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A PRIORI ESTIMATES FOR SOLUTIONS TO A FOUR POINT BOUNDARY VALUE PROBLEM FOR SINGULARLY PERTURBED SEMILINEAR DIFFERENTIAL EQUATIONS

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Abstract. This article concerns the existence and asymptotic behavior of solutions to a singularly perturbed second-order four-point boundary-value problem for nonlinear differential equations. Our analysis relies on the method of lower and upper solutions. We give accurate approximations of the solutions up to order $O(\epsilon)$.

1. Preliminaries

We consider the four (or three) point boundary value problem

$$\epsilon y'' + ky = f(t, y), \quad t \in [a, b],\ k < 0,\ 0 < \epsilon \ll 1, \tag{1.1}$$

$$y(c) - y(a) = 0, \quad y(b) - y(d) = 0, \quad a < c \le d < b. \tag{1.2}$$

We focus our attention on the existence and asymptotic behavior of the solutions $y_\epsilon(t)$ for (1.1), (1.2) and on an estimate of the difference between $y_\epsilon(t)$ and a solution $u(t)$ of the equation $ku = f(t, u)$.

The situation in the present case is complicated by the fact that there are inner points in the boundary conditions, in contrast to "standard" boundary conditions such as the Dirichlet, Neumann, Robin, or periodic problem [2, 3, 6, 7], for example. We note that the equation $\epsilon v_\epsilon'' - mv_\epsilon = 0$, $m > 0$, $0 < \epsilon$ such that $\tilde{v}_\epsilon(c) - \tilde{v}_\epsilon(a) = u(c) - u(a) > 0$ and $\tilde{v}_\epsilon(t) \to 0^+$ for $t \in (a, b]$ and $\epsilon \to 0^+$, which could be used to solve this problem by the method of lower and upper solutions. Instead we compose barrier functions (α, β) for two-endpoint boundary conditions to construct barrier functions for (1.1), (1.2), see e.g. [2].

In recent years multi-point boundary value problems have received a great deal of attention (see e.g. [1], [4] and the references therein). The reader is referred to [4] where a four-point boundary value problem with boundary conditions $y(c) - \nu_1 y(a) = 0, y(b) - \nu_2 y(d) = 0$ where the constants ν_1, ν_2 are not simultaneously equal to 1 and $\epsilon = 1$ is studied.

We apply the method of lower and upper solutions to prove the existence of a solution for problem (1.1), (1.2) and by taking $\epsilon \to 0^+$, the corresponding solutions

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converge uniformly on compact subsets of (a,b) to u, the solution of the reduced problem. Moreover, we prove that these solutions converge to u up to order $O(\epsilon)$.

As usual, we say that $\alpha_\epsilon \in C^2([a,b])$ is a lower solution for problem (1.1), (1.2) if for every $t\in(a,b)$ we have $\epsilon\alpha_\epsilon''(t)+k\alpha_\epsilon(t)\ge f(t,\alpha_\epsilon(t))$, and $\alpha_\epsilon(c)-\alpha_\epsilon(a)=0$, $\alpha_\epsilon(b)-\alpha_\epsilon(d)\le 0$. An upper solution $\beta_\epsilon\in C^2([a,b])$ satisfies $\epsilon\beta_\epsilon''(t)+k\beta_\epsilon(t)\le f(t,\beta_\epsilon(t))$ and $\beta_\epsilon(c)-\beta_\epsilon(a)=0$, $\beta_\epsilon(b)-\beta_\epsilon(d)\ge 0$ for every $t\in(a,b)$.

Lemma 1.1 ([5]). *If $\alpha_\epsilon,\beta_\epsilon$ are respectively lower and upper solutions for* (1.1), (1.2) *such that $\alpha_\epsilon\le\beta_\epsilon$, then there exists solution y_ϵ of* (1.1), (1.2) *with $\alpha_\epsilon\le y_\epsilon\le\beta_\epsilon$.*

Consider the set $\mathcal{H}(u)=\{(t,y): a\le t\le b, |y-u(t)|<d(t)\}$, where $d(t)$ is the positive continuous function on $[a,b]$ defined by

$$d(t)=\begin{cases}|u(c)-u(a)|+\delta & \text{for } a\le t\le a+\frac{\delta}{2}\\ \delta & \text{for } a+\delta\le t\le b-\delta\\ |u(b)-u(d)|+\delta & \text{for } b-\frac{\delta}{2}\le t\le b\end{cases}$$

where δ is a small positive constant and $u\in C^2$ is a solution of the reduced problem $ky=f(t,y)$ defined on $[a,b]$. We write $s(\epsilon)=O(r(\epsilon))$ when $0<\lim_{\epsilon\to 0^+}|\frac{s(\epsilon)}{r(\epsilon)}|<\infty$.

2. Main result

Theorem 2.1. *Let $f\in C^1(\mathcal{H}(u))$ satisfy the condition*

$$\Big|\frac{\partial f(t,y)}{\partial y}\Big|\le w<-k \quad \text{for every } (t,y)\in\mathcal{H}(u).$$

Then there exists ϵ_0 such that for every $\epsilon\in(0,\epsilon_0]$, problem (1.1), (1.2) *has a unique solution $y_\epsilon(t)$ satisfying the inequality*

$$-v_\epsilon^{(\mathrm{corr})}(t)-\hat v_\epsilon(t)-C\epsilon\le y_\epsilon(t)-(u(t)+v_\epsilon(t))\le\hat v_\epsilon(t)+C\epsilon \quad \text{if } u(c)-u(a)\ge 0$$

and

$$-\hat v_\epsilon(t)-C\epsilon\le y_\epsilon(t)-(u(t)+v_\epsilon(t))\le v_\epsilon^{(\mathrm{corr})}(t)+\hat v_\epsilon(t)+C\epsilon \quad \text{if } u(c)-u(a)\le 0$$

on $[a,b]$ where

$$\begin{aligned}
v_\epsilon(t)&=\frac{u(c)-u(a)}{D}\cdot\Big(e^{\sqrt{\frac{m}{\epsilon}}(b-t)}-e^{\sqrt{\frac{m}{\epsilon}}(t-b)}+e^{\sqrt{\frac{m}{\epsilon}}(t-d)}-e^{\sqrt{\frac{m}{\epsilon}}(d-t)}\Big),\\
\hat v_\epsilon(t)&=\frac{|u(b)-u(d)|}{D}\cdot\Big(e^{\sqrt{\frac{m}{\epsilon}}(t-a)}-e^{\sqrt{\frac{m}{\epsilon}}(a-t)}+e^{\sqrt{\frac{m}{\epsilon}}(c-t)}-e^{\sqrt{\frac{m}{\epsilon}}(t-c)}\Big),\\
D&=\Big(e^{\sqrt{\frac{m}{\epsilon}}(b-a)}+e^{\sqrt{\frac{m}{\epsilon}}(d-c)}+e^{\sqrt{\frac{m}{\epsilon}}(c-b)}+e^{\sqrt{\frac{m}{\epsilon}}(a-d)}\Big)\\
&\quad-\Big(e^{\sqrt{\frac{m}{\epsilon}}(a-b)}+e^{\sqrt{\frac{m}{\epsilon}}(c-d)}+e^{\sqrt{\frac{m}{\epsilon}}(b-c)}+e^{\sqrt{\frac{m}{\epsilon}}(d-a)}\Big),
\end{aligned}$$

$m=-k-w$, $C=\frac{1}{m}\max_{t\in[a,b]}|u''(t)|$ and the positive function

$$\begin{aligned}
v_\epsilon^{(\mathrm{corr})}(t)=\frac{w|u(c)-u(a)|}{\sqrt{m\epsilon}}\cdot\Big[&-O(1)\frac{v_\epsilon(t)}{(u(c)-u(a))}\\
&+O\big(e^{\sqrt{\frac{m}{\epsilon}}(a-d)}\big)\frac{\hat v_\epsilon(t)}{|u(b)-u(d)|}+tO\big(e^{\sqrt{\frac{m}{\epsilon}}\chi(t)}\big)\Big],
\end{aligned}$$

$\chi(t)<0$ for $t\in(a,b]$ and $v_\epsilon^{(\mathrm{corr})}(a)=v_\epsilon^{(\mathrm{corr})}(c)$.

Remark 2.2. The function $v_\epsilon(t)$ satisfies

(1) $\epsilon v_\epsilon'' - m v_\epsilon = 0$,
(2) $v_\epsilon(c) - v_\epsilon(a) = -(u(c) - u(a))$, $v_\epsilon(b) - v_\epsilon(d) = 0$,
(3) If $u(c)-u(a) \ge 0$ (≤ 0) then $v_\epsilon(t) \ge 0$ (≤ 0) and it is decreasing (increasing) for $a \le t \le \frac{b+d}{2}$ and increasing (decreasing) for $\frac{b+d}{2} \le t \le b$,
(4) If $\epsilon \to 0^+$ then $v_\epsilon(t)$ converges uniformly to 0 on every compact subset of $(a, b]$,
(5) $v_\epsilon(t) = (u(c) - u(a))O\big(e^{\sqrt{\frac{m}{\epsilon}}\chi(t)}\big)$, where $\chi(t) = a - t$ for $a \le t \le \frac{b+d}{2}$ and $\chi(t) = t - b + a - d$ for $\frac{b+d}{2} < t \le b$.

The function $\hat{v}_\epsilon(t)$ satisfies

(1) $\epsilon \hat{v}_\epsilon'' - m\hat{v}_\epsilon = 0$,
(2) $\hat{v}_\epsilon(c) - \hat{v}_\epsilon(a) = 0$, $\hat{v}_\epsilon(b) - \hat{v}_\epsilon(d) = |u(b) - u(d)|$,
(3) $\hat{v}_\epsilon(t) \ge 0$ and it is decreasing for $a \le t \le \frac{a+c}{2}$ and increasing for $\frac{a+c}{2} \le t \le b$,
(4) If $\epsilon \to 0^+$ then $\hat{v}_\epsilon(t)$ converges uniformly to 0 on every compact subset of $[a, b)$,
(5) $\hat{v}_\epsilon(t) = |u(b) - u(d)|O\big(e^{\sqrt{\frac{m}{\epsilon}}\hat{\chi}(t)}\big)$ where $\hat{\chi}(t) = c - b + a - t$ for $a \le t < \frac{a+c}{2}$ and $\hat{\chi}(t) = t - b$ for $\frac{a+c}{2} \le t \le b$.

The correction function $v_\epsilon^{(\mathrm{corr})}(t)$ is determined precisely in the next section.

3. The correction function

Consider the linear problem

$$\epsilon y'' - my = -2w|v_\epsilon(t)|, \quad t \in [a, b],\ \epsilon > 0 \tag{3.1}$$

with the boundary conditions (1.2). We apply the method of lower and upper solutions in order to obtain a solution. We define

$$\alpha_\epsilon(t) = 0$$

and

$$\beta_\epsilon(t) = \frac{2w}{m}\max\{|v_\epsilon(t)|, t \in [a, b]\} = \frac{2w}{m}|v_\epsilon(a)|.$$

Obviously, $|v_\epsilon(a)| = |u(c)-u(a)|\big(1+O\big(e^{\sqrt{\frac{m}{\epsilon}}(a-c)}\big)\big)$ and the constant functions α, β are, respectively, a lower and an upper solution for (3.1), (1.2). Thus, in view of Lemma 1.1, for every $\epsilon > 0$ there exists unique solution $v_\epsilon^{(\mathrm{corr})}(t)$ of linear problem (3.1), (1.2) such that

$$0 \le v_\epsilon^{(\mathrm{corr})}(t) \le \frac{2w}{m}|u(c) - u(a)|\Big(1 + O\big(e^{\sqrt{\frac{m}{\epsilon}}(a-c)}\big)\Big)$$

on $[a, b]$. We compute $v_\epsilon^{(\mathrm{corr})}(t)$ exactly:

$$v_\epsilon^{(\mathrm{corr})}(t) = -\frac{(\psi_\epsilon(a) - \psi_\epsilon(c))}{(u(c) - u(a))}v_\epsilon(t) + \frac{(\psi_\epsilon(d) - \psi_\epsilon(b))}{|u(b) - u(d)|}\hat{v}_\epsilon(t) + \psi_\epsilon(t),$$

where

$$\psi_\epsilon(t) = \frac{w|u(c) - u(a)|}{D\sqrt{m\epsilon}}t\left(e^{\sqrt{\frac{m}{\epsilon}}(b-t)} + e^{\sqrt{\frac{m}{\epsilon}}(t-b)} - e^{\sqrt{\frac{m}{\epsilon}}(d-t)} - e^{\sqrt{\frac{m}{\epsilon}}(t-d)}\right).$$

Hence

$$\begin{aligned}
&\psi_\epsilon(a)-\psi_\epsilon(c)\\
&=\frac{w|u(c)-u(a)|}{D\sqrt{m\epsilon}}a\left(e^{\sqrt{\frac{m}{\epsilon}}(b-a)}+e^{\sqrt{\frac{m}{\epsilon}}(a-b)}-e^{\sqrt{\frac{m}{\epsilon}}(d-a)}-e^{\sqrt{\frac{m}{\epsilon}}(a-d)}\right)\\
&\quad-\frac{w|u(c)-u(a)|}{D\sqrt{m\epsilon}}c\left(e^{\sqrt{\frac{m}{\epsilon}}(b-c)}+e^{\sqrt{\frac{m}{\epsilon}}(c-b)}-e^{\sqrt{\frac{m}{\epsilon}}(d-c)}-e^{\sqrt{\frac{m}{\epsilon}}(c-d)}\right)\\
&=\frac{w|u(c)-u(a)|}{\sqrt{m\epsilon}}O(1),
\end{aligned}$$

$$\begin{aligned}
\psi_\epsilon(d)-\psi_\epsilon(b)&=\frac{w|u(c)-u(a)|}{D\sqrt{m\epsilon}}d\left(e^{\sqrt{\frac{m}{\epsilon}}(b-d)}+e^{\sqrt{\frac{m}{\epsilon}}(d-b)}-2\right)\\
&\quad-\frac{w|u(c)-u(a)|}{D\sqrt{m\epsilon}}b\left(2-e^{\sqrt{\frac{m}{\epsilon}}(d-b)}-e^{\sqrt{\frac{m}{\epsilon}}(b-d)}\right)\\
&=\frac{w|u(c)-u(a)|}{\sqrt{m\epsilon}}O\left(e^{\sqrt{\frac{m}{\epsilon}}(a-d)}\right),
\end{aligned}$$

$$\psi_\epsilon(t)=\frac{w|u(c)-u(a)|}{\sqrt{m\epsilon}}O\left(e^{\sqrt{\frac{m}{\epsilon}}\chi(t)}\right).$$

Thus, we obtain

$$\begin{aligned}
v_\epsilon^{(\mathrm{corr})}(t)=\frac{w|u(c)-u(a)|}{\sqrt{m\epsilon}}\cdot\Big[&-O(1)\frac{v_\epsilon(t)}{(u(c)-u(a))}\\
&+O\left(e^{\sqrt{\frac{m}{\epsilon}}(a-d)}\right)\frac{\hat v_\epsilon(t)}{|u(b)-u(d)|}+tO\left(e^{\sqrt{\frac{m}{\epsilon}}\chi(t)}\right)\Big].
\end{aligned}\tag{3.2}$$

Hence, taking into consideration (3.2) and the fact that $v_\epsilon^{(\mathrm{corr})}(a)=v_\epsilon^{(\mathrm{corr})}(c)$, the correction function $v_\epsilon^{(\mathrm{corr})}$ converges uniformly to 0 on $[a,b]$ as $\epsilon\to 0^+$.

4. Proof of main theorem

First we analyze the case $u(c)-u(a)\ge 0$. Consider the lower solutions

$$\alpha_\epsilon(t)=u(t)+v_\epsilon(t)-v_\epsilon^{(\mathrm{corr})}(t)-\hat v_\epsilon(t)-\Gamma_\epsilon$$

and the upper solutions

$$\beta_\epsilon(t)=u(t)+v_\epsilon(t)+\hat v_\epsilon(t)+\Gamma_\epsilon.$$

Here $\Gamma_\epsilon=\epsilon\Delta/m$, where Δ is a constant to be defined below, $\alpha_\epsilon\le\beta_\epsilon$ on $[a,b]$ and they satisfy the correspondent prescribed boundary conditions.

Now we show that $\epsilon\alpha_\epsilon''(t)+k\alpha_\epsilon(t)\ge f(t,\alpha_\epsilon(t))$ and $\epsilon\beta_\epsilon''(t)+k\beta_\epsilon(t)\le f(t,\beta_\epsilon(t))$. Denoting $h(t,y)=f(t,y)-ky$, by the Taylor we have

$$\begin{aligned}
h(t,\alpha_\epsilon(t))&=h(t,\alpha_\epsilon(t))-h(t,u(t))\\
&=\frac{\partial h(t,\theta_\epsilon(t))}{\partial y}(v_\epsilon(t)-v_\epsilon^{(\mathrm{corr})}(t)-\hat v_\epsilon(t)-\Gamma_\epsilon),
\end{aligned}$$

where $\alpha_\epsilon(t) < \theta_\epsilon(t) < \beta_\epsilon(t)$ and $(t, \theta_\epsilon(t)) \in \mathcal{H}(u)$ for ϵ sufficiently small. Hence, from the inequalities $m \le \frac{\partial h(t,\theta_\epsilon(t))}{\partial y} \le m + 2w$ in $\mathcal{H}(u)$ we have

$$\begin{aligned}
&\epsilon\alpha_\epsilon''(t) - h(t, \alpha_\epsilon(t)) \\
&\ge \epsilon u''(t) + \epsilon v_\epsilon''(t) - \epsilon v_\epsilon^{(corr)''}(t) - \epsilon\hat{v}_\epsilon''(t) \\
&\quad - (m + 2w)v_\epsilon(t) + m v_\epsilon^{(\mathrm{corr})}(t) + m\hat{v}_\epsilon(t) + m\Gamma_\epsilon.
\end{aligned}$$

Since $v_\epsilon(t) = |v_\epsilon(t)|$, we have $-\epsilon v_\epsilon^{(corr)''}(t) - 2wv_\epsilon(t) + mv_\epsilon^{(\mathrm{corr})}(t) = 0$ and using ((3.1), we obtain

$$\epsilon\alpha_\epsilon''(t) - h(t, \alpha_\epsilon(t)) \ge \epsilon u''(t) + m\Gamma_\epsilon \ge -\epsilon|u''(t)| + \epsilon\Delta.$$

For $\beta_\epsilon(t))$ we have the inequality

$$\begin{aligned}
&h(t, \beta_\epsilon(t)) - \epsilon\beta_\epsilon''(t) \\
&= \frac{\partial h(t, \tilde{\theta}_\epsilon(t))}{\partial y}(v_\epsilon(t) + \hat{v}_\epsilon(t) + \Gamma_\epsilon) - \epsilon\beta_\epsilon''(t) \\
&= m(v_\epsilon(t) + \hat{v}_\epsilon(t) + \Gamma_\epsilon) - \epsilon(u''(t) + v_\epsilon''(t) + \hat{v}_\epsilon''(t)) \\
&\ge \epsilon\Delta - \epsilon|u''(t)|,
\end{aligned}$$

where $\alpha_\epsilon(t) < \tilde{\theta}_\epsilon(t) < \beta_\epsilon(t)$ and $(t, \tilde{\theta}_\epsilon(t)) \in \mathcal{H}(u)$ for ϵ sufficiently small.

Let us now analyse the case $u(c) - u(a) \le 0$: The lower solutions

$$\alpha_\epsilon(t) = u(t) + v_\epsilon(t) - \hat{v}_\epsilon(t) - \Gamma_\epsilon$$

and the upper solutions

$$\beta_\epsilon(t) = u(t) + v_\epsilon(t) + v_\epsilon^{(\mathrm{corr})}(t) + \hat{v}_\epsilon(t) + \Gamma_\epsilon$$

satisfy

$$\begin{aligned}
\epsilon\alpha_\epsilon'' - h(t, \alpha_\epsilon) &= \epsilon u'' + \epsilon v_\epsilon'' - \epsilon\hat{v}_\epsilon'' - \frac{\partial h}{\partial y}(v_\epsilon - \hat{v}_\epsilon - \Gamma_\epsilon) \\
&= \epsilon u'' + \epsilon v_\epsilon'' - \epsilon\hat{v}_\epsilon'' + \frac{\partial h}{\partial y}(-v_\epsilon + \hat{v}_\epsilon + \Gamma_\epsilon) \\
&\ge \epsilon u'' + \epsilon v_\epsilon'' - \epsilon\hat{v}_\epsilon'' + m(-v_\epsilon + \hat{v}_\epsilon + \Gamma_\epsilon) \\
&= \epsilon u'' + \epsilon\Delta \ge \epsilon\Delta - \epsilon|u''|
\end{aligned}$$

and

$$\begin{aligned}
h(t, \beta_\epsilon) - \epsilon\beta_\epsilon'' &= \frac{\partial h}{\partial y}\Big(v_\epsilon + v_\epsilon^{(\mathrm{corr})} + \hat{v}_\epsilon + \Gamma_\epsilon\Big) - \epsilon u'' - \epsilon v_\epsilon'' - \epsilon v_\epsilon^{(corr)''} - \epsilon\hat{v}_\epsilon'' \\
&\ge (m + 2w)v_\epsilon + m\Big(v_\epsilon^{(\mathrm{corr})} + \hat{v}_\epsilon + \Gamma_\epsilon\Big) - \epsilon u'' - \epsilon v_\epsilon'' - \epsilon v_\epsilon^{(corr)''} - \epsilon\hat{v}_\epsilon'' \\
&= -2w|v_\epsilon| + mv_\epsilon^{(\mathrm{corr})} - \epsilon v_\epsilon^{(corr)''} + \epsilon\Delta - \epsilon u'' \\
&= \epsilon\Delta - \epsilon u'' \ge \epsilon\Delta - \epsilon|u''|.
\end{aligned}$$

Now, if we choose a constant Δ such that $\Delta \ge |u''(t)|$, $t \in [a, b]$ then $\epsilon\alpha_\epsilon''(t) \ge h(t, \alpha_\epsilon(t))$ and $\epsilon\beta_\epsilon''(t) \le h(t, \beta_\epsilon(t))$ in $[a, b]$.

The existence of a solution for (1.1), (1.2) satisfying the above inequality follows from Lemma 1.1. The uniqueness follows from the fact that the function $h(t, y)$ is increasing in the variable y on the set $\mathcal{H}$.

Remark 4.1. Theorem 2.1 implies $y_\epsilon(t) = u(t) + O(\epsilon)$ on every compact subset of (a, b) and $\lim_{\epsilon\to 0^+} y_\epsilon(a) = u(c)$, $\lim_{\epsilon\to 0^+} y_\epsilon(b) = u(d)$. The boundary layer effect occurs at the point a (b) whenever $u(a) \neq u(c)$ ($u(b) \neq u(d)$).

5. Approximation of the solutions for (1.1), (1.2)

In this section we consider only the case $u(c) - u(a) \leq 0$ as the other case could be treated analogously. We define the approximate solution $\tilde{y}_\epsilon(t)$ of (1.1), (1.2) by

$$\tilde{y}_\epsilon(t) = \frac{1}{2}\left(\alpha_\epsilon(t) + \beta_\epsilon(t)\right) = u(t) + v_\epsilon(t) + \frac{v_\epsilon^{(\mathrm{corr})}(t)}{2}. \tag{5.1}$$

Taking into consideration the conclusions of Theorem 2.1, in both cases we obtain the following estimate for the solution y_ϵ of problem (1.1), (1.2)

$$|y_\epsilon(t) - \tilde{y}_\epsilon(t)| \leq \hat{v}_\epsilon(t) + \frac{v_\epsilon^{(\mathrm{corr})}(t)}{2} + \frac{\epsilon}{m}\max\{|u''(t)|, t \in [a, b]\}.$$

Example 5.1. Consider the nonlinear differential equation

$$\epsilon y'' + ky = y^2 + g(t), \quad k < 0,\ g \in C([a, b]) \tag{5.2}$$

subject to the boundary conditions (1.2). The assumptions of Theorem 2.1 are satisfied if and only if there exists $w > 0$ such that

$$\frac{1}{4}\big(k^2 - (w-k)^2\big) < g(t) < \frac{1}{4}\big(k^2 - (w+k)^2\big) \quad \text{on } [a, b], \tag{5.3}$$

$$|g(c) - g(a)| < \frac{1}{8}\big(w - k - \zeta(a)\big)\big(\zeta(a) + \zeta(c)\big), \tag{5.4}$$

$$|g(b) - g(d)| < \frac{1}{8}\big(w - k - \zeta(b)\big)\big(\zeta(b) + \zeta(d)\big), \tag{5.5}$$

$$|g(c) - g(a)| < \frac{1}{8}\big(w + k + \zeta(a)\big)\big(\zeta(a) + \zeta(c)\big), \tag{5.6}$$

$$|g(b) - g(d)| < \frac{1}{8}\big(w + k + \zeta(b)\big)\big(\zeta(b) + \zeta(d)\big), \tag{5.7}$$

where $\zeta(t) = \sqrt{k^2 - 4g(t)}$.

As an illustrative example we consider the problem (5.2), (1.2) with $k = -2$, $g(t) = t$, $a = 0$, $b = 1/2$, $c = d = 1/4$. It is not difficult to verify that the solution $u(t) = -1 + \sqrt{1-t}$ of the reduced problem satisfies conditions (5.3)–(5.7) for every $w \in \big(\frac{2}{\sqrt{2}+\sqrt{3}} + 2 - \sqrt{2}, 2\big)$. Thus, on the basis of Theorem 2.1, there exists $\epsilon_0 = \epsilon_0(w)$ such that for every $\epsilon \in (0, \epsilon_0]$ the problem $\epsilon y'' - 2y = y^2 + t$, (1.2) has a unique solution which is $O(\epsilon)$ close to approximate solution (5.1); i.e.,

$$\tilde{y}_\epsilon(t) = -1 + \sqrt{1-t} + v_\epsilon(t) + \frac{v_\epsilon^{(\mathrm{corr})}(t)}{2}.$$

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