = \theta, \quad y(1) = \sum_{i=1}^{m-2} k_i y(\xi_i) \tag{2.2}$$

has a unique solution

$$\begin{aligned} y(t) &= \frac{1}{\beta}\int_0^t \sin\beta(t-s)h(s)ds + \frac{\cos\beta t}{\beta a_0}\Big[\int_0^1 \sin\beta(1-s)h(s)ds \\ &\quad - \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \sin\beta(\xi_i - s)h(s)ds\Big] \\ &=: \int_0^1 G(t,s)h(s)ds, \end{aligned} \tag{2.3}$$

where

$$G(t,s) = \begin{cases} \frac{1}{\beta}\sin\beta(t-s) + \frac{\cos\beta t}{\beta a_0}[\sin\beta(1-s) - \sum_{j=i}^{m-2} k_j \sin\beta(\xi_j - s)], \\ \quad \text{if } \xi_{i-1} \le s \le \min\{t, \xi_i\}, ; i = 1, 2, \dots, m-1; \\ \frac{\cos\beta t}{\beta a_0}[\sin\beta(1-s) - \sum_{j=i}^{m-2} k_j \sin\beta(\xi_j - s)], \\ \quad \text{if } \max\{\xi_{i-1}, t\} \le s \le \xi_i, \ i = 1, 2, \dots, m-1. \end{cases}$$

The proof of the above lemma is easy, so we omit it.

Lemma 2.2. *There exist c_1, $c_2 > 0$ such that*

$$c_1(1-s) \le G(t,s) \le c_2(1-s), \quad t, s \in [0,1].$$

Proof. Take $H(t,s) = c(1-s) - G(t,s)$. We will prove that $H(t,s) \ge 0$, $t, s \in [0,1]$, when c is sufficiently large. For $t, s \in [0,1]$, we have

$$\begin{aligned} H(t,s) &\ge c(1-s) - \frac{1}{\beta}\sin\beta(t-s) - \frac{\cos\beta t}{\beta a_0}\sin\beta(1-s) \\ &\ge c(1-s) - \frac{1}{\beta}\sin\beta(1-s) - \frac{\sin\beta(1-s)}{\beta a_0} \end{aligned}$$

$$\begin{aligned}&= c(1-s) - \frac{1}{\beta}[1+\frac{1}{a_0}]\sin\beta(1-s)\\&\geq (c-1-\frac{1}{a_0})(1-s).\end{aligned}$$

Take $c_2 \geq 1 + \frac{1}{a_0}$, then $H(t,s) \geq 0$, $t, s \in [0,1]$.

Now, we prove $H(t,s) \leq 0$, $t, s \in [0,1]$, when c is sufficiently small. For $t \in [0,1]$, $s \in (\xi_1, 1]$, we have

$$\begin{aligned}H(t,s) &\leq c(1-s) - \frac{\cos\beta t}{\beta a_0}[\sin\beta(1-s) - \sum_{j=i}^{m-2} k_j \sin\beta(\xi_j - s)]\\&\leq c(1-s) - \frac{\cos\beta}{\beta a_0}[\sin\beta(1-s) - \sum_{j=2}^{m-2} k_j \sin\beta(\xi_j - s)]\\&\leq c(1-s) - \frac{k_1\cos\beta}{\beta a_0}\sin\beta(1-s).\end{aligned}$$

Since

$$g(x) = \begin{cases}\frac{\sin x}{x}, & 0 < x \leq \pi/2,\\ 1, & x = 0\end{cases}$$

is continuous on $[0, \pi/2]$. So, we obtain

$$\min_{x\in[0,\pi/2]} g(x) := m_0 > 0;$$

i.e., $\sin x \geq m_0 x$, $x \in [0, \pi/2]$. Therefore,

$$\begin{aligned}H(t,s) &\leq c(1-s) - \frac{m_0 k_1 \cos\beta}{a_0}(1-s)\\&= (c - \frac{m_0 k_1\cos\beta}{a_0})(1-s).\end{aligned}$$

For $t \in [0,1]$, $s \in [0, \xi_1]$, we obtain

$$\begin{aligned}H(t,s) &\leq c(1-s) - \frac{\cos\beta t}{\beta a_0}[\sin\beta(1-s) - \sum_{j=1}^{m-2} k_j \sin\beta(\xi_j - s)]\\&\leq c(1-s) - \frac{\cos\beta}{\beta a_0} 2\sum_{j=1}^{m-2} k_j \cos\frac{\beta(1+\xi_j-2s)}{2}\sin\frac{\beta(1-\xi_j)}{2}\\&\leq c - \frac{2\cos\beta}{\beta a_0}\sum_{i=1}^{m-2} k_i \cos\frac{\beta(1+\xi_i)}{2}\sin\frac{\beta(1-\xi_i)}{2}.\end{aligned}$$

Take

$$0 < c_1 \leq \min\big\{\frac{m_0 k_1\cos\beta}{a_0}, \frac{2\cos\beta(\sum_{i=1}^{m-2} k_i\cos\frac{\beta(1+\xi_i)}{2}\sin\frac{\beta(1-\xi_i)}{2})}{\beta a_0}\big\}.$$

Then we have $c_1(1-s) \leq G(t,s)$, $t, s \in [0,1]$. The proof is complete. □

Lemma 2.3. *Assume* (H1) *holds. If $h \in Q$, then the unique solution y of* (2.1)-(2.2) *satisfies $y(t) \geq \theta$, $t \in I$ and $y(t) \geq \gamma y(s)$ for all $t, s \in I$, where $\gamma = c_1/c_2$.*

Proof. Obviously, $y(t) \geq \theta$ for all $t \in I$. By Lemma 2.2, we obtain

$$y(t) \geq c_1 \int_0^1 (1-s)h(s)ds = \frac{c_1}{c_2} \int_0^1 c_2(1-s)h(s)ds \geq \gamma y(r), \quad \forall t,\ r \in I.$$

The proof is complete. $\square$

Define an operator $A : Q \to C[I, E]$ as follows

$$\begin{aligned} A(y(t)) := \frac{1}{\beta} \int_0^t \sin\beta(t-s)F(s, y(s))ds + \frac{\cos\beta t}{\beta a_0}[\int_0^1 \sin\beta(1-s)F(s, y(s))ds \\ - \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \sin\beta(\xi_i - s)F(s, y(s))ds]. \end{aligned} \tag{2.4}$$

By Lemmas 2.1 and 2.3, we obtain that $A : Q \to C^2[I, E] \cap Q$, and $y(t)$ is a positive solution of (1.1)-(1.2) if and only if $y(t) \in C^2[I, E] \cap Q$ and $y(t) \not\equiv \theta$ is a fixed point of the operator A.

Lemma 2.4. *Assume* (H1), (H2) *hold. Then, for any $r > 0$, the operator A is a strict set contraction on $Q \cap B_r$.*

Proof. Since $F(t, x)$ is uniformly continuous and bounded on $I \times (P \cap T_r)$, we see from (2.4) that A is continuous and bounded on $Q \cap B_r$. For any $S \subset Q \cap B_r$, by (2.4), we can easily get that functions $A(S) = \{Ay|y \in S\}$ are uniformly bounded and equicontinuous. By [12], we have

$$\alpha(A(S)) = \sup_{t \in I} \alpha(A(S(t))), \tag{2.5}$$

where $A(S(t)) = \{Ay(t) : y \in S,\ t \in I \text{ is fixed}\}$. For any $y \in C[I, E]$, $g \in C[I, I]$, by $\int_0^t g(s)y(s)ds \in \overline{\text{co}}(\{g(t)y(t)|t \in I\} \cup \{\theta\}) \subset \overline{\text{co}}(\{y(t)|t \in I\} \cup \{\theta\})$, we obtain

$$\begin{aligned} &\alpha(A(S(t))) \\ &= \alpha(\{\frac{1}{\beta} \int_0^t \sin\beta(t-s)F(s, y(s))ds + \frac{\cos\beta t}{\beta a_0}\Big[\int_0^1 \sin\beta(1-s)F(s, y(s))ds \\ &\quad - \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \sin\beta(\xi_i - s)F(s, y(s))ds\Big] : y \in S\}) \\ &\leq \frac{\sin\beta}{\beta}\alpha(\overline{\text{co}}(\{F(s, y(s)) : s \in I,\ y \in S\} \cup \{\theta\})) \\ &\quad + \frac{\sin\beta}{\beta a_0}\alpha(\overline{\text{co}}(\{F(s, y(s)) : s \in I,\ y \in S\} \cup \{\theta\})) \\ &\quad + \frac{\sum_{i=1}^{m-2} k_i \sin\beta\xi_i}{\beta a_0}\alpha(\overline{\text{co}}(\{F(s, y(s)) : s \in I,\ y \in S\} \cup \{\theta\})) \\ &= \frac{a_1}{\beta a_0}\alpha(\{F(s, y(s)) : s \in I,\ y \in S\}) \\ &\leq \frac{a_1}{\beta a_0}\alpha(F(I \times B)), \end{aligned}$$

where $B = \{y(s) : s \in I,\ y \in S\} \subset P \cap T_r$.

By (H2), we obtain

$$\alpha(A(S(t))) \leq \frac{a_1}{\beta a_0} L_r \alpha(B). \tag{2.6}$$

For any given $\varepsilon > 0$, there exists a partition $S = \cup_{j=1}^{l} S_j$ such that

$$\operatorname{diam}(S_j) < \alpha(S) + \frac{\varepsilon}{3}, \quad j = 1, 2, \dots, l. \tag{2.7}$$

Now, choose $y_j \in S_j$, $j = 1, 2, \dots, l$ and a partition $0 = t_0 < t_1 < \dots < t_k = 1$ such that

$$\|y_j(t) - y_j(\bar{t})\| < \frac{\varepsilon}{3}, \quad \forall t,\ \bar{t} \in [t_{i-1}, t_i],\ j = 1, 2, \dots, l,\ i = 1, 2, \dots, k. \tag{2.8}$$

Obviously, $B = \cup_{j=1}^{l} \cup_{i=1}^{k} B_{ij}$, where $B_{ij} = \{y(t) : y \in S_j, t \in [t_{i-1},\ t_i]\}$. For any $y(t), \overline{y}(\bar{t}) \in B_{ij}$, by (2.7) and (2.8), we obtain

$$\begin{aligned}
\|y(t) - \overline{y}(\bar{t})\| &\le \|y(t) - y_j(t)\| + \|y_j(t) - y_j(\bar{t})\| + \|y_j(\bar{t}) - \overline{y}(\bar{t})\| \\
&\le \|y - y_j\|_c + \frac{\varepsilon}{3} + \|y_j - \overline{y}\|_c \\
&\le 2 \operatorname{diam}(S_j) + \frac{\varepsilon}{3} < 2\alpha(S) + \varepsilon,
\end{aligned}$$

which implies $\operatorname{diam}(B_{ij}) \le 2\alpha(S) + \varepsilon$, and so, $\alpha(B) \le 2\alpha(S) + \varepsilon$. Since ε is arbitrary, we obtain

$$\alpha(B) \le 2\alpha(S). \tag{2.9}$$

It follows from (2.5), (2.6) and (2.9) that

$$\alpha(A(S)) \le \frac{2a_1}{\beta a_0} L_r \alpha(S), \quad \forall S \subset Q \cap B_r.$$

By (H2), we obtain that A is a strict set contraction on $Q \cap B_r$. $\square$

3. Main results

Let $K = \{y \in Q : y(t) \ge \gamma y(s),\ \forall t,\ s \in I\}$. Clearly, $K \subset Q$ is a cone of $C[I, E]$. By Lemmas 2.1 and 2.3, we obtain $AQ \subset K$. So, $AK \subset K$.

For convenience, for any $x \in P$ and $\varphi \in P^*$, we set

$$F^0 = \limsup_{\|x\| \to 0} \sup_{t \in I} \frac{\|F(t, x)\|}{\|x\|}, \quad F^\infty = \limsup_{\|x\| \to \infty} \sup_{t \in I} \frac{\|F(t, x)\|}{\|x\|},$$

$$F_0^\varphi = \liminf_{\|x\| \to 0} \inf_{t \in I} \frac{\varphi(F(t, x))}{\varphi(x)}, \quad F_\infty^\varphi = \liminf_{\|x\| \to \infty} \inf_{t \in I} \frac{\varphi(F(t, x))}{\varphi(x)}$$

and list the following assumptions:

- (H3) There exists $\varphi \in P^*$ such that $\varphi(x) > 0$ for any $x > \theta$ and $F_0^\varphi > \frac{\beta^2 a_0}{\gamma a_2}$.
- (H4) There exists $\varphi \in P^*$ such that $\varphi(x) > 0$ for any $x > \theta$ and $F_\infty^\varphi > \frac{\beta^2 a_0}{\gamma a_2}$.
- (H5) $F^0 < \frac{\beta^2 a_0}{N(1+a_0)(1-\cos\beta)}$.
- (H6) $F^\infty < \frac{\beta^2 a_0}{N(1+a_0)(1-\cos\beta)}$.
- (H7) There exists $r_0 > 0$ such that
$$\sup_{\substack{t \in I,\ x \in P \\ \gamma r_0/N \le \|x\| \le r_0}} \|F(t,\ x)\| < \frac{\beta^2 a_0}{N(1 + a_0)(1 - \cos\beta)} r_0.$$
- (H8) There exist $R_0 > 0$ and $\varphi \in P^*$ with $\varphi(x) > 0$ for any $x > \theta$ such that
$$\inf_{\substack{t \in I, x \in P \\ \gamma R_0/N \le \|x\| \le R_0}} \frac{\varphi(F(t, x))}{\varphi(x)} > \frac{\beta^2 a_0}{\gamma a_2}.$$

Theorem 3.1. *Assume* (H1), (H2) *hold. If one of the following conditions is satisfied:*

(i) (H4) *and* (H5) *hold.*
(ii) (H3) *and* (H6) *hold.*

Then the problem (1.1)-(1.2) *has at least one positive solution.*

Proof. (i) By (H4), we obtain that there exist constants $M > \frac{\beta^2 a_0}{\gamma a_2}$ and $r_1 > 0$ such that

$$\varphi(F(t,x)) \geq M\varphi(x), \quad \forall t \in I, x \in P, \quad \|x\| > r_1. \tag{3.1}$$

For any $R > Nr_1/\gamma$, we will show that

$$Ay \not\leq y, \quad \forall y \in K, \ \|y\|_c = R. \tag{3.2}$$

In fact, if not, there exists $y_0 \in K$, $\|y_0\|_c = R$ such that $Ay_0 \leq y_0$. By

$$y_0(t) \geq \gamma y_0(s) \geq \theta, \quad \forall t, \ s \in I, \tag{3.3}$$

we have

$$\|y_0(t)\| \geq \frac{\gamma}{N}\|y_0\|_c > r_1, \quad \forall t \in I. \tag{3.4}$$

By (2.4), for any $t \in I$, we have

$$\begin{aligned}
A(y_0(t)) &= \frac{1}{\beta}\int_0^t \sin\beta(t-s)F(s,y_0(s))ds + \frac{\cos\beta t}{\beta a_0}\Big[\int_0^1 \sin\beta(1-s)F(s,y_0(s))ds \\
&\quad - \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \sin\beta(\xi_i - s)F(s,y_0(s))ds\Big] \\
&\geq \frac{\cos\beta t}{\beta a_0}\sum_{i=1}^{m-2} k_i \int_{\xi_i}^1 \sin\beta(1-s)F(s,y_0(s))ds.
\end{aligned}$$

This inequality, (3.1), (3.3) and (3.4), imply

$$\begin{aligned}
\varphi(Ay_0(0)) &\geq \frac{1}{\beta a_0}\sum_{i=1}^{m-2} k_i \int_{\xi_i}^1 \sin\beta(1-s)M\gamma\varphi(y_0(0))ds \\
&= \frac{a_2}{\beta^2 a_0}M\gamma\varphi(y_0(0)).
\end{aligned}$$

Considering $Ay_0 \leq y_0$, we obtain

$$\varphi(y_0(0)) \geq \frac{\gamma a_2}{\beta^2 a_0}M\varphi(y_0(0)). \tag{3.5}$$

It is easy to see that $\varphi(y_0(0)) > 0$ (In fact, if $\varphi(y_0(0)) = 0$, by (3.3), we obtain $\varphi(y_0(0)) \geq \gamma\varphi(y_0(s)) \geq 0$ for all $s \in I$. So, we have $\varphi(y_0(s)) \equiv 0$ for all $s \in I$. That is, $y_0(s) \equiv \theta$. This is a contradiction with $\|y_0\|_c = R$). So, (3.5) contradicts with $M > \frac{\beta^2 a_0}{\gamma a_2}$. Therefore, (3.2) is true.

On the other hand, by (H5) and $F(t,\theta) = \theta$, we obtain that there exist constants $0 < \varepsilon < \frac{\beta^2 a_0}{N(1+a_0)(1-\cos\beta)}$ and $0 < r_2 < R$ such that

$$\|F(t,x)\| \leq \varepsilon\|x\|, \quad \forall t \in I, \ x \in P, \ \|x\| < r_2. \tag{3.6}$$

For any $0 < r < r_2$, we now prove that

$$Ay \not\geq y, \quad \forall y \in K, \ \|y\|_c = r. \tag{3.7}$$

In fact, if not, there exists $y_0 \in K$, $\|y_0\|_c = r$ such that $Ay_0 \geq y_0$. Since (2.4) implies

$$Ay_0(t) \leq \frac{a_0 + \cos\beta t}{\beta a_0} \int_0^1 \sin\beta(1-s)F(s, y_0(s))ds, \ \forall t \in I. \tag{3.8}$$

So, we have

$$\theta \leq y_0(t) \leq \frac{a_0 + \cos\beta t}{\beta a_0} \int_0^1 \sin\beta(1-s)F(s, y_0(s))ds, \quad \forall t \in I.$$

This, together with (3.6), imply

$$\|y_0(t)\| \leq \frac{N(1+a_0)\varepsilon}{\beta a_0} \int_0^1 \sin\beta(1-s)\|y_0(s)\|ds = \frac{N(1+a_0)(1-\cos\beta)\varepsilon\|y_0\|_c}{\beta^2 a_0},$$

for all $t \in I$. Therefore, we obtain $\varepsilon \geq \beta^2 a_0/N(1+a_0)(1-\cos\beta)$. This is a contradiction. So, (3.7) is true.

By (3.2), (3.7), Lemma 2.4 and Theorem 1.1, we obtain that the operator A has at least one fixed point $y \in K$ satisfying $r < \|y\|_c < R$.

(ii) By (H3), in the same way as establishing (3.2) we can assert that there exists $r_2 > 0$ such that for any $0 < r < r_2$,

$$Ay \not\leq y, \quad \forall y \in K, \ \|y\|_c = r. \tag{3.9}$$

On the other hand, by (H6), we obtain that there exist constants $r_1 > 0$ and ε, with $0 < \varepsilon < \frac{\beta^2 a_0}{N(1+a_0)(1-\cos\beta)}$, such that

$$\|F(t,x)\| \leq \varepsilon\|x\|, \quad \forall t \in I, \ x \in P, \ \|x\| > r_1.$$

By (H2), we obtain

$$\sup_{t\in I,\, x\in P\cap T_{r_1}} \|F(t,x)\| =: b < \infty.$$

So, we have

$$\|F(t,x)\| \leq \varepsilon\|x\| + b, \quad \forall t \in I, \ x \in P. \tag{3.10}$$

Take

$$R > \max\big\{r_2, \ \frac{Nb(1+a_0)(1-\cos\beta)}{\beta^2 a_0 - N\varepsilon(1+a_0)(1-\cos\beta)}\big\},$$

we will prove that

$$Ay \not\geq y, \quad \forall y \in K, \ \|y\|_c = R. \tag{3.11}$$

In fact, if there exists $y_0 \in K$, $\|y_0\|_c = R$ such that $Ay_0 \geq y_0$. Then, by (3.8) and (3.10), we obtain

$$\begin{aligned}
\|y_0(t)\| &\leq \frac{N(a_0 + \cos\beta t)}{\beta a_0} \int_0^1 \sin\beta(1-s)(\varepsilon\|y_0(s)\| + b)ds \\
&\leq \frac{N(1+a_0)(1-\cos\beta)}{\beta^2 a_0}(\varepsilon\|y_0\|_c + b), \quad \forall t \in I.
\end{aligned}$$

So, we have

$$\|y_0\|_c \leq \frac{Nb(1+a_0)(1-\cos\beta)}{\beta^2 a_0 - N\varepsilon(1+a_0)(1-\cos\beta)} < R.$$

A contradiction. Therefore, (3.11) holds.

By (3.9), (3.11), Lemma 2.4 and Theorem 1.1, we obtain that the operator A has at least one fixed point $y \in K$ satisfying $r < \|y\|_c < R$. The proof is complete. $\square$

Theorem 3.2. *Assume* (H1), (H2) *hold. If one of the following conditions is satisfied:*

(i) (H3), (H4), (H7) *hold.*
(ii) (H5), (H6), (H8) *hold.*

Then (1.1)-(1.2) *has at least two positive solutions.*

Proof. (i) By (H3), (H4) and the proof of Theorem 3.1, we obtain that there exist r, R with $0 < r < r_0 < R$ such that

$$Ay \not\leq y, \quad \forall y \in K, \ \|y\|_c = r,$$
$$Ay \not\leq y, \quad \forall y \in K, \ \|y\|_c = R.$$

Now, we prove that

$$Ay \not\geq y, \quad \forall y \in K, \ \|y\|_c = r_0. \tag{3.12}$$

In fact, if there exists $y_0 \in K$, $\|y_0\|_c = r_0$ such that $Ay_0 \geq y_0$. By (3.8) and (H7), we obtain

$$\|y_0\|_c < \frac{N(1+a_0)}{\beta a_0} \int_0^1 \sin\beta(1-s) \frac{\beta^2 a_0}{N(1+a_0)(1-\cos\beta)} r_0 ds = r_0.$$

A contradiction. So, (3.12) is true. By Lemma 2.4 and Theorem 1.1, we obtain that the operator A has at least two fixed points $y_1, \ y_2 \in K$ satisfying $r < \|y_1\|_c < r_0 < \|y_2\|_c < R$.

(ii) By (H5), (H6) and the proof of Theorem 3.1, we obtain that there exist r, R with $0 < r < R_0 < R$ such that

$$Ay \not\geq y, \quad \forall y \in K, \ \|y\|_c = r,$$
$$Ay \not\geq y, \quad \forall y \in K, \ \|y\|_c = R.$$

On the other hand, by (H8) and the same way as used in the proof of (3.2), we can prove that

$$Ay \not\leq y, \quad \forall y \in K, \ \|y\|_c = R_0. \tag{3.13}$$

By Lemma 2.4 and Theorem 1.1, we obtain that the operator A has at least two fixed points $y_1, y_2 \in K$ satisfying $r < \|y_1\|_c < R_0 < \|y_2\|_c < R$. The proof is complete. □

Similar to the proofs of Theorem 3.1 and Theorem 3.2, we can easily get the following corollaries.

Corollary 3.3. *Assume* (H1), (H2) *hold. If one of the following conditions is satisfied:*

(i) (H4), (H5), (H7), (H8) *hold with* $R_0 < \gamma r_0/N$.
(ii) (H3), (H6), (H7), (H8) *hold with* $r_0 < \gamma R_0/N$.

Then (1.1)-(1.2) *has at least three positive solutions.*

Corollary 3.4. *Assume* (H1), (H2) *hold. If one of the following conditions is satisfied:*

(i) (H5)–(H7) *hold, and there exist* $R_i > 0$, $\varphi_i \in P^*$ *with* $\varphi_i(x) > 0$ *for* $x > \theta$, $i = 1, 2$ *such that*

$$\inf_{t\in I,\, x\in P,\, \gamma R_i/N \leq \|x\| \leq R_i} \frac{\varphi_i(F(t,x))}{\varphi_i(x)} > \frac{\beta^2 a_0}{\gamma a_2}, \quad i = 1, 2,$$

where $R_1 < \gamma r_0/N$, $r_0 < \gamma R_2/N$.

(ii) (H3), (H4), (H8) *hold, and there exist* $r_1, r_2 > 0$ *such that*

$$\sup_{t\in I,\, x\in P,\, \gamma r_i/N\le\|x\|\le r_i} \|F(t,\ x)\| < \frac{\beta^2 a_0}{N(1+a_0)(1-\cos\beta)} r_i, \quad i=1,2,$$

where $r_1 < \gamma R_0/N$, $R_0 < \gamma r_2/N$.

Then (1.1)-(1.2) *has at least four positive solutions.*

We can prove easily the existence of multiple positive solutions for (1.1)-(1.2).

References

[1] N. P. Cac, J. A. Gatica; *Fixed point theorems for mappings in ordered Banach spaces*, J. Math. Anal. Appl., 71 (1979), 545-557.

[2] K. Demling; *Ordinary Differential Equations in Banach Spaces* Springer, Berlin, 1977.

[3] Z. Du, X. Lin, W. Ge; *Some higher-order multi-point boundary value problem at resonance*, J. Comput. Appl. Math. 177(2005), 55-65.

[4] B. Du, X. Hu; *A new continuation theorem for the existence of solutions to P-Lpalacian BVP at resonance*, Appl. Math. Comput. 208(2009), 172-176.

[5] M. Feng, D. Ji, W. Ge; *Positive solutions for a class of boundary-value problem with integral boundary conditions in Banach spaces*, J. Comput. Appl. Math., 222(2008), 351-363.

[6] D. Guo, V. Lakshmikantham, X. Liu; *Nonlinear Integral Equation in Abstract spaces*, Kluwer Academic Publishers, Dordrecht, 1996.

[7] D. Guo, V. Lakshmikantham; *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[8] D. Guo, V. Lakshmikantham; *Multiple solutions of two-point boundary value problem of ordinary differential equations in Banach space*, J. Math. Anal. Appl., 129(1988), 211-222.

[9] X. Han; *Positive solutions for a three-point boundary value problem at resonance*, J. Math. Anal. Appl., 336(2007) 556-568.

[10] N. Kosmatov; *A multi-point boundary value problem with two critical conditions*, Nonlinear Anal., 65(2006), 622-633.

[11] N. Kosmatov; *Multi-point boundary value problems on an unbounded domain at resonance*, Nonlinear Anal., 68(2008), 2158-2171.

[12] V. Lakshmikantham, S. Leela; *Nonlinear Differential Equations in Abstract Spaces*, Pergamon Press, Oxford, 1981.

[13] B. Liu; *Positive solutions of a nonlinear four-point boundary value problems in Banach spaces*, J. Math. Anal. Appl., 305(2005), 253-276.

[14] L. Liu, Z. Liu, Y. Wu; *Infinite boundary value problems for nth-order nonlinear impulsive integro-differential equations in Banach spaces*, Nonlinear Analysis, 67(2007), 2670-2679.

[15] Y. Liu, W. Ge; *Solvability of nonlocal boundary value problems for ordinary differential equations of higher order*, Nonlinear Anal., 57(2004), 435-458.

[16] S. Lu, W. Ge; *On the existence of m-point boundary value problem at resonance for higher order differential equation*, J. Math. Anal. Appl. 287(2003), 522-539.

[17] A. J. B. Potter; *A fixed point theorem for positive k-set contractions*, Proc. Edinburgh Math. Soc., 19(1974), 93-102.

[18] X. Zhang, M. Feng, W. Ge; *Existence and nonexistence of positive solutions for a class of nth-order three-point boundary value problems in Banach spaces*, Nonlinear Analysis, 70(2009), 584-597.

[19] X. Zhang, M. Feng, W. Ge; *Existence result of second-order differential equations with integral boundary conditions at resonance*, J. Math. Anal. Appl. 353(2009), 311-319.

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