Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 19, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

QUADRATIC FORMS AS LYAPUNOV FUNCTIONS IN THE STUDY OF STABILITY OF SOLUTIONS TO DIFFERENCE EQUATIONS

ALEXANDER O. IGNATYEV, OLEKSIY IGNATYEV

Abstract. A system of linear autonomous difference equations $x(n+1) = Ax(n)$ is considered, where $x \in \mathbb{R}^k$, A is a real nonsingular $k \times k$ matrix. In this paper it has been proved that if $W(x)$ is any quadratic form and m is any positive integer, then there exists a unique quadratic form $V(x)$ such that $\Delta_m V = V(A^m x) - V(x) = W(x)$ holds if and only if $\mu_i \mu_j \neq 1$ $(i = 1, 2 \dots k; j = 1, 2 \dots k)$ where $\mu_1, \mu_2, \dots, \mu_k$ are the roots of the equation $\det(A^m - \mu I) = 0$.

A number of theorems on the stability of difference systems have also been proved. Applying these theorems, the stability problem of the zero solution of the nonlinear system $x(n+1) = Ax(n) + X(x(n))$ has been solved in the critical case when one eigenvalue of a matrix A is equal to minus one, and others lie inside the unit disk of the complex plane.

1. Introduction and preliminaries

The theory of discrete dynamical systems has grown tremendously in the last decade. Difference equations can arise in a number of ways. They may be the natural model of a discrete process (in combinatoric, for example) or they may be a discrete approximation of a continuous process. The growth of the theory of difference systems has been strongly promoted by the advanced technology in scientific computation and the large number of applications to models in biology, engineering, and other physical sciences. For example, in papers [2, 7, 8, 10, 12, 19] systems of difference equations are applied as natural models of populations dynamics, in [13] difference equations are applied as a mathematical model in genetics.

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short-term perturbations which duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. It is known, for example, that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and

2000 *Mathematics Subject Classification.* 39A11, 34K20.
Key words and phrases. Difference equations; Lyapunov function.

Submitted February 1, 2010. Published February 3, 2011.

frequency modulated systems, do exhibit impulsive effects. Thus impulsive differential equations, that is, differential equations involving impulse effects, appear as a natural description of observed evolution phenomena of several real world problems [4, 5, 15, 20, 23, 34, 36, 37, 38, 39, 40, 43, 41, 42]. The early work on differential equations with impulse effect were summarized in monograph [36] in which the foundations of this theory were described. In recent years, the study of impulsive systems has received an increasing interest [31, 26, 21, 22, 3, 6, 9, 11, 25, 24, 27, 29]. In fact, an impulsive system consists of a continuous system which is governed by ordinary differential equations and a discrete system which is governed by difference equations. So the dynamics of impulsive systems essentially depends on properties of the corresponding difference systems, and this confirms the importance of studying the qualitative properties of difference systems.

The stability and asymptotic behaviour of solutions of these models that are especially important to many investigators. The stability of a discrete process is the ability of the process to resist *a priori* unknown small influences. A process is said to be stable if such disturbances do not change it. This property turns out to be of utmost importance since, in general, an individual predictable process can be physically realized only if it is stable in the corresponding natural sense. One of the most powerful methods, used in stability theory, is Lyapunov's direct method. This method consists in the use of an auxiliary function (the Lyapunov function).

Consider the system of difference equations

$$x(n+1) = f(n, x(n)), \quad f(n, 0) = 0, \tag{1.1}$$

where $n = 0, 1, 2, \dots$ is discrete time, $x(n) = (x_1(n), \dots, x_k(n))^T \in \mathbb{R}^k$, $f = (f_1, \dots, f_k)^T \in \mathbb{R}^k$. The function f we assume to be continuous and to satisfy Lipschitz condition in x. System (1.1) admits the trivial solution

$$x(n) = 0. \tag{1.2}$$

Denote $x(n, n_0, x^0)$ the solution of (1.1) coinciding with $x^0 = (x_1^0, x_2^0, \dots, x_k^0)^T$ for $n = n_0$. We also denote $\mathbb{Z}_+$ the set of nonnegative real integers, $\mathbb{N}_{n_0} = \{n \in \mathbb{Z}_+ : n \ge n_0\}$, $\mathbb{N} = \{n \in \mathbb{Z}_+ : n \ge 1\}$, $B_r = \{x \in \mathbb{R}^k : \|x\| \le r\}$.

By analogy to ordinary differential equations, let us introduce the following definitions.

Definition 1.1. The trivial solution of system (1.1) is said to be stable if for any $\varepsilon > 0$ and $n_0 \in \mathbb{Z}_+$ there exists a $\delta = \delta(\varepsilon, n_0) > 0$ such that $\|x^0\| < \delta$ implies $\|x(n, n_0, x^0)\| < \varepsilon$ for $n \in \mathbb{N}_{n_0}$. Otherwise the trivial solution of system (1.1) is called unstable. If in this definition δ can be chosen independent of n_0 (i.e. $\delta = \delta(\varepsilon)$), then the zero solution of system (1.1) is said to be uniformly stable.

Definition 1.2. Solution (1.2) of system (1.1) is said to be attracting if for any $n_0 \in \mathbb{Z}_+$ there exists an $\eta = \eta(n_0) > 0$ such that for any $\varepsilon > 0$ and $x^0 \in B_\eta$ there exists an $N = N(\varepsilon, n_0, x^0) \in \mathbb{N}$ such that $\|x(n, n_0, x^0)\| < \varepsilon$ for all $n \in \mathbb{N}_{n_0+N}$.

In other words, solution (1.2) of system (1.1) is called attracting if

$$\lim_{n\to\infty} \|x(n, n_0, x^0)\| = 0. \tag{1.3}$$

Definition 1.3. The trivial solution of system (1.1) is said to be uniformly attracting if for some $\eta > 0$ and for each $\varepsilon > 0$ there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that $\|x(n, n_0, x^0)\| < \varepsilon$ for all $n_0 \in \mathbb{Z}_+$, $x^0 \in B_\eta$, and $n \ge n_0 + N$.

In other words, solution (1.2) of system (1.1) is called uniformly attracting if (1.3) holds uniformly in $n_0 \in \mathbb{Z}_+$ and $x^0 \in B_\eta$.

Definition 1.4. The zero solution of system (1.1) is called:

- asymptotically stable if it is both stable and attracting;
- uniformly asymptotically stable if it is both uniformly stable and uniformly attracting.

Definition 1.5. *The trivial solution of system* (1.1) *is said to be exponentially stable if there exist* $M > 0$ *and* $\eta \in (0,1)$ *such that* $\|x(n, n_0, x^0)\| < M\|x^0\|\eta^{n-n_0}$ *for* $n \in \mathbb{N}_{n_0}$.

A great number of papers is devoted to investigation of the stability of solution (1.2) of system (1.1). The general theory of difference equations and the base of the stability theory are stated in [1, 16, 32, 14, 33]. It has been proved in [30] that if system (1.1) is autonomous (i.e. f does not depend explicitly in n) or periodic (i.e. there exists $\omega \in \mathbb{N}$ such that $f(n,x) \equiv f(n+\omega, x)$), then from the stability of solution (1.2) it follows its uniform stability, and from its asymptotic stability it follows its uniform asymptotic stability. Papers [18, 28, 35] deal with the stability investigation of the zero solution of system (1.1) when this system is periodic or almost periodic.

Let us formulate the main theorems of Lyapunov's direct method about the stability of the zero solution of the system of autonomous difference equations

$$x(n+1) = F(x(n)) \tag{1.4}$$

where $x, F \in \mathbb{R}^n$, F is a continuous function; $F(0) = 0$. These statements have been mentioned in [16, Theorems 4.20 and 4.27]. They are connected with the existence of an auxiliary function $V(x)$; the analog of its derivative is the variation of V relative to (1.4) which is defined as $\Delta V(x) = V(F(x)) - V(x)$.

Theorem 1.6. *If there exists a positive definite continuous function* $V(x)$ *such that* $\Delta V(x)$ *relative to* (1.4) *is negative semi-definite function or identically equals to zero, then the trivial solution of system* (1.4) *is stable.*

Theorem 1.7. *If there exists a positive definite continuous function* $V(x)$ *such that* $\Delta V(x)$ *relative to* (1.4) *is negative definite, then the trivial solution of system* (1.4) *is asymptotically stable.*

Theorem 1.8. *If there exists a continuous function* $V(x)$ *such that* $\Delta V(x)$ *relative to* (1.4) *is negative definite, and the function* V *is not positive semi-definite, then the trivial solution of system* (1.4) *is unstable.*

Consider the autonomous system

$$x(n+1) = Ax(n) + X(x(n)), \tag{1.5}$$

where A is a $k \times k$ nonsingular matrix, X is a function such that

$$\lim_{\|x\| \to 0} \frac{\|X(x)\|}{\|x\|} = 0. \tag{1.6}$$

Recall that for a real $k \times k$ matrix $A = (a_{ij})$, an eigenvalue of A is a real or complex number λ such that

$$\det(A - \lambda I_k) = 0 \tag{1.7}$$

where I_k is the unit $k \times k$ matrix. Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be eigenvalues of A. According to [16, p.175], let us denote $\rho(A) = \max_{1\le i\le k} |\lambda_i|$. In [16] the following theorems have been proved.

Theorem 1.9. *If $\rho(A) < 1$, then the zero solution of system* (1.5) *is asymptotically stable (moreover, the exponential stability holds in this case).*

Theorem 1.10. *Let $\rho(A) \le 1$ and modulus of some eigenvalues of A are equal to one. Then a function $X(x)$ in system* (1.5) *can be chosen such that the zero solution of system (1.5) is either stable or unstable.*

The goal of this paper is to extend Theorems 1.6, 1.7, 1.8 and to apply the obtained theorems for the study of the stability of the zero solution of system (1.5) in critical case $\lambda = -1$. The paper is organized as following. In chapter 2, Theorems 1.6, 1.7, and 1.8 are extended, and the theorems on the instability are proved. In chapter 3, the problem on the possibility to construct Lyapunov function in the form of quadratic polynomial is considered. In chapter 4, the problem of the stability of the zero solution of system (1.5) is considered in the critical case when equation (1.7) has a root $\lambda = -1$ and other roots lie in the unit disk of the complex plane.

2. Some general theorems extending Theorems 1.6, 1.7, 1.8

Consider system of difference equations (1.1) and a function $V : \mathbb{Z}_+ \times B_H \to \mathbb{R}$, continuous in B_H and satisfying the equality $V(n, 0) = 0$. We remind that the function f in (1.1) is Lipschitzian in x, so there is a constant L such that $\|f(n, x) - f(n, y)\| \le L\|x - y\|$. Denote the m-th variation of V at the moment n

$$\Delta_m V(n, x(n)) = V(n + m, x(n + m)) - V(n, x(n))$$

where $m \in \mathbb{N}$.

Definition 2.1. *A function $r : \mathbb{R}_+ \to \mathbb{R}_+$ is called a Hahn's function if it is continuous, increasing and $r(0) = 0$. The class of Hahn's functions will be denoted $\mathcal{K}$.*

Theorem 2.2. *If system* (1.1) *is such that there exist $m \in \mathbb{N}$, a function $a \in \mathcal{K}$, and a function $V : \mathbb{Z}_+ \times B_H \to \mathbb{R}$ such that $V(n, 0) = 0$,*

$$V(n, x) \ge a(\|x\|), \tag{2.1}$$

and

$$\Delta_m V \le 0, \tag{2.2}$$

then the trivial solution of system (1.1) *is stable.*

Proof. Let $n_0 \in \mathbb{Z}_+$ and $\varepsilon \in (0, H)$. We shall show that there exists a $\delta = \delta(\varepsilon, n_0) > 0$ such that $x^0 \in B_\delta$ implies $\|x(n, n_0, x^0)\| < \varepsilon$ for $n \in \mathbb{N}_{n_0}$. First we shall show that this inequality is true for $n = n_0 + sm$ where $s \in \mathbb{Z}_+$. Since V is continuous and $V(n_0, 0) = 0$, there is a $\delta = \delta(\varepsilon, n_0) > 0$ such that

$$V(n_0, x^0) < a\Big(\frac{\varepsilon}{1 + L + L^2 + \dots + L^{m-1}}\Big) \tag{2.3}$$

for all $x^0 \in B_\delta$. From conditions (2.1), (2.2), and (2.3) it follows

$$\begin{aligned} a(\|x(n_0 + sm, n_0, x^0)\|) &\le V(n_0 + sm, x(n_0 + sm, n_0, x^0)) \\ &\le V(n_0, x^0) < a\Big(\frac{\varepsilon}{1 + L + L^2 + \dots + L^{m-1}}\Big); \end{aligned}$$

therefore,

$$\|x(n_0+sm,n_0,x^0)\| < \frac{\varepsilon}{1+L+L^2+\dots+L^{m-1}}.$$

Estimate the value of $\|x(n_0+sm+1,n_0,x^0)\|$:

$$\begin{aligned}\|x(n_0+sm+1,n_0,x^0)\| &= \|f(n_0+sm,x(n_0+sm,n_0,x^0))\|\\ &\le L\|x(n_0+sm,n_0,x^0)\|\\ &< \frac{L\varepsilon}{1+L+L^2+\dots+L^{m-1}} < \varepsilon.\end{aligned}$$

Similarly we obtain

$$\begin{gathered}\|x(n_0+sm+2,n_0,x^0)\| < \frac{L^2\varepsilon}{1+L+L^2+\dots+L^{m-1}} < \varepsilon,\dots,\\ \|x(n_0+sm+m-1,n_0,x^0)\| < \frac{L^{m-1}\varepsilon}{1+L+L^2+\dots+L^{m-1}} < \varepsilon.\end{gathered}$$

Hence the zero solution of system (1.1) is stable. □

Theorem 2.3. *If the conditions of the previous theorem are satisfied, and there exists $b \in \mathcal{K}$ such that*

$$V(n,x) \le b(\|x\|), \tag{2.4}$$

then the zero solution of system (1.1) *is uniformly stable.*

Proof. Under condition (2.4), the value δ can be chosen independent of n_0. Set $\delta = b^{-1}(a(\varepsilon))$, where b^{-1} is the function inverted to b. In this case

$$\begin{aligned}a(\|x(n_0+sm,n_0,x^0)\|) &\le V(n_0+sm,x(n_0+sm,n_0,x^0)) \le V(n_0,x^0)\\ &\le b(\|x^0\|) < b\Big(b^{-1}\Big(a\big(\frac{\varepsilon}{1+L+L^2+\dots+L^{m-1}}\big)\Big)\Big)\\ &= a\Big(\frac{\varepsilon}{1+L+L^2+\dots+L^{m-1}}\Big),\end{aligned}$$

whence it follows $\|x(n,n_0,x^0)\| < \varepsilon$ for $n \in \mathbb{N}_{n_0}$. This completes the proof. □

Theorem 2.4. *If system* (1.1) *is such that there exist $m \in \mathbb{N}$, functions $a,b,c \in \mathcal{K}$, and a continuous function $V : \mathbb{Z}_+ \times B_H \to \mathbb{R}$ such that inequalities* (2.1),(2.4), *and*

$$\Delta_m V(n,x) \le -c(\|x\|) \tag{2.5}$$

hold, then the zero solution of system (1.1) *is uniformly asymptotically stable.*

Proof. Let $h \in (0,H)$ and $\eta > 0$ be such that $\|x(n,n_0,x^0)\| < h$ whenever $x^0 \in B_\eta, n_0 \in \mathbb{Z}_+, n \in \mathbb{N}_{n_0}$. The existence of such η follows from the uniform stability of solution (1.2) of system (1.1). Let $\varepsilon \in (0,\eta)$ be small enough, and $\delta = \delta(\varepsilon)$ be a number chosen by correspondence to definition of the uniform stability: if $\|x^0\| < \delta$, then $\|x(n,n_0,x^0)\| < \varepsilon$ for $n_0 \in \mathbb{Z}_+, n \ge n_0$. Take arbitrary $x^0 \in B_\eta$ and $n_0 \in \mathbb{Z}_+$. Estimate the interval of the discrete time, during which the trajectory $x(n,n_0,x^0)$ may lie in the set $B_h \setminus \delta(\varepsilon)$. According to (2.5), for $x \in B_h \setminus \delta(\varepsilon)$ we have $\Delta_m V \le -c(\delta(\varepsilon))$, whence we obtain

$$V(n_0+sm,x(n_0+sm,n_0,x^0)) - V(n_0,x^0) \le -sc(\delta(\varepsilon)),$$

whence

$$s \le \frac{V(n_0,x^0) - V(n_0+sm,x(n_0+sm,n_0,x^0))}{c(\delta(\varepsilon))} < \frac{b(h)}{c(\delta(\varepsilon))}.$$

So choosing $N = N(\varepsilon) = [\frac{b(h)}{c(\delta(\varepsilon))}] + 1$, we obtain that there exists s_0 such that $s_0 m \leq N(\varepsilon)$ and $x(n_0 + s_0 m, n_0, x^0) \in B_{\delta(\varepsilon)}$, therefore due the uniform stability of the zero solution we have $x(n, n_0, x^0) \in B_\varepsilon$ for $n \geq n_0 + N$. This completes the proof. □

Theorem 2.5. *If system* (1.1) *is such that there exist $m \in \mathbb{N}$ and a continuous bounded function $V : \mathbb{Z}_+ \times B_H \to \mathbb{R}$ such that $\Delta_m V$ is positive definite and V is not negative semidefinite, then the zero solution of system* (1.1) *is unstable.*

Proof. Since $\Delta_m V$ is positive definite, there exists a $c \in \mathcal{K}$ such that

$$\Delta_m V(n, x) \geq c(\|x\|) \tag{2.6}$$

holds. Let $\varepsilon \in (0, H)$ be an arbitrary number and $n_0 \in \mathbb{Z}_+$. We shall show that for each $\delta > 0$ there exist $x^0 \in B_\delta$ and $n \geq n_0$ such that $\|x(n, n_0, x^0)\| \geq \varepsilon$. Let δ be a positive number as small as desired. As an initial value, we take x^0 such that $0 < \|x^0\| < \delta$ and $V(n_0, x^0) = V_0 > 0$. Let us show that there exists an $n \in \mathbb{N}_{n_0}$ such that inequality $\|x(n, n_0, x^0)\| \geq \varepsilon$ holds. Suppose the contrary:

$$\|x(n, n_0, x^0)\| < \varepsilon \tag{2.7}$$

is valid for all $n \in \mathbb{N}_{n_0}$. From (2.6) it follows that $V(n_0 + m, x(n_0 + m, n_0, x^0)) \geq V_0 + c(\|x_0\|)$, $V(n_0 + 2m, x(n_0 + 2m, n_0, x^0)) \geq V_0 + 2c(\|x_0\|)$, ...,

$$V(n_0 + sm, x(n_0 + sm, n_0, x^0)) \geq V_0 + sc(\|x_0\|). \tag{2.8}$$

Inequality (2.8) contradicts the boundedness of V in $\mathbb{Z}_+ \times B_H$. Thus, assuming the validity of (2.7) we have the contradiction. The obtained contradiction completes the proof. □

Theorem 2.6. *If system* (1.1) *is such that there exist $m \in \mathbb{N}$, positive constants α_1, α_2, and a function $V(n, x)$, bounded in $\mathbb{Z}_+ \times B_H$, such that $\Delta_m V$ has the form*

$$\Delta_m V = \alpha_1 V(n, x) + \alpha_2 W(n, x) \tag{2.9}$$

where W is positive semidefinite and V is not negative semidefinite, then the zero solution of system (1.1) *is unstable.*

Proof. From (2.9) it follows

$$\Delta_m V(n, x) \geq \alpha_1 V(n, x). \tag{2.10}$$

Let $0 < \varepsilon < H$ and $n_0 \in \mathbb{Z}_+$. Choose the initial value x^0 such that $\|x^0\| < \delta$ and $V(n_0, x^0) = v_0 > 0$, where δ is a positive number, as small as desired. Let us show that there exists $n > n_0$ such that $\|x(n, n_0, x^0)\| \geq \varepsilon$. Suppose the contrary:

$$\|x(n, n_0, x^0)\| < \varepsilon \tag{2.11}$$

holds for all $n \in \mathbb{N}_{n_0}$. Inequality (2.10) is true for all $n \in \mathbb{N}_{n_0}$, and since $V(n_0, x^0) > 0$, the value $\Delta_m V$ is positive for all $m \in \mathbb{N}$. Therefore the sequence $\{V(n_0 + sm, x(n_0 + sm, n_0, x^0))\}_{s=0}^{\infty}$ is increasing. From (2.10) we find that

$$\Delta_m V(n_0 + sm, x(n_0 + sm, n_0, x^0)) \geq \alpha_1 V(n_0 + sm, x(n_0 + sm, n_0, x^0)) \geq \alpha_1 v_0,$$

hence $V(n_0 + sm, x(n_0 + sm, n_0, x^0)) \geq \alpha_1 v_0 s$. But this is impossible because of the boundedness of the function V in B_ε. The obtained contradiction shows that assumption (2.11) is false. This completes the proof. □

3. Lyapunov functions for linear autonomous systems

Side by side with system (1.5), let us consider the system of linear difference equations

$$x(n+1) = Ax(n), \tag{3.1}$$

whence we obtain

$$x(n+m) = A^m x(n). \tag{3.2}$$

To study the stability properties of the zero solution of system (3.1), Elaydi [16, 17] suggested to use quadratic forms

$$V(x) = \sum_{\substack{i_1+i_2+\dots+i_k=2,\\ i_j\ge 0\,(j=1,\dots,k)}} b_{i_1,i_2,\dots,i_k} x_1^{i_1} x_2^{i_2} \dots x_k^{i_k} \tag{3.3}$$

as Lyapunov functions. Let

$$W(x) = \sum_{\substack{i_1+i_2+\dots+i_k=2,\\ i_j\ge 0(j=1,\dots,k)}} q_{i_1,i_2,\dots,i_k} x_1^{i_1} x_2^{i_2} \dots x_k^{i_k} \tag{3.4}$$

be an arbitrary real quadratic form. Let us clarify the conditions under which there exists a quadratic form (3.3) such that

$$\Delta_m V(x) = V(A^m x) - V(x) = W(x). \tag{3.5}$$

Theorem 3.1. *If the roots $\mu_1, \mu_2, \dots, \mu_k$ of the polynomial*

$$\det(A^m - \mu I_k) = 0 \tag{3.6}$$

are such that

$$\mu_i\mu_j \neq 1 \quad (i = 1, \dots, k; j = 1, \dots, k), \tag{3.7}$$

then for any quadratic form (3.4) *there exists the unique quadratic form* (3.3) *such that equality* (3.5) *holds.*

Proof. Denote N the number of terms of a quadratic form in $x_1, x_2, \dots, x_k$. It is obvious that this number is equal to the number of different systems of nonnegative integers $i_1, i_2, \dots, i_k$ constrained by the condition $i_1+i_2+\dots+i_k = 2$. This number is equal to

$$N = \frac{k(k+1)}{2}.$$

Let us enumerate the coefficients of forms $V(x)$ and $W(x)$ and denote them by letters $b_1, b_2, \dots, b_N$ and $q_1, q_2, \dots, q_N$ respectively:

$$\begin{gathered} b_{2,0,\dots,0} = b_1, \quad b_{1,1,\dots,0} = b_2, \quad b_{1,0,\dots,1} = b_k, \\ b_{0,2,\dots,0} = b_{k+1}, \quad b_{0,1,1,\dots,0} = b_{k+2}, \dots, b_{0,1,\dots,1} = b_{2k-1}, \dots, \\ b_{0,0,\dots,2,0} = b_{N-2}, \quad b_{0,0,\dots,1,1} = b_{N-1}, \quad b_{0,0,\dots,0,2} = b_N, \\ q_{2,0,\dots,0} = q_1, \quad q_{1,1,\dots,0} = q_2, \quad q_{1,0,\dots,1} = q_k, \\ q_{0,2,\dots,0} = q_{k+1}, \quad q_{0,1,1,\dots,0} = q_{k+2}, \dots, q_{0,1,\dots,1} = q_{2k-1}, \dots, \\ q_{0,0,\dots,2,0} = q_{N-2}, \quad q_{0,0,\dots,1,1} = q_{N-1}, \quad q_{0,0,\dots,0,2} = q_N. \end{gathered}$$

Denote $b = (b_1, b_2, \dots, b_N)^T$, $q = (q_1, q_2, \dots, q_N)^T$. The left-hand and the right-hand sides of equality (3.5) represent quadratic forms with respect to $x_1, x_2, \dots, x_k$.

Equating coefficients corresponding to products $x_1^{i_1}x_2^{i_2}\dots x_k^{i_k}$, we obtain the system of linear equations with respect to $b_1, b_2, \dots, b_N$. This system has the form

$$Rb = q, \tag{3.8}$$

where $R = (r_{ij})_{i,j=1}^N$; elements r_{ij} of the matrix R can be expressed via elements of the matrix A. System (3.8) has the unique solution for any vector q if and only if

$$\det R \neq 0. \tag{3.9}$$

Let us show that condition (3.9) holds if inequalities (3.7) are valid. To do this, let us introduce new variable $z = (z_1, \dots, z_k)^T$ by the linear transformation $x = Gz$ with a nonsingular matrix G such that in new variables system (3.2) has the form

$$z(n + m) = Pz(n), \tag{3.10}$$

where $P = (p_{ij})_{i,j=1}^k$; p_{ii} are the eigenvalues of the matrix A^m, $p_{i,i+1}$ are equal to 0 or 1, and all other elements of the matrix P are equal to zero. According to [16, Theorem 3.23], such transformation does exist. In general case, if the matrix A^m has complex eigenvalues, the variables $z_1, \dots, z_k$ and elements of the matrix G are also complex. Polynomials (3.3) and (3.4) have the following forms in variables $z_1, z_2, \dots, z_k$:

$$V(z) = \sum_{\substack{i_1+i_2+\dots+i_k=2,\\ i_j\geq 0 (j=1,\dots,k)}} c_{i_1,i_2,\dots,i_k} z_1^{i_1} z_2^{i_2} \dots z_k^{i_k}, \tag{3.11}$$

$$W(z) = \sum_{\substack{i_1+i_2+\dots+i_k=2,\\ i_j\geq 0 (j=1,\dots,k)}} d_{i_1,i_2,\dots,i_k} z_1^{i_1} z_2^{i_2} \dots z_k^{i_k}. \tag{3.12}$$

The quadratic form $W(z)$ is real, hence in relation (3.12), side by side with any non-real summand $d_{i_1,i_2,\dots,i_k} z_1^{i_1} z_2^{i_2} \dots z_k^{i_k}$ there is the summand $d_{i_1^*,i_2^*,\dots,i_k^*} z_1^{i_1^*} z_2^{i_2^*} \dots z_k^{i_k^*}$ such that

$$d_{i_1^*,i_2^*,\dots,i_k^*} z_1^{i_1^*} z_2^{i_2^*} \dots z_k^{i_k^*} = \overline{d}_{i_1,i_2,\dots,i_k} \overline{z}_1^{i_1} \overline{z}_2^{i_2} \dots \overline{z}_k^{i_k}$$

where the over line means the complex conjugate symbol. Enumerating $d_{i_1,\dots,i_k}$ and $c_{i_1,\dots,i_k}$ as follows

$$\begin{gathered}
d_{2,0,\dots,0} = d_1, \quad d_{1,1,\dots,0} = d_2, \quad d_{1,0,\dots,1} = d_k,\\
d_{0,2,\dots,0} = d_{k+1}, \quad d_{0,1,1,\dots,0} = d_{k+2}, \dots, d_{0,1,\dots,1} = d_{2k-1}, \dots,\\
d_{0,0,\dots,2,0} = d_{N-2}, \quad d_{0,0,\dots,1,1} = d_{N-1}, \quad d_{0,0,\dots,0,2} = d_N,\\
c_{2,0,\dots,0} = c_1, \quad c_{1,1,\dots,0} = c_2, \quad c_{1,0,\dots,1} = c_k,\\
c_{0,2,\dots,0} = c_{k+1}, \quad c_{0,1,1,\dots,0} = c_{k+2}, \dots, c_{0,1,\dots,1} = c_{2k-1}, \dots,\\
c_{0,0,\dots,2,0} = c_{N-2}, \quad c_{0,0,\dots,1,1} = c_{N-1}, \quad c_{0,0,\dots,0,2} = c_N,
\end{gathered}$$

and denoting $c = (c_1, \dots, c_N)^T$, $d = (d_1, \dots, d_N)^T$, let us rewrite equality (3.5) in variables $z_1, \dots, z_k$:

$$V(Pz) - V(z) = W(z). \tag{3.13}$$

The left-hand and right-hand sides of equality (3.13) represent quadratic forms with respect to $z_1, \dots, z_k$. Equating the coefficients corresponding to the products $z_1^2, z_1z_2, \dots, z_1z_k$, $z_2^2, \dots, z_{k-1}z_k, z_k^2$, we obtain the system of linear algebraic equations with respect to $c_1, \dots, c_N$, which we write in the matrix form

$$Uc = d, \tag{3.14}$$

where $U = (u_{ij})_{i,j=1}^{N}$. The matrix U has the triangular form

$$U = \begin{pmatrix} p_{11}^2 - 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 2p_{11}p_{12} & p_{11}p_{22} - 1 & \dots & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & p_{11}p_{kk} - 1 & 0 & \dots & 0 & 0 \\ p_{12}^2 & p_{12}p_{22} & \dots & 0 & p_{22}^2 - 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & p_{k-1,k-1}p_{kk} - 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & p_{k-1,k}p_{kk} & p_{kk}^2 - 1 \end{pmatrix}$$

System (3.14) has a unique solution if and only if $\det U \neq 0$. Taking into account that $u_{ij} = 0$ for $j > i$, we obtain that $\det U$ is equal to the product of diagonal elements of the matrix U:

$$\det U = \prod_{i=1,2,\dots,k;j=i,i+1,\dots,k} (p_{ii}p_{jj} - 1).$$

Bearing in mind that $p_{ii} = \mu_i$ and returning in (3.13) from variables $z_1, \dots, z_k$ to variables $x_1, \dots, x_k$ by means of the transformation $z = G^{-1}x$, we obtain that a quadratic form V satisfying (3.5) exists and is unique if and only if $\mu_i\mu_j \neq 1 \quad (i, j = 1, \dots, k)$. The proof is complete. □

In the case $m = 1$ we have the following corollary.

Corollary 3.2. *If the eigenvalues $\lambda_1, \dots, \lambda_k$ of the matrix A are such that*

$$\lambda_i\lambda_j \neq 1 \quad (i = 1, \dots, k; j = 1, \dots, k), \tag{3.15}$$

then for any quadratic form (3.4) *there exists the unique quadratic form* (3.3) *such that*

$$\Delta V = V(Ax) - V(x) = W(x)\,. \tag{3.16}$$

Theorem 3.3. *If for some $m \in \mathbb{N}$, the roots $\mu_1, \dots, \mu_k$ of characteristic equation* (3.6) *satisfy conditions*

$$|\mu_i| < 1 \quad (i = 1, \dots, k), \tag{3.17}$$

then for any positive definite quadratic form $W(x)$ there exists the unique negative definite quadratic form $V(x)$ such that

$$\Delta_m V(x) = W(x).$$

Proof. According to [16], the sets $\{\mu_1, \mu_2, \dots, \mu_k\}$ and $\{\lambda_1^m, \lambda_2^m, \dots, \lambda_k^m\}$ are identical, hence from (3.17) it follows

$$|\lambda_i| < 1 \quad (i = 1, \dots, k). \tag{3.18}$$

Let $W(x)$ be an arbitrary positive definite quadratic form. If (3.17) holds, then (3.7) is valid. Therefore, there exists a unique quadratic form $V(x)$ such that (3.5) holds. Let us show that $V(x)$ is negative definite. Suppose the contrary: there is a nonzero x^0 such that $V(x^0) \geq 0$. In this case, we have that $V(x^1) = V(A^m x^0) = V(x^0) + W(x^0) > 0$, and according to Theorem 2.5, the zero solution of system (3.1) is unstable. But on the other hand, (3.18) and Theorem 1.9 imply that the zero solution of system (3.1) is asymptotically stable. The obtained contradiction completes the proof. □

Theorem 3.4. *If for some $m \in \mathbb{N}$, the roots $\mu_1, \dots, \mu_k$ of the characteristic equation* (3.6) *are such that*

$$\rho(A) > 1 \tag{3.19}$$

and conditions (3.7) *hold, then for any positive definite quadratic form $W(x)$ there exists a unique quadratic form $V(x)$ satisfying* (3.5)*, and this form is not negative semidefinite (in particular, negative definite).*

Proof. Let $W(x)$ be a positive definite quadratic form. By virtue of Theorem 3.1, there exists a unique quadratic form $V(x)$ which satisfies (3.5). To complete the proof of Theorem 3.4, all we need is to show that $V(x)$ can be neither negative definite nor negative semidefinite. If $V(x)$ is negative definite, then by virtue of Theorem 2.4, the zero solution of system (3.1) is asymptotically stable, and therefore $\rho(A) < 1$, but it contradicts to (3.19). On the other hand, $V(x)$ cannot be negative semidefinite no matter of values of $|\mu_i|$. To verify this, consider any solution of system (3.1) with the initial condition $x^0 \neq 0$ vanishing V: $V(x^0) = 0$. Hence $V(A^m x^0) = W(x^0) > 0$, but this contradicts to its negative semidefiniteness. The obtained contradiction completes the proof. □

Remark 3.5. Conditions (3.7) (or (3.15) for $m = 1$) in Theorem 3.4 are essential because if at least one of these conditions is not valid, then, in general, Theorem 3.4 is not true.

To show this, let us consider the system $x(n+1) = Ax(n)$, where $A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$. Here $\rho(A) = 3 > 1$; for all $m \in \mathbb{N}$ we have $\mu_1 = 3^m, \mu_2 = 1$. Conditions (3.7) are not satisfied because $\mu_2 \cdot \mu_2 = 1$. For **any** quadratic form $V = ax_1^2 + bx_1x_2 + cx_2^2$ we obtain

$$V(A^m x) - V(x) = a\big(3^{2m} - 1\big)x_1^2 + b\big(3^m - 1\big)x_1x_2.$$

This form cannot be positive definite; so there is no quadratic form V such that (3.5) holds.

Consider now the case when at least one of conditions (3.7) is not satisfied but $\rho(A) > 1$. Let us show that in this case the zero solution of system (3.1) is also unstable.

Theorem 3.6. *If the matrix A in system* (3.1) *is such that $\rho(A) > 1$ and at least one of conditions* (3.7) *is not satisfied, then for any positive definite quadratic form $W(x)$ there exists a quadratic form $V(x)$ and positive numbers α_1, α_2 such that $\Delta_m V = \alpha_1 V + \alpha_2 W$ holds, and $V(x)$ is not negative semidefinite.*

Proof. Side by side with system (3.1), let us consider the system

$$x(n+1) = \alpha A x(n) \tag{3.20}$$

where $\alpha > 0$. From system (3.20) we obtain

$$x(n+m) = \alpha^m A^m x(n). \tag{3.21}$$

The roots $\sigma_1, \sigma_2, \dots, \sigma_k$ of its characteristic equation

$$\det(\alpha^m A^m - \sigma I_k) = 0$$

continuously depend on α, and for $\alpha = 1$ they coincide with the roots $\mu_1, \mu_2, \dots, \mu_k$ of the characteristic equation (3.6) of system (3.2). Moreover, there exist values of α, close to the value $\alpha = 1$ such that σ_i satisfy inequalities

$$\sigma_i \sigma_j \neq 1 \quad (i, j = 1, \dots, k)$$

and $\rho(\alpha^m A^m) > 1$. Let $W(x)$ be an arbitrary positive definite quadratic form. According to Theorem 3.4, there exists the unique quadratic form $V(x)$ such that

$$\Delta_m V(x)\big|_{(3.20)} = V(\alpha^m A^m x) - V(x) = W(x), \tag{3.22}$$

and $V(x)$ is not negative semidefinite. On the other hand, it is easy to check that

$$\begin{aligned} \Delta_m V(x)\big|_{(3.20)} &= (V(\alpha^m A^m x) - V(\alpha^m x)) + (V(\alpha^m x) - V(x)) \\ &= \alpha^{2m} \Delta_m V(x)\big|_{(3.1)} + (\alpha^{2m} - 1)V(x). \end{aligned} \tag{3.23}$$

Comparing (3.22) and (3.23) we obtain

$$\Delta_m V(x)\big|_{(3.1)} = \alpha_1 V(x) + \alpha_2 W(x), \quad \text{where} \quad \alpha_1 = \frac{1-\alpha^{2m}}{\alpha^{2m}}, \alpha_2 = \frac{1}{\alpha^{2m}}.$$

Choosing $0 < \alpha < 1$ we have $\alpha_1 > 0$, $\alpha_2 > 0$. This completes the proof. □

So now we can formulate the well-known criterion of the instability by linear approximation (see for example [1]) as the following corollary of the above theorems.

Corollary 3.7. *From Theorems 2.6, 3.4, and 3.6 it follows that if $\rho(A) > 1$, then the trivial solution of system* (3.1) *is unstable.*

4. Critical case $\lambda = -1$

In this section, we consider the critical case when one root of the characteristic equation (1.7) is equal to minus one; i.e., we shall assume that (1.7) has one root $\lambda_1 = -1$, and other roots satisfy the conditions $|\lambda_i| < 1 \quad (i = 2, 3, \dots, k)$. The function $X = (X_1, \dots, X_k)^T$ is supposed to be holomorphic, and its expansion into Maclaurin series begins with terms of the second order of smallness. So system (1.5) takes the form

$$\begin{aligned} x_j(n+1) &= a_{j1}x_1(n) + a_{j2}x_2(n) + \dots + a_{jk}x_k(n) \\ &\quad + X_j(x_1(n), \dots, x_k(n)) \quad (j = 1, \dots, k). \end{aligned} \tag{4.1}$$

Henceforth we shall consider the critical case when the characteristic equation of the system of the first approximation

$$x_j(n+1) = a_{j1}x_1(n) + a_{j2}x_2(n) + \dots + a_{jk}x_k(n) \quad (j = 1, \dots, k) \tag{4.2}$$

has one root, equal to minus one, and other $k-1$ roots which modules are less then one.

From (4.1) we obtain

$$\begin{aligned} x_j(n+2) &= A_{j1}x_1(n) + A_{j2}x_2(n) + \dots + A_{jk}x_k(n) \\ &\quad + X_j^*(x_1(n), \dots, x_k(n)) \quad (j = 1, \dots, k). \end{aligned} \tag{4.3}$$

Here $\mathcal{A} = (A_{ij})_{i,j=1}^k = A^2$ and $X^* = (X_1^*, \dots, X_k^*)^T$ is a vector all of whose components are power series in the components of x lacking constant and first degree terms and convergent for $\|x\|$ sufficiently small. Let us introduce in system (4.2) the variable y instead of one variable x_j by means of the substitution

$$y = \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_k x_k, \tag{4.4}$$

where $\beta_j \quad (j = 1, \dots, k)$ are some constants which we choose such that

$$y(n+1) = -y(n). \tag{4.5}$$

From (4.4) and (4.5) we obtain

$$\begin{aligned} y(n+1) &= \beta_1 x_1(n+1) + \beta_2 x_2(n+1) + \dots + \beta_k x_k(n+1) \\ &= \beta_1[a_{11}x_1(n) + a_{12}x_2(n) + \dots + a_{1k}x_k(n)] \\ &\quad + \beta_2[a_{21}x_1(n) + a_{22}x_2(n) + \dots + a_{2k}x_k(n)] + \dots \\ &\quad + \beta_k[a_{k1}x_1(n) + a_{k2}x_2(n) + \dots + a_{kk}x_k(n)] \\ &= -(\beta_1 x_1(n) + \beta_2 x_2(n) + \dots + \beta_k x_k(n)). \end{aligned}$$

Equating the coefficients corresponding to $x_j(n)$ $(j = 1, 2, \dots, k)$, we obtain the system of linear homogeneous algebraic equations with respect to β_j $(j = 1, \dots, k)$:

$$a_{1j}\beta_1 + a_{2j}\beta_2 + \dots + a_{kj}\beta_k = -\beta_j, \tag{4.6}$$

or in the matrix form

$$(A^T + I_k)\beta = 0,$$

where $\beta = (\beta_1, \dots, \beta_k)^T$. Since the equation $\det(A^T + \lambda I_k) = 0$ has the root $\lambda = -1$, the determinant of system (4.6) is equal to zero. Therefore this system has a solution in which not all constants $\beta_1, \dots, \beta_k$ are equal to zero. To be definite, let us assume that $\beta_k \neq 0$. Then we can use the variable y instead of the variable x_k. Other variables x_j $(j = 1, \dots, k-1)$ we preserve without change. Denoting

$$\nu_{ji} = a_{ji} - \frac{\beta_i}{\beta_k}a_{jk}, \quad \nu_j = \frac{a_{jk}}{\beta_k} \quad (i, j = 1, 2, \dots, k-1),$$

we transform equations (4.2) to the form

$$\begin{gathered} x_j(n+1) = \nu_{j1}x_1(n) + \nu_{j2}x_2(n) + \dots + \nu_{j,k-1}x_{k-1}(n) + \nu_j y(n) \\ (j = 1, \dots, k-1), \end{gathered} \tag{4.7}$$

$$y(n+1) = -y(n), \tag{4.8}$$

where ν_{ji} and ν_j are constants.

The characteristic equation of system (4.7) and (4.8) reduces to two equations: $\lambda + 1 = 0$ and

$$\det(\Upsilon - \lambda I_{k-1}) = 0, \tag{4.9}$$

where $\Upsilon = (\nu_{ij})_{i,j=1}^{k-1}$. Since a characteristic equation is invariant with respect to linear transformations and in this case has $k-1$ roots, whose modules are less then one, then equation (4.9) has $k-1$ roots, and modules of all these roots are less then one. Denote

$$x_j = y_j + l_j y \quad (j = 1, \dots, k-1), \tag{4.10}$$

where l_j $(j = 1, \dots, k-1)$ are constants which we choose such that right-hand sides of system (4.7) do not contain $y(n)$. In this designations, taking into account (4.8), system (4.7) takes the form

$$\begin{aligned} y_j(n+1) &= \nu_{j1}y_1(n) + \nu_{j2}y_2(n) + \dots + \nu_{j,k-1}y_{k-1}(n) \\ &\quad + [\nu_{j1}l_1 + \nu_{j2}l_2 + \dots + (\nu_{jj} - 1)l_j + \dots + \nu_{j,k-1}l_{k-1} + \nu_j]y(n), \end{aligned}$$

$(j = 1, \dots, k-1)$. We choose constants l_j such that

$$\nu_{j1}l_1 + \nu_{j2}l_2 + \dots + (\nu_{jj} + 1)l_j + \dots + \nu_{j,k-1}l_{k-1} = -\nu_j \quad (j = 1, \dots, k-1). \tag{4.11}$$

Minus one is not a root of the characteristic equation (4.9), hence the determinant of system (4.11) is not equal to zero, therefore this system has the unique solution

$(l_1, \dots, l_{k-1})$. As a result of change (4.10), system (4.7) and (4.8) transforms to the form

$$\begin{gathered} y_j(n+1) = \nu_{j1}y_1(n) + \nu_{j2}y_2(n) + \dots + \nu_{j,k-1}y_{k-1}(n) \\ (j = 1, \dots, k-1), \\ y(n+1) = -y(n), \end{gathered} \tag{4.12}$$

and nonlinear system (4.1) takes the form

$$\begin{aligned} y_j(n+1) &= \nu_{j1}y_1(n) + \nu_{j2}y_2(n) + \dots + \nu_{j,k-1}y_{k-1}(n) \\ &\quad + \Psi_j(y_1(n), \dots, y_{k-1}(n), y(n)) \quad (j = 1, \dots, k-1), \\ y(n+1) &= -y(n) + \Psi(y_1(n), \dots, y_{k-1}(n), y(n)), \end{aligned} \tag{4.13}$$

where $\Psi_j \quad (j = 1, \dots, k-1)$ and Ψ are holomorphic functions of $y_1, \dots, y_{k-1}, y$ whose expansions in power series lack constant and first degree terms:

$$\begin{gathered} \Psi_j(y_1, y_2, \dots, y_{k-1}, y) = \sum_{i_1+i_2+\dots+i_{k-1}+i_k=2}^{\infty} \psi^{(j)}_{i_1,i_2,\dots,i_{k-1},i_k} y_1^{i_1} y_2^{i_2} \dots y_{k-1}^{i_{k-1}} y^{i_k} \\ (j = 1, \dots, k-1), \\ \Psi(y_1, y_2, \dots, y_{k-1}, y) = \sum_{i_1+i_2+\dots+i_{k-1}+i_k=2}^{\infty} \psi_{i_1,i_2,\dots,i_{k-1},i_k} y_1^{i_1} y_2^{i_2} \dots y_{k-1}^{i_{k-1}} y^{i_k}. \end{gathered}$$

By (4.10) it is clear that the problem of the stability of the trivial solution of system (4.1) is equivalent to the problem of stability of the zero solution of system (4.13). Further, form (4.13) will be basic for the study of the stability of the zero solution in the case when this problem can be solved by means of terms of the first and second powers in expansions of Ψ_j $(j = 1, \dots, k-1)$ and Ψ.

From equations (4.13) we find

$$\begin{aligned} y_j(n+2) &= c_{j1}y_1(n) + c_{j2}y_2(n) + \dots + c_{j,k-1}y_{k-1}(n) \\ &\quad + Y_j(y_1(n), \dots, y_{k-1}(n), y(n)) \quad (j = 1, \dots, k-1), \end{aligned} \tag{4.14}$$

$$y(n+2) = y(n) + Y(y_1(n), \dots, y_{k-1}(n), y(n)), \tag{4.15}$$

where $c_{ij} = \sum_{s=1}^{k-1} \nu_{is}\nu_{sj}$; Y_j $(j = 1, \dots, k-1)$ and Y are holomorphic functions of $y_1, \dots, y_{k-1}, y$ whose expansions in power series lack constant and first degree terms:

$$\begin{gathered} Y_j(y_1, y_2, \dots, y_{k-1}, y) = \sum_{i_1+i_2+\dots+i_{k-1}+i_k=2}^{\infty} v^{(j)}_{i_1,i_2,\dots,i_{k-1},i_k} y_1^{i_1} y_2^{i_2} \dots y_{k-1}^{i_{k-1}} y^{i_k} \\ (j = 1, \dots, k-1), \\ Y(y_1, y_2, \dots, y_{k-1}, y) = \sum_{i_1+i_2+\dots+i_{k-1}+i_k=2}^{\infty} v_{i_1,i_2,\dots,i_{k-1},i_k} y_1^{i_1} y_2^{i_2} \dots y_{k-1}^{i_{k-1}} y^{i_k}. \end{gathered}$$

Theorem 4.1. *If the function Y is such that the coefficient $v_{0,0,\dots,0,2}$ is not equal to zero, then the solution*

$$y_1 = 0, \quad y_2 = 0, \quad \dots, \quad y_{k-1} = 0, \quad y = 0$$

of system (4.13) is unstable.

Proof. Let

$$V_1(y_1,\dots,y_{k-1})=\sum_{s_1+s_2+\dots+s_{k-1}=2} B_{s_1,s_2,\dots,s_{k-1}} y_1^{s_1}y_2^{s_2}\dots y_{k-1}^{s_{k-1}}$$

be the quadratic form such that

$$\begin{aligned}\Delta_2 V_1\big|_{(4.12)} &= V_1(c_{11}y_1+\dots+c_{1,k-1}y_{k-1},\dots,c_{k-1,1}y_1+\dots\\ &\quad +c_{k-1,k-1}y_{k-1})-V_1(y_1,\dots,y_{k-1})\\ &= y_1^2+y_2^2+\dots+y_{k-1}^2.\end{aligned} \tag{4.16}$$

Since modules of all eigenvalues of matrix $\mathcal{C}=(c_{ij})_{I,j=1}^{k-1}$ are less then one, then according to [16, Theorem 4.30] such quadratic form is unique and negative definite. Consider the Lyapunov function

$$V(y_1,\dots,y_{k-1},y)=V_1(y_1,\dots,y_{k-1})+\alpha y, \tag{4.17}$$

where $\alpha=const$. Let us find $\Delta_2 V$:

$$\begin{aligned}\Delta_2 V\big|_{(4.13)} &= \sum_{s_1+\dots+s_{k-1}=2} B_{s_1,\dots,s_{k-1}}\{[c_{11}y_1+\dots+c_{1,k-1}y_{k-1}\\ &\quad +Y_1(y_1,\dots,y_{k-1},y)]^{s_1}\times\dots\times[c_{k-1,1}y_1+\dots+c_{k-1,k-1}y_{k-1}\\ &\quad +Y_{k-1}(y_1,\dots,y_{k-1},y)]^{s_{k-1}}-y_1^{s_1}\dots y_{k-1}^{s_{k-1}}\}+\alpha Y(y_1,\dots,y_{k-1},y).\end{aligned}$$

Taking into account (4.16), $\Delta_2 V$ can be written in the form

$$\Delta_2 V\big|_{(4.13)}=W(y_1,\dots,y_{k-1},y)+W_*(y_1,\dots,y_{k-1},y),$$

where

$$\begin{aligned}W &= (y_1^2+y_2^2+\dots+y_{k-1}^2)+\alpha v_{0,0,\dots,0,2}y^2\\ &\quad +\alpha(v_{2,0,\dots,0}y_1^2+v_{1,1,\dots,0}y_1y_2+\dots+v_{1,0,\dots,1,0}y_1y_{k-1}\\ &\quad +v_{1,0,\dots,0,1}y_1y+v_{0,2,\dots,0}y_2^2+\dots+v_{0,0,\dots,1,1}y_{k-1}y),\end{aligned}$$

and W_* is a holomorphic function whose Maclaurin-series expansion begins with terms of the third power in $y_1,\dots,\ y_{k-1},\ y$. We choose the sign of α such that $\alpha v_{0,\dots,0,2}>0$. Let us show that $|\alpha|$ can be chosen so small that the quadratic form W is positive definite. To do this, let us show that α can be chosen such that principal minors of the matrix

$$\begin{pmatrix} 1+\alpha v_{2,0,\dots,0} & \frac12\alpha v_{1,1,\dots,0} & \frac12\alpha v_{1,0,1,\dots,0} & \dots & \frac12\alpha v_{1,0,\dots,1,0} & \frac12\alpha v_{1,0,\dots,0,1}\\ \frac12\alpha v_{1,1,\dots,0} & 1+\alpha v_{0,2,\dots,0} & \frac12\alpha v_{0,1,1,\dots,0} & \dots & \frac12\alpha v_{0,1,\dots,1,0} & \frac12\alpha v_{0,1,\dots,0,1}\\ \frac12\alpha v_{1,0,1,\dots,0} & \frac12\alpha v_{0,1,1,\dots,0} & 1+\alpha v_{0,0,2,\dots,0} & \dots & \frac12\alpha v_{0,0,1,\dots,1,0} & \frac12\alpha v_{0,0,1,\dots,0,1}\\ \dots & \dots & \dots & \dots & \dots & \dots\\ \frac12\alpha v_{1,0,\dots,1,0} & \frac12\alpha v_{0,1,\dots,1,0} & \frac12\alpha v_{0,0,1,\dots,1,0} & \dots & 1+\alpha v_{0,\dots,0,2,0} & \frac12\alpha v_{0,\dots,0,1,1}\\ \frac12\alpha v_{1,0,\dots,0,1} & \frac12\alpha v_{0,1,\dots,0,1} & \frac12\alpha v_{0,0,1,\dots,0,1} & \dots & \frac12\alpha v_{0,0,\dots,1,1} & \frac12\alpha v_{0,0,\dots,0,2}\end{pmatrix}$$

are positive. In fact, any principal minor Ω_s of this matrix is a continuous function of α: $\Omega_s=\Omega_s(\alpha)$. Note that $\Omega_s(0)=1$ for $s=1,2,\dots,k-1$. Thus there exists $\alpha_*>0$ such that for $|\alpha|<\alpha_*$ we have $\Omega_s(\alpha)\ge\frac12\quad(s=1,2,\dots,k-1)$. Let us prove that the inequality $\Omega_k>0$ holds for sufficiently small $|\alpha|$. To do this, let us expand Ω_k in terms of the elements of the last row. We obtain $\Omega_k=\frac12\alpha v_{0,0,\dots,0,2}\Omega_{k-1}+\alpha^2\Omega_*$ where Ω_* is a polynomial with respect to α and $v_{i_1,i_2,\dots,i_k}$ $(i_1+i_2+\dots+i_k=2, i_j\ge 0)$. Hence we have $\Omega_k>0$ for sufficiently small $|\alpha|$. So for α which absolute value is small enough and the sign of which coincides with the sign of $v_{0,0,\dots,2}$, the quadratic form W is positive definite. Therefore the sum $W+W_*$ is also positive definite in sufficiently small neighbourhood of the origin. At the same time, the function V of form (4.17) is alternating. Hence by virtue of Theorem 2.5, the zero solution of system (4.13) is unstable. □

Remark 4.2. It is impossible to construct a Lyapunov function V such that its first variation $\Delta_1 V = \Delta V$ relative to system (4.13) is positive (or negative) definite, so we cannot apply Theorem 1.8 and have to apply Theorem 2.5 for $m = 2$.

Thus in the case $v_{0,0,\dots,2} \neq 0$, the stability problem has been solved independently of the terms whose degrees are higher then two. Consider now the case $v_{0,0,\dots,2} = 0$. We shall transform system (4.13) to the form where $v^{(j)}_{0,0,\dots,2} = 0$ ($j = 1, 2, \dots, k-1$). Denote

$$y_j = \xi_j + m_j y^2 \quad (j = 1, 2, \dots, k-1), \tag{4.18}$$

where m_j are constants. In these designations, system (4.13) has the form

$$\begin{aligned} \xi_j(n+1) &= \nu_{j1}\xi_1(n) + \nu_{j2}\xi_2(n) + \dots + \nu_{j,k-1}\xi_{k-1}(n) \\ &\quad + y^2(n)(\nu_{j1}m_1 + \nu_{j2}m_2 + \dots + \nu_{j,k-1}m_{k-1}) \\ &\quad + \Psi_j(\xi_1(n) + m_1 y^2(n), \dots, \xi_{k-1}(n) + m_{k-1}y^2(n), y(n)) \\ &\quad - m_j\big[y^2(n) - 2y(n)\Psi(\xi_1(n) + m_1 y^2(n), \dots, \xi_{k-1}(n) \\ &\quad + m_{k-1}y^2(n), y(n)) \\ &\quad + \Psi^2(\xi_1(n) + m_1 y^2(n), \dots, \xi_{k-1}(n) + m_{k-1}y^2(n), y(n))\big], \end{aligned} \tag{4.19}$$

$$y(n+1) = -y(n) + \Psi(\xi_1(n) + m_1 y^2(n), \dots, \xi_{k-1}(n) + m_{k-1}y^2(n), y(n)). \tag{4.20}$$

Choose constants $m_1, \dots, m_{k-1}$ such that the coefficients corresponding to $y^2(n)$ in right-hand sides of system (4.19), are equal to zero.

Equating to zero the corresponding coefficients, we obtain the system of linear algebraic equations with respect to $m_1, \dots, m_{k-1}$:

$$\nu_{j1}m_1 + \nu_{j2}m_2 + \dots + \nu_{j,k-1}m_{k-1} = m_j - \psi^{(j)}_{0,0,\dots,2} \quad (j = 1, 2, \dots, k-1).$$

This system has a unique solution because one is not an eigenvalue of the matrix Υ. Substituting the obtained values $m_1, \dots, m_{k-1}$ to (4.19) and (4.20), we obtain the system

$$\begin{aligned} \xi_j(n+1) &= \nu_{j1}\xi_1(n) + \nu_{j2}\xi_2(n) + \dots + \nu_{j,k-1}\xi_{k-1}(n) \\ &\quad + \Phi_j(\xi_1(n), \dots, \xi_{k-1}(n), y(n)) \quad (j = 1, \dots, k-1), \end{aligned} \tag{4.21}$$

$$y(n+1) = -y(n) + \Phi(\xi_1(n), \dots, \xi_{k-1}(n), y(n)), \tag{4.22}$$

where

$$\begin{aligned} \Phi_j(\xi_1, \dots, \xi_{k-1}, y) &= \Psi_j(\xi_1 + m_1 y^2, \dots, \xi_{k-1} + m_{k-1}y^2, y) \\ &\quad + 2m_j y\Psi(\xi_1 + m_1 y^2, \dots, \xi_{k-1} + m_{k-1}y^2, y) \\ &\quad - m_j\Psi^2(\xi_1 + m_1 y^2, \dots, \xi_{k-1} + m_{k-1}y^2, y) - \psi^{(j)}_{0,0,\dots,2}y^2, \\ \Phi(\xi_1, \dots, \xi_{k-1}, y) &= \Psi(\xi_1 + m_1 y^2, \dots, \xi_{k-1} + m_{k-1}y^2, y). \end{aligned}$$

Expansions of Φ_j and Φ in power series begin with terms of the second degree, and coefficients corresponding to y^2 in expansions of Φ_j and Φ are equal to zero. System (4.21) and (4.22) will be basic in our further investigation of the stability of the zero solution

$$\xi_1 = 0, \quad \xi_2 = 0, \quad \dots, \quad \xi_{k-1} = 0, \quad y = 0. \tag{4.23}$$

Side by side with system (4.21) and (4.22), let us consider the system

$$\begin{aligned} \xi_j(n+2) = c_{j1}\xi_1(n) + c_{j2}\xi_2(n) + \dots + c_{j,k-1}\xi_{k-1}(n) \\ + \Xi_j(\xi_1(n), \dots, \xi_{k-1}(n), y(n)) \quad (j = 1, \dots, k-1), \end{aligned} \tag{4.24}$$

$$y(n+2) = y(n) + Y_*(\xi_1(n), \dots, \xi_{k-1}(n), y(n)), \tag{4.25}$$

where expansions of Ξ_j and Y_* in power series begin with terms of the second degree, and expansions of Ξ_j do not include terms corresponding to $y^2(n)$.

Denote by $\Xi_j^{(0)}(y)$ $(j = 1, \dots, k-1)$ and $Y_*^{(0)}(y)$ the sum of all terms in functions Ξ_j and Y_* respectively, which do not include $\xi_1, \dots, \xi_{k-1}$, so

$$\Xi_j^{(0)}(y) = \Xi_j(0, \dots, 0, y) = h_j y^3 + \sum_{s=4}^{\infty} h_j^{(s)} y^s,$$

$$Y_*^{(0)}(y) = Y_*(0, \dots, 0, y) = hy^3 + \sum_{s=4}^{\infty} h^{(s)} y^s,$$

where $h, h_j, h^{(s)}, h_j^{(s)}$ $(j = 1, \dots, k-1;\ s = 4, 5, \dots)$ are constants.

Theorem 4.3. *The solution* (4.23) *of system* (4.21) *and* (4.22) *is asymptotically stable if $h < 0$ and unstable if $h > 0$.*

Proof. We shall show that there exists a Lyapunov function V such that it depends on $\xi_1, \dots, \xi_{k-1}, y$, and $\Delta_2 V$ is positive definite. Consider the system of linear equations

$$\xi_j(n+1) = \nu_{j1}\xi(n) + \nu_{j2}\xi_2(n) + \dots + \nu_{j,k-1}\xi_{k-1}(n) \quad (j = 1, \dots, k-1). \tag{4.26}$$

Let $W = \sum_{i_1+\dots+i_{k-1}=2} w_{i_1,\dots,i_{k-1}} \xi_1^{i_1} \dots \xi_{k-1}^{i_{k-1}}$ be a quadratic form of variables $\xi_1, \dots, \xi_{k-1}$, such that

$$\Delta_2 W\big|_{(4.26)} = \xi_1^2 + \dots + \xi_{k-1}^2. \tag{4.27}$$

Since all eigenvalues of the matrix Υ are inside of the unit disk, the form W satisfying (4.27), exists, is unique and negative definite [16, Theorem 4.30].

If functions Ξ_j $(j = 1, \dots, k-1)$ do not depend on y, then the second variation Δ_2 of the function W along system (4.21); i.e., the expression

$$\begin{aligned} \sum_{i_1+\dots+i_{k-1}=2} w_{i_1,\dots,i_{k-1}} \big\{ [c_{11}\xi_1 + c_{12}\xi_2 + \dots + c_{1,k-1}\xi_{k-1} + \Xi_1]^{i_1} \dots \\ [c_{k-1,1}\xi_1 + \dots + c_{k-1,k-1}\xi_{k-1} + \Xi_{k-1}]^{i_{k-1}} - \xi_1^{i_1} \dots \xi_{k-1}^{i_{k-1}} \big\} \end{aligned} \tag{4.28}$$

is a positive definite function on the variables $\xi_1, \dots, \xi_{k-1}$ for $\xi_1, \dots, \xi_{k-1}$ sufficiently small.

On the other hand, if the function Y_* does not depend on $\xi_1, \dots, \xi_{k-1}$ (i.e. if $Y_* = Y_*^{(0)}$), then the second variation Δ_2 of the function $\frac{1}{2}hy^2$ is equal to

$$\Delta_2\big(\frac{1}{2}hy^2\big) = \frac{1}{2}h\left[2yY_*^{(0)} + {Y_*^{(0)}}^2\right] = h^2y^4 + hh^{(4)}y^5 + o(y^5), \tag{4.29}$$

and this variation is a positive definite function with respect to y for sufficiently small $|y|$. Therefore, under these conditions, the variation Δ_2 of the function $V_1 = \frac{1}{2}hy^2 + W(\xi_1, \dots, \xi_{k-1})$ along the total system (4.21) and (4.22) is a positive definite function of all variables $\xi_1, \dots, \xi_{k-1}, y$ in some neighbourhood of the origin. Taking into account (4.27) and (4.29), this variation can be represented in the form

$$(h^2 + g_1)y^4 + \xi_1^2 + \dots + \xi_{k-1}^2 + \sum_{i,j=1}^{k-1} g_{ij}^{(1)} \xi_i \xi_j, \tag{4.30}$$

where g_1 is a holomorphic function of the variable y, vanishing for $y = 0$, and $g_{ij}^{(1)}$ are holomorphic functions of variables $\xi_1, \dots, \xi_{k-1}$, vanishing for $\xi_1 = \dots = \xi_{k-1} = 0$.

But since the functions $\Xi_j \quad (j = 1, \dots, k-1)$ include y, and the function Y_* includes $\xi_1, \dots, \xi_{k-1}$, the variation Δ_2 of the function V_1 along system (4.21) and (4.22), in general, is not positive definite. In this difference, there appear the terms breaking the positive definiteness.

Note that expression (4.30) remains positive definite if the function g_1 includes not only the variable y, but also the variables $\xi_1, \dots, \xi_{k-1}$, and functions $g_{ij}^{(1)}$ include not only variables $\xi_1, \dots, \xi_{k-1}$, but also the variable y. It is only important the functions g_1 and $g_{ij}^{(1)}$ to vanish for $\xi_1 = \dots = \xi_{k-1} = y = 0$. Taking into account this fact, let us write the second variation of the function V_1 along (4.21) and (4.22) in the form

$$\begin{aligned} \Delta_2 V_1 &= \Delta_2\Big(\frac{1}{2} h y^2\Big) + \Delta_2 W = h y Y_* + \frac{1}{2} h Y_*^2 \\ &\quad + \sum_{i_1 + \dots + i_{k-1} = 2} w_{i_1,\dots,i_{k-1}} \{[c_{11}\xi_1 + c_{12}\xi_2 + \dots + c_{1,k-1}\xi_{k-1} + \Xi_1]^{i_1} \times \dots \\ &\quad \times [c_{k-1,1}\xi_1 + \dots + c_{k-1,k-1}\xi_{k-1} + \Xi_{k-1}]^{i_{k-1}} - \xi_1^{i_1} \dots \xi_{k-1}^{i_{k-1}}\} \\ &= [h^2 + g_1(\xi_1, \dots, \xi_{k-1}, y)] y^4 + \xi_1^2 + \dots + \xi_{k-1}^2 \\ &\quad + \sum_{i,j=1}^{k-1} g_{ij}^{(1)}(\xi_1, \dots, \xi_{k-1}, y) \xi_i \xi_j + Q(\xi_1, \dots, \xi_{k-1}, y), \end{aligned} \tag{4.31}$$

where functions g_1 and $g_{ij}^{(1)} \quad (i, j = 1, \dots, k-1)$ vanish for $\xi_1 = \dots = \xi_{k-1} = y = 0$, and Q is the sum of all terms, which can be included neither to the expression

$$g_1(\xi_1, \dots, \xi_{k-1}, y) y^4, \tag{4.32}$$

nor to the expression

$$\sum_{i,j=1}^{k-1} g_{ij}^{(1)}(\xi_1, \dots, \xi_{k-1}, y) \xi_i \xi_j. \tag{4.33}$$

All terms which are included to the expression Q, can be divided into next four groups: the terms free of $\xi_1, \dots, \xi_{k-1}$, the terms linear with respect to $\xi_1, \dots, \xi_{k-1}$, the terms quadratic with respect to $\xi_1, \dots, \xi_{k-1}$, and the terms having degree higher than two with respect to $\xi_1, \dots, \xi_{k-1}$. It is evident that all terms of the last group can be included into expression (4.33); therefore we shall consider only first three groups of terms.

All terms, free of $\xi_1, \dots, \xi_{k-1}$, are obviously included in expressions (4.29) (where they have been written explicitly) and in $\sum_{i_1+\dots+i_{k-1}=2} w_{i_1,\dots,i_{k-1}} \Xi_1^{(0)\,i_1} \dots \Xi_{k-1}^{(0)\,i_{k-1}}$ (where there are summands of the sixth and higher degrees with respect to y). All these summands can be included into expression (4.32). Hence the function Q does not include the terms, free of $\xi_1, \dots, \xi_{k-1}$.

Terms, linear with respect to $\xi_1,\dots,\xi_{k-1}$, are included into expression (4.31) both by means of summands from $hyY_* + \frac{1}{2}hY_*^2$ and from (4.28). If these terms have order not less than fourth with respect to y, then it is clear that they can be included into expression (4.32). Thus the function Q has only those terms, linear with respect to $\xi_1,\dots,\xi_{k-1}$, which have degrees two and three with respect to y.

Finally, consider the terms, quadratic with respect to $\xi_1,\dots,\xi_{k-1}$. If these terms have the total degree higher than two, then they can be included into expression (4.33) and therefore they are not contained in the function Q. All quadratic terms with respect to $\xi_1,\dots,\xi_{k-1}$ having the second degree (i.e. the terms with constant coefficients) are contained in the expression

$$\begin{aligned}&\sum_{i_1+\dots+i_{k-1}=2} w_{i_1,\dots,i_{k-1}}\{[c_{11}\xi_1 + c_{12}\xi_2 + \dots + c_{1,k-1}\xi_{k-1}]^{i_1} \times \dots\\ &\times [c_{k-1,1}\xi_1 + \dots + c_{k-1,k-1}\xi_{k-1}]^{i_{k-1}} - \xi_1^{i_1}\dots\xi_{k-1}^{i_{k-1}}\}\\ &= \xi_1^2 + \dots + \xi_{k-1}^2,\end{aligned}$$

and hence are not contained in the function Q. Thus the function Q has the form

$$Q = y^2 Q_2(\xi_1,\dots,\xi_{k-1}) + y^3 Q_3(\xi_1,\dots,\xi_{k-1}), \tag{4.34}$$

where Q_2 and Q_3 are linear forms with respect to $\xi_1,\dots,\xi_{k-1}$:

$$Q_2 = q_1^{(2)}\xi_1 + q_2^{(2)}\xi_2 + \dots + q_{k-1}^{(2)}\xi_{k-1}, \quad Q_3 = q_1^{(3)}\xi_1 + q_2^{(3)}\xi_2 + \dots + q_{k-1}^{(3)}\xi_{k-1}.$$

The presence of summand (4.34) in (4.31) breaks the positive definiteness of $\Delta_2 V_1$. To get rid of the summand $y^2Q_2(\xi_1,\dots,\xi_{k-1})$, let us add the the summand $y^2P_2(\xi_1,\dots,\xi_{k-1}) = y^2(p_1^{(2)}\xi_1 + p_2^{(2)}\xi_2 + \dots + p_{k-1}^{(2)}\xi_{k-1})$, to the function V_1. Here $p_j^{(2)}\quad (j = 1,\dots,k-1)$ are constants. In other words, consider the function

$$V_2 = \frac{1}{2}hy^2 + W(\xi_1,\dots,\xi_{k-1}) + y^2P_2(\xi_1,\dots,\xi_{k-1}) \tag{4.35}$$

instead of the function V_1. The term $y^2P_2(\xi_1,\dots,\xi_{k-1})$ brings the following summands to $\Delta_2 V_1$:

$$\begin{aligned}&\Delta_2(y^2P_2(\xi_1,\dots,\xi_{k-1}))\\ &= [y^2 + 2yY_*(\xi_1,\dots,\xi_{k-1},y) + Y_*^2(\xi_1,\dots,\xi_{k-1},y)]\\ &\times \sum_{j=1}^{k-1} p_j^{(2)}[c_{j,1}\xi_1 + c_{j,2}\xi_2 + \dots + c_{j,k-1}\xi_{k-1} + \Xi_j(\xi_1,\dots,\xi_{k-1},y)]\\ &\quad - y^2[p_1^{(2)}\xi_1 + p_2^{(2)}\xi_2 + \dots + p_{k-1}^{(2)}\xi_{k-1}]\\ &= y^2\Big[\sum_{j=1}^{k-1} p_j^{(2)}(c_{j1}\xi_1 + c_{j2}\xi_2 + \dots + c_{j,k-1}\xi_{k-1} - \xi_j)\Big] + G(\xi_1,\dots,\xi_{k-1},y).\end{aligned}$$

Here the function G is the sum of summands every of which can be included either to expression (4.32) or to (4.33). Let us choose constants $p_1^{(2)},\dots,p_{k-1}^{(2)}$ such that the equality

$$\sum_{j=1}^{k-1} p_j^{(2)}(c_{j1}\xi_1 + c_{j2}\xi_2 + \dots + c_{j,k-1}\xi_{k-1} - \xi_j) = -\sum_{j=1}^{k-1} q_j^{(2)}\xi_j \tag{4.36}$$

holds. To do this, let us equate the coefficients corresponding to $\xi_j \quad (j=1,\dots,k-1)$ in the right-hand and left-hand sides of equality (4.36). We obtain the system of linear equations with respect to $p_j^{(2)} \quad (j=1,\dots,k-1)$:

$$c_{1j}p_1^{(2)} + c_{2j}p_2^{(2)} + \dots + (c_{jj}-1)p_j^{(2)} + \dots + c_{k-1,j}p_{k-1}^{(2)} = -q_j^{(2)} \quad (j=1,\dots,k-1). \tag{4.37}$$

The determinant of this system is not equal to zero because all eigenvalues of $\mathcal{C}$ are inside the unit disk. Therefore system (4.37) has the unique solution. Substituting the obtained values $p_1^{(2)},\dots,p_{k-1}^{(2)}$ into the expression $P_2(\xi_1,\dots,\xi_{k-1})$, we obtain

$$\begin{aligned}\Delta_2 V_2 &= [h^2 + g_2(\xi_1,\dots,\xi_{k-1},y)]y^4 + (\xi_1^2+\dots+\xi_{k-1}^2)\\ &\quad + \sum_{i,j=1}^{k-1} g_{ij}^{(2)}(\xi_1,\dots,\xi_{k-1},y)\xi_i\xi_j + y^3Q_3(\xi_1,\dots,\xi_{k-1}),\end{aligned} \tag{4.38}$$

where g_2 and $g_{ij}^{(2)}$ are functions, vanishing for $\xi_1=\xi_2=\dots=\xi_{k-1}=y=0$.

Similarly, one can show that it is possible to be rid of the summand $y^3Q_3(\xi_1,\dots,\xi_{k-1})$ in expression (4.38). To do this, all we need is to add to the function V_2 the summand

$$y^3P_3(\xi_1,\dots,\xi_{k-1}) = y^3(p_1^{(3)}\xi_1 + p_2^{(3)}\xi_2 + \dots + p_{k-1}^{(3)}\xi_{k-1}),$$

where $p_j^{(3)}$ $(j=1,\dots,k-1)$ are constants. In other words, consider the function

$$V = \frac{1}{2}hy^2 + W(\xi_1,\dots,\xi_{k-1}) + y^2P_2(\xi_1,\dots,\xi_{k-1}) + y^3P_3(\xi_1,\dots,\xi_{k-1}) \tag{4.39}$$

instead of the function V_2. Its difference Δ_2 along system (4.21) and (4.22) is

$$\begin{aligned}\Delta_2 V &= [h^2 + g(\xi_1,\dots,\xi_{k-1},y)]y^4 + (\xi_1^2+\dots+\xi_{k-1}^2)\\ &\quad + \sum_{i,j=1}^{k-1} g_{ij}(\xi_1,\dots,\xi_{k-1},y)\xi_i\xi_j,\end{aligned} \tag{4.40}$$

where g and g_{ij} are functions vanishing for $\xi_1=\xi_2=\dots=\xi_{k-1}=y=0$.

It follows from (4.40) that $\Delta_2 V$ is positive definite in sufficiently small neighbourhood of the origin, and the function V of the form (4.39) is negative definite for $h<0$ and changes its sign for $h>0$. Hence according to Theorems 2.4 and 2.5, we can conclude that the solution (4.23) of system (4.21) and (4.22) is asymptotically stable for $h<0$ and unstable for $h>0$. This completes the proof. □

Remark 4.4. Obviously, that substitutions (4.4), (4.10), and (4.18) are such that the investigation of the stability of solution (4.23) of system (4.21) and (4.22) is equivalent to the investigation of the stability of the zero solution of system (4.1).

Remark 4.5. In Theorems 4.1 and 4.3 there are conditions under which the problem of the stability of the zero solution of system (4.1) can be solved in the critical case when one eigenvalue of the linearized system is equal to minus one. The obtained criteria do not depend on nonlinear terms with degrees of smallness more than three. If we obtain $h=0$, then the stability problem cannot be solved by terms of the first, second, and third degrees of smallness in the expansions of the right-hand sides of the system of difference equations. To solve this problem, it is necessary to consider also the terms of higher degrees.

References

[1] R. P. Agarwal. *Difference Equations and Inequalities*. Marcel Dekker, New York, 2nd edition, 2000.

[2] R. P. Agarwal, D. O'Regan, and P. J. Y. Wong. Dynamics of epidemics in homogeneous/heterogeneous populations and the spreading of multiple inter-related infectious deseases: constant-sign periodic solutions for the discrete model. *Nonlinear Anal. Real World Appl.*, 8(4):1040–1061, 2007.

[3] N. U. Ahmed. *Dynamic Systems and Control with Applications*. World Scientific, Singapore, 2006.

[4] J. Angelova and A. Dishliev. Optimization problems for one-impulsive models from population dynamics. *Nonlinear Analysis*, 39:483–497, 2000.

[5] J. Angelova, A. Dishliev, and S. Nenov. Comparison of zero-solutions of systems ODE via asymptotical stability. *Nonlinear Analysis*, 42:339–350, 2000.

[6] D. D. Bainov and P. S. Simeonov. *Systems with impulse effect: stability, theory and applications*. Halsted Press, New York – Chichester – Brisbane – Toronto, 1989.

[7] F. Brauer and C. Castillo-Chavez. *Mathematical Models in Population Biology and Epidemiology*. Springer, New York, 2001.

[8] E. Braverman. On a discrete model of population dynamics with impulsive harvesting or recruitment. *Nonlinear Analysis*, 63(5-7):e751–e759, 2005.

[9] A. Cabada and E. Liz. Discontinuous impulsive differential equations with nonlinear boundary conditions. *Nonlinear Analysis*, 28:1491 – 1497, 1997.

[10] M. L. Castro and D. A. R. J. A. L. Silva, J. A. L. and D. A. R. Justo. Stability in an age-structured metapopulation model. *J. Math. Biol.*, 52(2):183–208, 2006.

[11] V. Chellaboina, S. P. Bhat, and W. M. Haddad. An invariance principle for nonlinear hybrid and impulsive dynamical systems. *Nonlinear Analysis*, 53:527–550, 2003.

[12] F. Chen. Permanence and global attractivity of a discrete multispecies Lotka-Volterra competition predador-prey systems. *Applied Mathematics and Computation*, 182(1):3–12, 2006.

[13] R. Continho, B. Fernandez, R. Lima, and A. Meyroneinc. Discrete time piecewise affine models of genetic regulatory networks. *J. Math. Biol.*, 52(4):524–570, 2006.

[14] P. Cull, M. Flahive, and R. Robson. *Difference Equations. From Rabbits to Chaos*. Springer, New York, 2005.

[15] A. D'Onofrio. Pulse vaccination strategy in the SIR epidemic model: Global asymptotic stable eradication in presence of vaccine failures. *Mathematical and Computer Modelling*, 36:473 – 489, 2002.

[16] S. Elaydi. *An Introduction to Difference Equations*. Springer, New York, 3rd edition, 2005.

[17] S. Elaydi. *Discrete Chaos*. Chapman & Hall/CRC, Boca Raton, 2nd edition, 2008.

[18] J. E. Franke and J. F. Selgrade. Attractors for discrete periodic dynamical systems. *J. Math. Anal. Appl.*, 286(1):64–79, 2003.

[19] J. E. Franke and A. A. Yakubu. Globally attracting attenuant versus resonant cycles in periodic compensatory leslie models. *Mathematical Biosciences*, 204(1):1–20, 2006.

[20] A. K. Gelig and A. N. Churilov. *Stability and Oscillations of Nonlinear Pulse-Modulated Systems*. Birkhauser, Boston, 1998.

[21] R. I. Gladilina and A. O. Ignatyev. On necessary and sufficient conditions for the asymptotic stability of impulsive systems. *Ukrainian Mathematical Journal*, 55(8):1254–1264, 2003.

[22] R. I. Gladilina and A. O. Ignatyev. On the stability of periodic impulsive systems. *Mathematical Notes*, 76(1):41–47, 2004.

[23] Z.-H. Guan, J. Lam, and G. Chen. On impulsive autoassociative neural networks. *Neural Networks*, 13:63–69, 2000.

[24] W. M. Haddad, V. Chellaboina, and S. G. Nersesov. Hybrid nonnegative and compartmental dynamical systems. *Mathematical Problems in Engineering*, 8(6):493–515, 2002.

[25] W. M. Haddad, V. Chellaboina, and S. G. Nersesov. *Impulsive and hybrid dynamical systems: stability, dissipativity, and control*. Princeton University Press, Princeton, 2006.

[26] A. O. Ignat'ev, O. A. Ignat'ev, and A. A. Soliman. Asymptotic stability and instability of the solutions of systems with impulse action. *Mathematical Notes*, 80(4):491–499, 2006.

[27] A. O. Ignatyev. Method of Lyapunov functions in problems of stability of solutions of systems of differential equations with impulse action. *Sbornik: Mathematics*, 194(10):117–132, 2003.

[28] A. O. Ignatyev. Stability of the zero solution of an almost periodic system of finite-difference equations. *Differential Equations*, 40(1):98–103, 2004.
[29] A. O. Ignatyev. Asymptotic stability and instability with respect to part of variables for solutions to impulsive systems. *Siberian Mathematical Journal*, 49(1):102–108, 2008.
[30] A. O. Ignatyev and O. A. Ignatyev. On the stability in periodic and almost periodic difference systems. *Journal of Mathematical Analysis and Applications*, 313(2):678–688, 2006.
[31] A. O. Ignatyev and O. A. Ignatyev. Investigation of the asymptotic stability of solutions of systems with impulse effect. *International Journal of Mathematics, Game Theory and Algebra*, 17(3):141–164, 2008.
[32] W. G. Kelley and A. C. Peterson. *Difference equations: an introduction with applications*. Academic Press, New York, 2nd edition, 2000.
[33] V. Lakshmikantham and D. Trigiante. *Theory of Difference Equations: Numerical Methods and Applications*. Marcel Dekker, New York, 2nd edition, 2002.
[34] S. I. Nenov. Impulsive controllability and optimization problems in population dynamics. *Nonlinear Analysis*, 36:881–890, 1999.
[35] M. Sadkane and L. Grammont. A note on the Lyapunov stability of periodic discrete-time systems. *Journal of Computational and Applied Mathematics*, 176:463–466, 2005.
[36] A. M. Samoilenko and N. A. Perestyuk. *Impulsive differential equations*. World Scientific, Singapure–New Jersey–London, 1995.
[37] R. J. Smith and L. M. Wahl. Distinct effects of protease and reverse transcriptase inhibition in an immunological model of HIV-1 infection with impulsive drug effects. *Bulletin of Mathematical Biology*, 66:1259–1283, 2004.
[38] R. J. Smith and L. M. Wahl. Drug resistence in an immunological model of HIV - 1 infection with impulsive drug effects. *Bulletin of Mathematical Biology*, 67:783–813, 2005.
[39] S. Tang and R. A. Cheke. Density-dependent birth rate, birth pulses and their population dynamic consequences. *Journal of Mathematical Biology*, 44:185–199, 2002.
[40] S. Tang and R. A. Cheke. State-dependent impulsive models of integrated pest management (IPM) strategies and their consequences. *Journal of Mathematical Biology*, 50:257–292, 2005.
[41] X. Zhang, Z. Shuai, and K. Wang. Optimal impulsive harvesting policy for single population. *Nonlinear Analysis: Real World Applications*, 4:639–651, 2003.
[42] Y. Zhang, B. Liu, and L. Chen. Extinction and permanence of a two-prey one-predator system with impulsive effect. *Mathematical Medicine and Biology*, 20:309–325, 2003.
[43] Y. Zhou and H. Liu. Stability of periodic solutions for an SIS model with pulse vaccination. *Mathematical and Computer Modelling*, 38:299 – 308, 2003.

Alexander O. Ignatyev
Institute for Applied Mathematics and Mechanics, R. Luxemburg Street,74, Donetsk-83114, Ukraine
E-mail address: aoignat@mail.ru, ignat@iamm.ac.donetsk.ua

Oleksiy Ignatyev
Department of Statistics and Probability, Michigan State University, A408 Wells Hall, East Lansing, MI 48824-1027, USA
E-mail address: ignatyev@stt.msu.edu, aignatye@math.kent.edu