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NONLOCAL BOUNDARY-VALUE PROBLEMS FOR N-TH ORDER ORDINARY DIFFERENTIAL EQUATIONS BY MATCHING SOLUTIONS

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Abstract. We are concerned with the existence and uniqueness of solutions to nonlocal boundary-value problems on an interval $[a, c]$ for the differential equation $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$, where $n \geq 3$. We use the method of matching solutions, with some monotonicity conditions on f.

1. Introduction

In this article, we are concerned with the existence and uniqueness of solutions of boundary-value problems (BVP's) for the differential equation

$$y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x)), \quad n \geq 3, \ x \in [a, c], \tag{1.1}$$

$$\begin{gathered} y(a) - \sum_{i=1}^{s} \alpha_i y(\xi_i) = y_1, \quad y^{(i)}(b) = y_{i+2}, \quad 0 \leq i \leq n-3, \\ \sum_{j=1}^{t} \beta_j y(\eta_j) - y(c) = y_n, \end{gathered} \tag{1.2}$$

where $a < \xi_1 < \xi_2 < \dots < \xi_s < b < \eta_1 < \eta_2 < \dots < \eta_t < c$, $s, t \in \mathbb{N}$, $\alpha_i > 0$ for $1 \leq i \leq s$, $\beta_j > 0$ for $1 \leq j \leq t$, $\sum_{i=1}^{s} \alpha_i = 1$, $\sum_{j=1}^{t} \beta_j = 1$, and $y_1, y_2, \dots, y_n \in \mathbb{R}$.

It is assumed throughout that $f : [a, c] \times \mathbb{R}^n \to \mathbb{R}$ is continuous and that solutions for the initial value problems (IVP's) for (1.1) are unique and exist on $[a, c]$. Moreover $a < \xi_1 < \xi_2 < \dots < \xi_s < b < \eta_1 < \eta_2 < \dots < \eta_t < c$ are fixed throughout.

Consider the following boundary conditions:

$$y(a) - \sum_{i=1}^{s} \alpha_i y(\xi_i) = y_1, \quad y^{(i)}(b) = y_{i+2}, \ 0 \leq i \leq n-3, \quad y^{(n-2)}(b) = m, \tag{1.3}$$

$$y(a) - \sum_{i=1}^{s} \alpha_i y(\xi_i) = y_1, \quad y^{(i)}(b) = y_{i+2}, \ 0 \leq i \leq n-3, \quad y^{(n-1)}(b) = m, \tag{1.4}$$

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$$y^{(i)} = y_{i+2},\ 0 \le i \le n-3, \quad y^{(n-2)}(b) = m, \quad \sum_{j=1}^{t} \beta_j y(\eta_j) - y(c) = y_n, \tag{1.5}$$

$$y^{(i)} = y_{i+2},\ 0 \le i \le n-3, \quad y^{(n-1)}(b) = m, \quad \sum_{j=1}^{t} \beta_j y(\eta_j) - y(c) = y_n, \tag{1.6}$$

where $m \in \mathbb{R}$. We show that (1.1)-(1.2) has a unique solution by matching solutions of the BVP's (1.1)-(1.3) on $[a, b]$ and (1.1)-(1.5) on $[b, c]$, or (1.1)-(1.4) on $[a, b]$ and (1.1)-(1.6) on $[b, c]$.

The method of matching solutions was first used by Bailey et al. [1]. They considered the solutions of two-point boundary value problems for the second order differential equation $y'' = f(x, y, y')$ by matching solutions of initial value problems. After that, in 1973, Barr and Sherman [2] applied the solution matching techniques to third order equations and generalized to equations of arbitrary order. A monotonicity condition on f played an important role. In 1981, Rao et al. [10] generalized the monotonicity of f of third order differential equations and introduced an auxiliary monotone function g. In 1983, Henderson [4] generalized to nth order BVP's and considered more general boundary conditions. Since then there has been a lot of literature dealing with solutions of third order BVP's or higher order BVP's by using matching solutions; see [3, 5, 6, 7, 8, 9], etc.

In this article, we consider the n-th order BVP's with nonlocal boundary conditions (1.1)-(1.2) and use a weaker condition on the auxiliary function g. In Section 2, we give some preliminary results, and in Section 3, we prove the existence and uniqueness of solutions of (1.1)-(1.2). In Section 4, we generalize our results to BVP's with more general boundary conditions:

$$\begin{gathered} y^{(\tau)}(a) - \sum_{i=1}^{s} \alpha_i y^{(\tau)}(\xi_i) = y_1, \quad y^{(i)}(b) = y_{i+2},\ 0 \le i \le n-3, \\ \sum_{j=1}^{t} \beta_j y^{(\sigma)}(\eta_j) - y^{(\sigma)}(c) = y_n, \end{gathered} \tag{1.7}$$

with $\tau, \sigma \in \{0, 1, \dots, n-3\}$ fixed.

We assume there is a continuous function $g : [a, c] \times \mathbb{R}^n \to \mathbb{R}$ and that f and g satisfy the following conditions:

(A) For $u, v \in \mathbb{R}$, $f(x, v_0, v_1, \dots, v_{n-2}, v) - f(x, u_0, u_1, \dots, u_{n-2}, u) > g(x, v_0 - u_0, v_1 - u_1, \dots, v_{n-2} - u_{n-2}, v - u)$ when $x \in (a, b]$, $(-1)^{n-i} v_i \ge (-1)^{n-i} u_i$, $0 \le i \le n-3$, and $v_{n-2} > u_{n-2}$; or when $x \in [b, c)$, $v_i \ge u_i$, $0 \le i \le n-3$, and $v_{n-2} > u_{n-2}$.

(B) There exists $\delta_1 > 0$, such that for all $0 < \delta < \delta_1$, the IVP

$$z^{(n)} = g(x, z, z', \dots, z^{(n-1)}), \tag{1.8}$$

$$z^{(i)}(b) = 0, \quad 0 \le i \le n-1, \quad i \ne n-2, \quad z^{(n-2)}(b) = \delta \tag{1.9}$$

has a solution z on $[a, c]$ such that $z^{(n-2)}(x) \ge 0$ on $[a, c]$.

(C) There exists $\delta_2 > 0$, such that for all $0 < \delta < \delta_2$, the IVP

$$z^{(n)} = g(x, z, z', \dots, z^{(n-1)}), \tag{1.10}$$

$$z^{(i)}(b) = 0, \quad 0 \le i \le n-2, \quad z^{(n-1)}(b) = \delta, (-\delta) \tag{1.11}$$

has a solution z on $[b,c]$ ($[a,b]$) such that $z^{(n-2)}(x)\geq 0$ on $[b,c]$, ($z^{(n-2)}(x)\geq 0$ on $[a,b]$).

(D) For each $w\in\mathbb{R}$, $g(x,v_0,v_1,\dots,v_{n-2},w)\geq g(x,u_0,u_1,\dots,u_{n-2},w)$ when $x\in(a,b]$, $(-1)^{n-i}(v_i-u_i)\geq 0$, $i=0,1,\dots,n-3$, and $v_{n-2}>u_{n-2}\geq 0$, or when $x\in[b,c)$, $v_i\geq u_i$, $i=0,1,\dots,n-3$, and $v_{n-2}>u_{n-2}\geq 0$.

2. Preliminaries

In this section, we give two lemmas which show the relationship between the value of the $n-2$nd order and the $n-1$st order of two solutions of (1.1) at b that satisfy the boundary conditions (2), respectively, on the interval $[a,b]$ and the interval $[b,c]$. All of the results in Section 3 are based on two lemmas. We basically prove the lemmas by using contradiction.

Lemma 2.1. *Suppose p and q are solutions of* (1.1) *on $[a,b]$ and $w=p-q$ satisfies the following boundary conditions:*

$$w(a)-\sum_{i=1}^{s}\alpha_i w(\xi_i)=0,\quad w^{(i)}(b)=0,\quad 0\leq i\leq n-3.$$

Then, $w^{(n-2)}(b)=0$ if and only if $w^{(n-1)}(b)=0$. Also, $w^{(n-2)}(b)>0$ if and only if $w^{(n-1)}(b)>0$.

Proof. ($\Rightarrow$) The necessity of the first part. Suppose $w^{(n-2)}(b)=0$ and $w^{(n-1)}(b)\neq 0$. Without loss of generosity, we assume $w^{(n-1)}(b)<0$.

Since $0=w(a)-\sum_{i=1}^{s}\alpha_i w(\xi_i)=\sum_{i=1}^{s}\alpha_i(w(a)-w(\xi_i))$ and $\alpha_i>0$, for some i_1, $w(a)\geq w(\xi_{i_1})$, and for some i_2, $w(a)\leq w(\xi_{i_2})$. Hence, there exists $r_1\in(a,b)$ such that $w'(r_1)=0$ and $(-1)^{n-1}w'(x)>0$ on (r_1,b).

By repeated applications of Rolle's Theorem, there exists $r_2\in(r_1,b)$ such that $w^{(n-2)}(r_2)=0$ and $w^{(n-2)}(x)>0$, for $x\in(r_2,b)$. Hence, $(-1)^{n-j}w^{(j)}(x)>0$, for $j=0,1,\dots,n-2$, on (r_2,b).

Let $\delta\in\mathbb{R}$ with $0<\delta<\min\{\delta_2,-w^{(n-1)}(b)\}$. Then, by Condition (C), we have a solution z of (1.10)-(1.11) on $[a,b]$, such that $z^{(i)}(b)=0$, $0\leq i\leq n-2$, $z^{(n-1)}(b)=-\delta$, and $z^{(n-2)}(x)\geq 0$ on $[a,b]$.

Let $h=w-z$. Then, we have

$$\begin{gathered}
h^{(n)}=f(x,p,p',\dots,p^{(n-1)})-f(x,q,q',\dots,q^{(n-1)})-g(x,z,z',\dots,z^{(n-1)}),\\
h^{(i)}(b)=0,\ 0\leq i\leq n-2,\quad h^{(n-1)}(b)=w^{(n-1)}(b)-z^{(n-1)}(b)<0.
\end{gathered}$$

Notice $h^{(n-2)}(r_2)=w^{(n-2)}(r_2)-z^{(n-2)}(r_2)\leq 0$, $h^{(n-2)}(b)=0$ and $h^{(n-1)}(b)<0$. So there exists $r_3\in[r_2,b)$ such that $h^{(n-2)}(r_3)=0$ and $h^{(n-2)}(x)>0$ for $x\in(r_3,b)$. Then, it follows that $(-1)^{n-j}h^{(j)}(x)>0$ on (r_3,b), for $j=0,1,\dots,n-2$. Therefore, by Rolle's Theorem, there is $r_4\in(r_3,b)$ such that $h^{(n-1)}(r_4)=0$. Since $h^{(n-1)}(b)<0$, there is $r_5\in[r_4,b)$ such that $h^{(n-1)}(r_5)=0$ and $h^{(n-1)}(x)<0$ for $x\in(r_5,b)$. Then,

$$h^{(n)}(r_5)=\lim_{x\to r_5^+}\frac{h^{(n-1)}(x)-h^{(n-1)}(r_5)}{x-r_5}\leq 0,$$

whereas by Conditions (A) and (D), (note that $[r_5,b)\subset(r_3,b)\subset(r_2,b)$),

$$\begin{aligned}
h^{(n)}(r_5)&=f(r_5,p,p',\dots,p^{(n-1)})-f(r_5,q,q',\dots,q^{(n-1)})-g(r_5,z,z',\dots,z^{(n-1)})\\
&>g(r_5,w,w',\dots,w^{(n-1)})-g(r_5,z,z',\dots,z^{(n-1)})\geq 0,
\end{aligned}$$

which is a contradiction. Therefore, $w^{(n-1)}(b)=0$.

($\Leftarrow$) The sufficiency of the first part. Suppose $w^{(n-1)}(b)=0$ and $w^{(n-2)}(b)\neq 0$. Without loss of generality, we assume $w^{(n-2)}(b)>0$.

Since $0=w(a)-\sum_{i=1}^{s}\alpha_i w(\xi_i)=\sum_{i=1}^{s}\alpha_i(w(a)-w(\xi_i))$ and $\alpha_i>0$, there exists $r_1\in(a,b)$ such that $w'(r_1)=0$, and $(-1)^{n-1}w'(x)>0$ on (r_1,b).

By repeated applications of Rolle's Theorem, there exists $r_2\in(r_1,b)$ such that $w^{(n-2)}(r_2)=0$ and $w^{(n-2)}(x)>0$ for $x\in(r_2,b)$. Hence, $(-1)^{n-j}w^{(j)}(x)>0$, for $j=0,1,\dots,n-2$, on (r_2,b).

Now let $0<\delta<\min\{\delta_1,w^{(n-2)}(b)\}$, and let z be a solution of (1.8)-(1.9) satisfying Condition (B) and $z^{(n-2)}(x)\geq 0$ on $[a,b]$. Let $h=w-z$. Then,

$$h^{(n)}=f(x,p,p',\dots,p^{(n-1)})-f(x,q,q',\dots,q^{(n-1)})-g(x,z,z',\dots,z^{(n-1)}),$$
$$h^{(i)}(b)=0,\ 0\leq i\leq n-1,\ i\neq n-2,\quad h^{(n-2)}(b)=w^{(n-2)}(b)-z^{(n-2)}(b)>0.$$

Note that $h^{(n-2)}(r_2)=w^{(n-2)}(r_2)-z^{(n-2)}(r_2)\leq 0$. Hence, there is $r_3\in[r_2,b)$ such that $h^{(n-2)}(r_3)=0$, $h^{(n-2)}(x)>0$ on (r_3,b). By Rolle's Theorem, there is $r_4\in(r_3,b)$ such that $h^{(n-1)}(r_4)>0$ and $(-1)^{n-j}h^{(j)}(x)>0$ on (r_4,b), for $j=0,1,\dots,n-2$.

By Conditions (A) and (D),

$$\begin{aligned}h^{(n)}(b)&=f(b,p,p',\dots,p^{(n-1)})-f(b,q,q',\dots,q^{(n-1)})-g(b,z,z',\dots,z^{(n-1)})\\&>g(b,w,w',\dots,w^{(n-1)})-g(b,z,z',\dots,z^{(n-1)})\geq 0.\end{aligned}$$

Together with $h^{(n-1)}(b)=0$, we have that $h^{(n-1)}(x)<0$ on a left neighborhood of b. Since $h^{(n-1)}(r_4)>0$, there is $r_5\in(r_4,b)$ such that $h^{(n-1)}(r_5)=0$ and $h^{(n-1)}(x)<0$ on (r_5,b). Hence, $h^{(n)}(r_5)\leq 0$.

However, (note that $[r_5,b)\subset(r_4,b)\subset(r_2,b)$),

$$\begin{aligned}h^{(n)}(r_5)&=f(r_5,p,p',\dots,p^{(n-1)})-f(r_5,q,q',\dots,q^{(n-1)})-g(r_5,z,z',\dots,z^{(n-1)})\\&>g(r_5,w,w',\dots,w^{(n-1)})-g(r_5,z,z',\dots,z^{(n-1)})\geq 0,\end{aligned}$$

which is a contradiction. Hence, our assumption is false.

($\Rightarrow$) The necessity of the second part. Assume $w^{(n-1)}(b)<0$ and $w^{(n-2)}(b)>0$. Similar to the proof of the first part, we have $r_1\in(a,b)$ such that $w^{(n-2)}(r_1)=0$ and $w^{(n-2)}(x)>0$, for $x\in(r_1,b)$ and $(-1)^{n-j}w^{(j)}(x)>0$ on (r_1,b), for $j=0,1,\dots,n-2$.

Now let $0<\delta<\min\{\delta_1,w^{(n-2)}(b)\}$, and let z be a solution of (1.8)-(1.9) satisfying Condition (B) and $z^{(n-2)}(x)\geq 0$ on $[a,b]$. Let $h=w-z$. Then,

$$h^{(n)}=f(x,p,p',\dots,p^{(n-1)})-f(x,q,q',\dots,q^{(n-1)})-g(x,z,z',\dots,z^{(n-1)}),$$
$$h^{(i)}(b)=0,\ 0\leq i\leq n-3,\quad h^{(n-2)}(b)=w^{(n-2)}(b)-z^{(n-2)}(b)>0.$$

Note that $h^{(n-1)}(b)=w^{(n-1)}(b)-z^{(n-1)}(b)=w^{(n-1)}(b)<0$, $h^{(n-2)}(b)>0$ and $h^{(n-2)}(r_1)=w^{(n-2)}(r_1)-z^{(n-2)}(r_1)=-z^{(n-2)}(r_1)\leq 0$. So there exists $r_2\in[r_1,b)$ such that $h^{(n-2)}(r_2)=0$, $h^{(n-2)}(x)>0$, for $x\in(r_2,b)$. It follows that $(-1)^{n-j}h^{(j)}(x)>0$ on (r_2,b), for $j=0,1,\dots,n-2$.

By Rolle's Theorem and $h^{(n-1)}(b)<0$, there is $r_3\in(r_2,b)$ such that $h^{(n-1)}(r_3)=0$ and $h^{(n-1)}(x)<0$ on (r_3,b). Therefore, $h^{(n)}(r_3)\leq 0$, whereas by Conditions (A) and (D), (note that $[r_3,b)\subset(r_2,b)\subset(r_1,b)$),

$$h^{(n)}(r_3)=f(r_3,p,p',\dots,p^{(n-1)})-f(r_3,q,q',\dots,q^{(n-1)})-g(r_3,z,z',\dots,z^{(n-1)})$$

$$> g(r_3, w, w', \dots, w^{(n-1)}) - g(r_3, z, z', \dots, z^{(n-1)}) \geq 0,$$

which is a contradiction.

($\Leftarrow$) The sufficiency of the second part. We assume that $w^{(n-1)}(b) > 0$ and $w^{(n-2)}(b) < 0$. Then, we get the same situation as the proof of necessity with opposite signs of $w^{(n-1)}(b)$ and $w^{(n-2)}(b)$, which also implies a contradiction. Hence, the sufficiency is true. □

Lemma 2.2. *Suppose p and q are solutions of* (1.1) *on $[b, c]$ and $w = p-q$ satisfies the following boundary conditions:*

$$w^{(i)}(b) = 0,\ 0 \leq i \leq n-3, \quad \sum_{j=1}^{t} \beta_j w(\eta_j) - w(c) = 0.$$

Then, $w^{(n-2)}(b) = 0$ if and only if $w^{(n-1)}(b) = 0$. Also, $w^{(n-2)}(b) > 0$ if and only if $w^{(n-1)}(b) < 0$.

Proof. ($\Rightarrow$) The necessity of the first part. Assume $w^{(n-2)}(b) = 0$ and for contradiction, without loss of generality, we assume $w^{(n-1)}(b) > 0$.

By $\sum_{j=1}^{t} \beta_j w(\eta_j) - w(c) = 0$, there exists $r_1 \in (b, c)$ such that $w'(r_1) = 0$. By repeated applications of Rolle's Theorem, there exists $r_2 \in (b, r_1)$ such that $w^{(n-2)}(r_2) = 0$ and $w^{(n-2)}(x) > 0$ on (b, r_2). It follows that $w^{(j)}(x) > 0$ on (b, r_2), for $j = 0, 1, \dots, n-2$.

Let $0 < \delta < \min\{\delta_2, w^{(n-1)}(b)\}$. Then, by Condition (C), we have a solution z of (1.10)-(1.11) on $[b, c]$ such that $z^{(i)}(b) = 0$, $0 \leq i \leq n-2$, $z^{(n-1)}(b) = \delta$, and $z^{(n-2)}(x) \geq 0$ on $[b, c]$. Then, $z^{(j)}(x) \geq 0$, for $j = 0, 1, \dots, n-2$, on $[b, c]$.

Let $h = w - z$. Then,

$$\begin{gathered} h^{(n)} = f(x, p, p', \dots, p^{(n-1)}) - f(x, q, q', \dots, q^{(n-1)}) - g(x, z, z', \dots, z^{(n-1)}), \\ h^{(i)}(b) = 0,\ 0 \leq i \leq n-2,\ h^{(n-1)}(b) = w^{(n-1)}(b) - z^{(n-1)}(b) > 0. \end{gathered}$$

Note that $h^{(n-2)}(r_2) = w^{(n-2)}(r_2) - z^{(n-2)}(r_2) \leq 0$. Hence, there is $r_3 \in (b, r_2]$ such that $h^{(n-2)}(r_3) = 0$, $h^{(n-2)}(x) > 0$ on (b, r_3), and hence, $h^{(j)}(x) > 0$, for $j = 0, 1, \dots, n-2$ on (b, r_3). By $h^{(n-2)}(b) = 0$, Rolle's Theorem, and $h^{(n-1)}(b) > 0$, there exists $r_4 \in (b, r_3)$ such that $h^{(n-1)}(r_4) = 0$ and $h^{(n-1)}(x) > 0$ on (b, r_4). Hence, $h^{(n)}(r_4) \leq 0$, but by Conditions (A) and (D) and $(b, r_4] \subset (b, r_3) \subset (b, r_2)$, we have

$$\begin{aligned} h^{(n)}(r_4) &= f(r_4, p, p', \dots, p^{(n-1)}) - f(r_4, q, q', \dots, q^{(n-1)}) - g(r_4, z, z', \dots, z^{(n-1)}) \\ &> g(r_4, w, w', \dots, w^{(n-1)}) - g(r_4, z, z', \dots, z^{(n-1)}) \geq 0, \end{aligned}$$

which is a contradiction.

($\Leftarrow$) The sufficiency of the first part. Suppose $w^{(n-1)}(b) = 0$ and $w^{(n-2)}(b) > 0$. Similar to the above, we have $r_1 \in (b, c)$ such that $w^{(n-2)}(r_1) = 0$ and $w^{(j)}(x) > 0$ on (b, r_1) for $j = 0, 1, \dots, n-2$.

Let $0 < \delta < \min\{\delta_1, w^{(n-2)}(b)\}$. Then, by Condition (B), we have a solution z of (1.8)-(1.9) on $[b, c]$ such that $z^{(i)}(b) = 0$, $0 \leq i \leq n-1$, $i \neq n-2$, $z^{(n-2)}(b) = \delta$, and $z^{(n-2)}(x) \geq 0$ on $[b, c]$. Then, $z^{(j)}(x) \geq 0$, for $j = 0, 1, \dots, n-2$, on $[b, c]$.

Let $h = w - z$. Then,

$$\begin{gathered} h^{(n)} = f(x, p, p', \dots, p^{(n-1)}) - f(x, q, q', \dots, q^{(n-1)}) - g(x, z, z', \dots, z^{(n-1)}) \\ h^{(i)}(b) = 0,\ 0 \leq i \leq n-1,\ i \neq n-1, \quad h^{(n-2)}(b) = w^{(n-2)}(b) - z^{(n-2)}(b) > 0. \end{gathered}$$

Note that $h^{(n-2)}(r_1) = w^{(n-2)}(r_1) - z^{(n-2)}(r_1) = -z^{(n-2)}(r_1) \le 0$. So there is $r_2 \in (b, r_1]$ such that $h^{(n-2)}(r_2) = 0$ and $h^{(n-2)}(x) > 0$, for $x \in (b, r_2)$, and $h^{(j)}(x) > 0$ on (b, r_2), for $j = 0, 1, \dots, n-2$. By Rolle's Theorem, there is $r_3 \in (b, r_2)$ such that $h^{(n-1)}(r_3) < 0$.

Note that

$$\begin{aligned} h^{(n)}(b) &= f(b, p, p', \dots, p^{(n-1)}) - f(b, q, q', \dots, q^{(n-1)}) - g(b, z, z', \dots, z^{(n-1)}) \\ &> g(b, w, w', \dots, w^{(n-1)}) - g(b, z, z', \dots, z^{(n-1)}) \ge 0. \end{aligned}$$

Hence, there is $r_4 \in (b, r_3)$ such that $h^{(n-1)}(r_4) = 0$ and $h^{(n-1)}(x) > 0$ on (b, r_4), which implies $h^{(n)}(r_4) \le 0$. But by $(b, r_4] \subset (b, r_2) \subset (b, r_1)$ and Conditions (A) and (D), we have that

$$\begin{aligned} h^{(n)}(r_4) &= f(r_4, p, p', \dots, p^{(n-1)}) - f(r_4, q, q', \dots, q^{(n-1)}) - g(r_4, z, z', \dots, z^{(n-1)}) \\ &> g(r_4, w, w', \dots, w^{(n-1)}) - g(r_4, z, z', \dots, z^{(n-1)}) \ge 0, \end{aligned}$$

which is a contradiction.

($\Rightarrow$) The necessity of the second part. Suppose $w^{(n-2)}(b) > 0$ and $w^{(n-1)}(b) > 0$. Similar to the proof of the necessity of the first part, we also can get a contradiction. Hence, we omit the proof. Therefore, if $w^{(n-2)}(b) > 0$, then $w^{(n-1)}(b) < 0$.

($\Leftarrow$) The sufficiency of the second part. Suppose $w^{(n-1)}(b) < 0$. If $w^{(n-2)}(b) < 0$, then similar to the proof of necessity, we can get $w^{(n-1)}(b) > 0$, which is a contradiction. Hence, the sufficiency is also true. □

3. Existence and uniqueness of solutions to (1.1)-(1.2)

Before discussing existence and uniqueness for (1.1)-(1.2), we consider the uniqueness of solutions to each of the BVP's for (1.1) satisfying any of (1.3), (1.4), (1.5), or (1.6).

Lemma 3.1. *Let $y_1, y_2, \dots, y_n \in \mathbb{R}$ be given and assume Conditions* (A)–(D) *are satisfied. Then, given $m \in \mathbb{R}$, each of the BVP's for* (1.1) *satisfying any of conditions* (1.3)*,* (1.4)*,* (1.5)*, or* (1.6) *has at most one solution.*

Proof. The case of (1.1)-(1.3): Suppose there are two distinct solutions $p(x)$ and $q(x)$ for some $m \in \mathbb{R}$. Let $w = p - q$. Then, w satisfies

$$\begin{gathered} w^{(n)} = f(x, p, p', \dots, p^{(n-1)}) - f(x, q, q', \dots, q^{(n-1)}), \\ w(a) - \sum_{i=1}^{s} \alpha_i w(\xi_i) = 0, \quad w^{(i)}(b) = 0, \quad 0 \le i \le n-2. \end{gathered}$$

By Lemma 2.1, we can get that $w^{(n-1)}(b) = 0$. Then, by the uniqueness of solutions of IVP's for (1.1), we can conclude that $p \equiv q$ on $[a, b]$. Hence, (1.1)-(1.3) has at most one solution on $[a, b]$.

The other cases: By using similar ideas and Lemma 2.1 and Lemma 2.2, we resolve the other cases. □

Lemma 3.2. *Let $y_1, y_2, \dots, y_n \in \mathbb{R}$ be given. Assume Conditions (A)-(D) are satisfied. Then, the BVP* (1.1)-(1.2) *has at most one solution.*

Proof. We argue by contradiction. Suppose for some values $y_1, y_2, \dots, y_n \in \mathbb{R}$, there exist distinct solutions p and q of (1.1)-(1.2). Also, let $w = p - q$. Then, from Lemma 2.1 and Lemma 2.2, we get $w^{(n-2)}(b) \neq 0$, $w^{(n-1)}(b) \neq 0$.

Without loss of generality, we suppose $w^{(n-2)}(b) > 0$. Then, by Lemma 2.1, $w^{(n-1)}(b) > 0$. But by Lemma 2.2, $w^{(n-1)}(b) < 0$. This is a contradiction. Hence, $p \equiv q$ on $[a, c]$. □

Next, we show that solutions of (1.1) satisfying each of (1.3), (1.4), (1.5), or (1.6) respectively are monotone functions of m at b. For notation purposes, given any $m \in \mathbb{R}$, let $\alpha(x, m)$, $u(x, m)$, $\beta(x, m)$, $v(x, m)$ denote the solutions, when they exist, of the boundary value problems of (1.1) satisfying (1.3), (1.4), (1.5), or (1.6), respectively.

Lemma 3.3. *Suppose that* (A)–(D) *are satisfied and that for each $m \in \mathbb{R}$, there exist solutions of* (1.1) *satisfying each of the conditions* (1.3), (1.4), (1.5), (1.6), *respectively. Then, $\alpha^{(n-1)}(b, m)$ and $\beta^{(n-1)}(b, m)$ are, respectively, strictly increasing and strictly decreasing functions of m with ranges all of $\mathbb{R}$.*

Proof. The proof of $\{\alpha^{(n-1)}(b, m) | m \in \mathbb{R}\} = \mathbb{R}$ is the same as that in [4, Theorem 2.4]. We omit it here. □

Similarly, we obtain monotonicity conditions on $u^{(n-2)}(b, m)$ and $v^{(n-2)}(b, m)$.

Lemma 3.4. *Under the assumption of Lemma 3.3, the functions $u^{(n-2)}(b, m)$ and $v^{(n-2)}(b, m)$ are, respectively, strictly increasing and strictly decreasing functions of m, with ranges all $\mathbb{R}$.*

Finally, we arrive at our existence result for (1.1)-(1.2), which is obtained by solution matching.

Theorem 3.5. *Assume the hypotheses of Lemma 3.3. Then,* (1.1)-(1.2) *has a unique solution.*

Proof. We prove the existence from either Lemma 3.3 or Lemma 3.4. Making use of Lemma 3.4, there exists a unique $m_0 \in \mathbb{R}$ such that $u^{(n-2)}(b, m_0) = v^{(n-2)}(b, m_0)$. Then,

$$y(x) = \begin{cases} u(x, m_0), & a \le x \le b, \\ v(x, m_0), & b \le x \le c, \end{cases}$$

is a solution of (1.1)-(1.2) and by Lemma 3.2, $y(x)$ is the unique solution. □

4. Existence and uniqueness of solutions to (1.1)-(1.7)

We can obtain analogous results to those of Section 3 for (1.1)-(1.7) with $\tau, \sigma \in \{0, 1, \dots, n-3\}$ fixed. We obtain solutions to (1.1)-(1.7) by matching solutions satisfying the following types of boundary conditions:

$$y^{(\tau)}(a) - \sum_{i=1}^{s} \alpha_i y^{(\tau)}(\xi_i) = y_1, \quad y^{(i)}(b) = y_{i+2},\ 0 \le i \le n-3, \quad y^{(n-2)}(b) = m, \tag{4.1}$$

$$y^{(\tau)}(a) - \sum_{i=1}^{s} \alpha_i y^{(\tau)}(\xi_i) = y_1, \quad y^{(i)}(b) = y_{i+2},\ 0 \le i \le n-3, \quad y^{(n-1)}(b) = m, \tag{4.2}$$

$$y^{(i)}(b) = y_{i+2},\ 0 \le i \le n-3, \quad y^{(n-2)}(b) = m, \quad \sum_{j=1}^{t} \beta_j y^{(\sigma)}(\eta_j) - y^{(\sigma)}(c) = y_n, \tag{4.3}$$

$$y^{(i)}(b) = y_{i+2},\ 0 \le i \le n-3, \quad y^{(n-1)}(b) = m, \quad \sum_{j=1}^{t} \beta_j y^{(\sigma)}(\eta_j) - y^{(\sigma)}(c) = y_n, \tag{4.4}$$

where $m \in \mathbb{R}$, $a < \xi_1 < \xi_2 < \dots < \xi_s < b < \eta_1 < \eta_2 < \dots < \eta_t < c$, $s, t \in \mathbb{N}$, $\alpha_i > 0$ for $1 \le i \le s$, $\beta_j > 0$ for $1 \le j \le t$, $\sum_{i=1}^{s} \alpha_i = 1$, $\sum_{j=1}^{t} \beta_j = 1$ and $y_1, y_2, \dots, y_n \in \mathbb{R}$.

We omit the proofs of the following results since they are essentially the same as those presented in Section 2 with only small modifications.

Lemma 4.1. *Let $y_1, y_2, \dots, y_n \in \mathbb{R}$ be given and assume conditions* (A)–(D) *are satisfied. Then, given $m \in \mathbb{R}$, each of the BVP's for* (1.1) *satisfying any of conditions* (4.1)*,* (4.2)*,* (4.3)*, or* (4.4) *has at most one solution.*

Lemma 4.2. *Let $y_1, y_2, \dots, y_n \in \mathbb{R}$ be given and assume conditions (A)-(D) are satisfied. Then* (1.1)-(1.7) *has at most one solution.*

Now, given any $m \in \mathbb{R}$, let $\theta(x, m)$, $l(x, m)$, $\vartheta(x, m)$, $o(x, m)$ denote the solutions, when they exist, of the boundary value problems of (1.1) satisfying (4.1), (4.2), (4.3), (4.4), respectively.

Lemma 4.3. *Suppose that* (A)–(D) *are satisfied and that for each $m \in \mathbb{R}$, there exist solutions of* (1.1) *satisfying each of the conditions* (4.1)*,* (4.2)*,* (4.3)*,* (4.4)*. Then, $\theta^{(n-1)}(b, m)$ and $\vartheta^{(n-1)}(b, m)$ are respectively strictly increasing and strictly decreasing functions of m with ranges all of $\mathbb{R}$. Also, $l^{(n-2)}(b, m)$ and $o^{(n-2)}(b, m)$ are respectively strictly increasing and strictly decreasing functions of m with ranges all of $\mathbb{R}$.*

Theorem 4.4. *Assume the hypotheses of Lemma 4.3. Then* (1.1)-(1.7) *has a unique solution.*

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