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SOLUTION TO A NONLINEAR BLACK-SCHOLES EQUATION

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Abstract. Option pricing with transaction costs leads to a nonlinear Black-Scholes type equation where the nonlinear term reflects the presence of transaction costs. Under suitable conditions, we prove the existence of weak solutions in a bounded domain and we extend the results to the whole domain using a diagonal process.

1. Introduction

In a complete financial market without transaction costs, the celebrated Black-Scholes model (1973) [4] provides not only a rational option pricing formula, but also a hedging portfolio that replicates the contingent claim. In the Black-Scholes analysis, it is assumed that hedging takes place continuously, and therefore, in a market with proportional transaction costs, it tends to be infinitely expensive. So the requirement of replicating the value of the option continuously has to be relaxed. The first model in that direction was presented by Leland (1985) [14]. He assumes that the portfolio is rebalanced at a discrete time δt fixed and that the transaction costs are proportional to the value of the underlying; that is the costs incurred at each step is $\kappa|\nu|S$, where ν is the number of shares of the underlying bought ($\nu > 0$) or sold ($\nu < 0$) at price S and κ is a constant depending on individual investors. Leland derived an option price formula that is the Black-Scholes formula with an adjusted volatility

$$\hat{\sigma} = \sigma\Big(1 + \sqrt{\frac{2}{\pi}}\frac{\kappa}{\sigma\sqrt{\delta t}}\Big)^{1/2}.$$

Hoggard, Whalley and Wilmott [10] derived a model for portfolios of options in the presence of transaction costs in 1994. We will outline the steps that they followed.

Let $C(S,t)$ be the value of the option and Π be the value of the hedge portfolio. Assume that the value of the underlying follows the random walk

$$\delta S = \mu S\delta t + \sigma S\phi\delta t^{1/2},$$

where ϕ is drawn from a normal distribution, μ is a measure of the average rate of growth of the asset price also known as the drift, and σ is a measure of the

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fluctuation (risk) in the asset prices, it corresponds to the diffusion coefficient. Then the change in the value of the portfolio over the time step δt is given by

$$\delta\Pi = \sigma S\big(\frac{\partial C}{\partial S} - \Delta\big)\phi\delta t^{1/2} + \Big(\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}\phi^2 + \mu S\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} - \mu\Delta S\Big)\delta t - \kappa S|\nu|$$

Let us consider the delta hedging strategy; that is choose the number of assets held short at time t to be $\Delta = \frac{\partial C}{\partial S}(S,t)$. Therefore the number of assets to be traded after δt is given by

$$\nu = \frac{\partial C}{\partial S}(S+\delta S, t+\delta t) - \frac{\partial C}{\partial S}(S,t) \simeq \frac{\partial^2 C}{\partial S^2}\sigma S\phi\delta t^{1/2}.$$

So the expected transaction cost over a time step is

$$E[\kappa S|\nu|] = \sqrt{\frac{2}{\pi}}\kappa\sigma S^2\big|\frac{\partial^2 C}{\partial S^2}\big|\delta t^{1/2},$$

and the expected change in the value of the portfolio is

$$E(\delta\Pi) = \Big(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} - \kappa\sigma S^2\sqrt{\frac{2}{\pi\delta t}}\big|\frac{\partial^2 C}{\partial S^2}\big|\Big)\delta t.$$

If the holder of the option expects to make as much from his portfolio as from a bank account at a riskless interest rate r (no arbitrage), then

$$E(\delta\Pi) = r\big(C - S\frac{\partial C}{\partial S}\big)\delta t.$$

Hence the Hoggard, Whalley and Wilmott model for option pricing with transaction costs is given by

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial^2 S} + rS\frac{\partial C}{\partial S} - rC - \kappa\sigma S^2\sqrt{\frac{2}{\pi\delta t}}\big|\frac{\partial^2 C}{\partial S^2}\big| = 0, \tag{1.1}$$

for $(S,t) \in (0,\infty) \times (0,T)$, and the final condition

$$C(S,T) = \max(S-E,0), \quad S \in (0,\infty)$$

for European call options with strike price E.

Note that equation (1.1) contents the usual Black-Scholes terms with an additional nonlinear term modelling the presence of transaction costs. Setting

$$x = \log(S/E), \quad t = T - \tau/\frac{1}{2}\sigma^2, \quad C = EV(X,\tau),$$

equation (1.1) becomes

$$-\frac{\partial V}{\partial \tau} + \frac{\partial^2 V}{\partial x^2} + (k-1)\frac{\partial V}{\partial x} - kV = \kappa^*\big|\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x}\big|, \tag{1.2}$$

for $(x,\tau) \in \mathbb{R} \times (0,T^*)$, with the initial condition

$$V(x,0) = \max(e^x - 1, 0), \quad x \in \mathbb{R},$$

where $k = r/(\sigma^2/2)$, $\kappa^* = \kappa\sqrt{8/(\pi\sigma^2\delta t)}$ and $T^* = \sigma^2 T/2$. Next set

$$V(x,\tau) = e^x U(x,\tau).$$

Then (1.2) yields

$$-\frac{\partial U}{\partial \tau} + \frac{\partial^2 U}{\partial x^2} + (k+1)\frac{\partial U}{\partial x} = \kappa^*\big|\frac{\partial^2 U}{\partial x^2} + \frac{\partial U}{\partial x}\big|, \quad (x,\tau) \in \mathbb{R} \times (0,T^*) \tag{1.3}$$

with the initial condition

$$U(x,0) = \max(1 - e^{-x}, 0).$$

The previous discussion motivates us to consider the following problem that includes cost structures that go beyond proportional transaction costs

$$-\frac{\partial U}{\partial t} + \frac{\partial^2 U}{\partial x^2} + \alpha \frac{\partial U}{\partial x} = \beta F\big(\frac{\partial U}{\partial x}, \frac{\partial^2 U}{\partial x^2}\big), \quad (x,t) \in \mathbb{R} \times (0,T) \tag{1.4}$$

and

$$U(x,0) = U_0(x), \quad x \in \mathbb{R}, \tag{1.5}$$

where α and β are nonnegative constants.

Publications related to the above problem can be found in [2, 6, 16]. It is also worth noting that such problems can be solved using the techniques used in [18, 19]. The goal of this paper is to show that the theoretical problem (1.4)-(1.5) has a strong solution where the derivatives are understood in the distribution sense. We use the following assumptions:

- (H1) $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function,
- (H2) $F(p,q) \le |p| + |q|$,
- (H3) For $U \in H^2_{\rm loc}(\mathbb{R})$, $\frac{\partial}{\partial x} F(U, \frac{\partial U}{\partial x}) \in L^2(0,T; L^2_{\rm loc}(\mathbb{R}))$. Let $B_R = \{x \in \mathbb{R} : |x| < R\}$. Then if $w_k \to w$ in $L^2(0,T; H^1_0(B_R))$, then $\frac{\partial}{\partial x} F(w_k, \frac{\partial w_k}{\partial x}) \to \frac{\partial}{\partial x} F(w, \frac{\partial w}{\partial x})$ in $L^2(0,T; L^2(B_R))$.
- (H4) $U_0 \in H^1_{\rm loc}(\mathbb{R})$,
- (H5) $\beta < 1$.

This work is organized as follows: In section 2 we review some notions of functional analysis that will be used later, then in section 3 we solve a similar problem in a ball and finally in section 4 we construct a solution in the whole domain using a diagonal process.

2. Definitions and notion

Function Spaces. Let $\Omega \in \mathbb{R}^p$, $p \in \mathbb{N}$, be an open subset.

Definition 2.1. Suppose u and $v \in L^1_{\rm loc}(\Omega)$, and α is a multi-index constant. v is said to be the α^{th}-weak partial derivative of u, denoted $D^\alpha u = v$, if

$$\int_\Omega u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega v \phi \, dx$$

for any test function $\phi \in C_c^\infty$.

Note that a weak partial derivative of u, when it exists, is unique up to a set of measure zero. The Sobolev space

$$H^m(\Omega) = \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega), \text{ for any multiindex } \alpha \text{ with } |\alpha| \le m\},$$

where the derivatives are taken in the weak sense, is a Hilbert space when endowed with the inner product

$$(u,v)_{H^m(\Omega)} = \sum_{|\alpha| \le m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}.$$

Let $H^1_0(\Omega) = \{u \in H^1$ such that $u = 0$ on $\partial\Omega\}$ be the closure of C_c^∞ in $H^1(\Omega)$. Let the space $H^{-1}(\Omega)$ is the topological dual of $H^1_0(\Omega)$.

Let X be a Banach space and let T be a nonnegative integer. The space $L^2(0,T;X)$ consists of all measurable functions $u:(0,T)\to X$ with

$$\|u\|_{L^2(0,T;X)} := \Big(\int_0^T \|u(t)\|_X^2 dt\Big)^{1/2} < \infty.$$

Note that $L^2(0,T;X)$ is a Banach space endowed with the norm $\|u\|_{L^2(0,T;X)}$.

The space $C([0,T];X)$ consists of all continuous functions $u:[0,T]\to X$ with

$$\|u\|_{C([0,T];X)} := \max_{0\le t\le T} \|u(t)\|_X < \infty.$$

Note that $C([0,T];X)$ is a Banach space endowed with the norm $\|u\|_{C([0,T];X)}$.

Schaefer's fixed point theorem. Let X be a real Banach space.

Definition 2.2. A nonlinear mapping $A:X\to X$ is said to be compact if for each bounded sequence $\{u_k\}_{k=1}^\infty$, the sequence $\{A[u_k]\}_{k=1}^\infty$ is precompact; that is, there exists a subsequence $\{u_{k_j}\}_{j=1}^\infty$ such that $\{A[u_{k_j}]\}_{j=1}^\infty$ converges in X.

Theorem 2.3 (Schaefer's fixed point Theorem)**.** *Suppose $A:X\to X$ is a continuous and compact mapping. Assume further that the set*

$$\{u\in X \text{ such that } u=\lambda A[u] \text{ for some } 0\le\lambda\le 1\}$$

is bounded. Then A has a fixed point.

We will use the Schaefer's fixed point theorem in order to show the existence of a solution in a ball.

3. Solutions in bounded domains

Let $B_R=\{x\in\mathbb{R}:|x|<R\}$ be the open ball centered at the origin with radius R. Assume that U_0 is suitable cut into bounded functions defined on B_R and such that (H1)-(H5) are satisfied in $B_R\times[0,T]$. Set $w=\frac{\partial U}{\partial x}$ and consider an analogous problem in $B_R\times[0,T]$ with zero Dirichlet condition on the lateral boundary.

$$-\frac{\partial w}{\partial t}+\frac{\partial^2 w}{\partial x^2}+\alpha\frac{\partial w}{\partial x}=\beta\frac{\partial}{\partial x}F\big(w,\frac{\partial w}{\partial x}\big)\quad (x,t)\in B_R\times(0,T) \tag{3.1}$$

$$w(x,0)=w_0(x)\quad x\in B_R \tag{3.2}$$

$$w(x,t)=0,\quad (x,t)\in\partial B_R\times[0,T]. \tag{3.3}$$

Definition 3.1. A function w is said to be a weak solution of (3.1)-(3.3) if $w\in L^2(0,T;H_0^1(B_R))$, $\frac{\partial w}{\partial t}\in L^2(0,T;H^{-1}(B_R))$ and

$$\int_{B_R}\Big(\frac{\partial w}{\partial t}\phi+\frac{\partial w}{\partial x}\frac{\partial\phi}{\partial x}+\alpha w\frac{\partial\phi}{\partial x}\Big)\,dx=-\beta\int_{B_R}F\big(w,\frac{\partial w}{\partial x}\big)\frac{\partial\phi}{\partial x}\,dx \tag{3.4}$$

for all $\phi\in H_0^1(B_R)$.

Remark 3.2 ([5, Thm. 3, sec. 5.9.2])**.** If w belongs to $L^2(0,T;H_0^1(B_R))$ and $\frac{\partial w}{\partial t}$ belongs to $L^2(0,T;H^{-1}(B_R))$, then:

(i) $w\in C([0,T];L^2(B_R))$;
(ii) the mapping $t\to\|w(t)\|^2_{L^2(B_R)}$ is absolutely continuous with

$$\frac{d}{dt}\|w(t)\|^2_{L^2(B_R)}=2\int_{B_R}\frac{\partial w}{\partial t}w dt\quad \text{a.e. } 0\le t\le T \tag{3.5}$$

Theorem 3.3 (A priori estimate)**.** *If w is a weak solution of* (3.1)-(3.3)*, then there exists a positive constant C independent of w such that*

$$\max_{0\le t\le T}\|w(t)\|_{L^2(B_R)}+\|w\|_{L^2(0,T;H_0^1(B_R))}+\|\frac{\partial w}{\partial t}\|_{L^2(0,T;H^{-1}(B_R))}\le C\|w_0\|_{L^2(B_R)}.$$

Proof. Choosing $w(t)\in H_0^1(B_R)$ as the test function in (3.4), we obtain

$$\int_{B_R}\Big(\frac{\partial w}{\partial t}w+\frac{\partial w}{\partial x}\frac{\partial w}{\partial x}+\alpha w\frac{\partial w}{\partial x}\Big)\,dx=-\beta\int_{B_R}F\big(w,\frac{\partial w}{\partial x}\big)\frac{\partial w}{\partial x}\,dx.$$

Therefore, by (3.5),

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(B_R)}+\|\frac{\partial w}{\partial x}\|^2_{L^2(B_R)}+\frac{1}{2}\alpha\int_{B_R}\frac{\partial w^2}{\partial x}\,dx=-\beta\int_{B_R}F\big(w,\frac{\partial w}{\partial x}\big)\frac{\partial w}{\partial x}\,dx$$

and from (3.3),

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(B_R)}+\|\frac{\partial w}{\partial x}\|^2_{L^2(B_R)}\le\beta\int_{B_R}\big|F\big(w,\frac{\partial w}{\partial x}\big)\frac{\partial w}{\partial x}\big|\,dx.$$

Using (H2), we obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(B_R)}+\|\frac{\partial w}{\partial x}\|^2_{L^2(B_R)}\le\beta\int_{B_R}\Big(|w||\frac{\partial w}{\partial x}|+|\frac{\partial w}{\partial x}|^2\Big)\,dx.$$

By the Cauchy-Schwartz inequality with $\epsilon>0$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(B_R)}+\|\frac{\partial w}{\partial x}\|^2_{L^2(B_R)}\\&\le\beta\Big(\int_{B_R}|\frac{\partial w}{\partial x}|^2\,dx+\epsilon\int_{B_R}|\frac{\partial w}{\partial x}|^2\,dx+\frac{1}{4\epsilon}\int_{B_R}|w|^2\,dx\Big).\end{aligned}\tag{3.6}$$

Since $\beta<1$, choosing $\epsilon\ll1$, yields

$$\frac{d}{dt}\|w(t)\|^2_{L^2(B_R)}+C_1\|w\|^2_{H_0^1(B_R)}\le C_2\|w\|^2_{L^2(B_R)}\tag{3.7}$$

for a.e. $0\le t\le T$, and appropriate positive constants C_1 and C_2.

Next we write $\eta(t):=\|w(t)\|^2_{L^2(B_R)}$, then by (3.7),

$$\eta'(t)\le C_2\eta(t),\ \text{for a.e.}0\le t\le T.$$

The differential form of Gronwall inequality implies

$$\eta(t)\le e^{C_2t}\eta(0)\quad\text{a.e.}0\le t\le T.$$

Since $\eta(0)=\|w(0)\|^2_{L^2(B_R)}=\|w_0\|^2_{L^2(B_R)}$,

$$\|w(t)\|^2_{L^2(B_R)}\le e^{C_2t}\|w_0\|^2_{L^2(B_R)}.$$

Hence

$$\max_{0\le t\le T}\|w(t)\|_{L^2(B_R)}\le C_{11}\|w_0\|^2_{L^2(B_R)},\tag{3.8}$$

where C_{11} is some constant. To obtain a bound for the second term, we consider (3.7), and integrate from 0 to T, obtaining

$$\|w(T)\|^2_{L^2(B_R)}-\|w_0\|^2_{L^2(B_R)}+C_1\int_0^T\|w\|^2_{H_0^1(B_R)}dt\le C_2\int_0^T\|w\|^2_{L^2(B_R)}dt.$$

Therefore,

$$\|w(T)\|^2_{L^2(B_R)}+C_1\int_0^T\|w\|^2_{H_0^1(B_R)}dt\le C_2\int_0^T\|w\|^2_{L^2(B_R)}dt+\|w_0\|^2_{L^2(B_R)}.$$

Hence

$$C_1 \int_0^T \|w\|^2_{H_0^1(B_R)} dt \le C_2 \int_0^T \|w\|^2_{L^2(B_R)} dt + \|w_0\|^2_{L^2(B_R)}.$$

Using (3.8) we conclude that

$$\|w\|_{L^2(0,T;H_0^1(B_R))} \le C_{22} \|w_0\|_{L^2(B_R)}, \tag{3.9}$$

where C_{22} is a constant. Finally, to obtain a bound for the third term, by fixing $v \in H_0^1(B_R)$ with $\|v\|_{H_0^1(B_R)} \le 1$. By (3.4), we have

$$\int_{B_R} \frac{\partial w}{\partial t} v\, dx + \int_{B_R} \Big(\frac{\partial w}{\partial x}\frac{\partial v}{\partial x} + \alpha w \frac{\partial v}{\partial x}\Big)\, dx = -\beta \int_{B_R} F\big(w, \frac{\partial w}{\partial x}\big)\frac{\partial v}{\partial x}\, dx.$$

Thus

$$\Big|\int_{B_R} \frac{\partial w}{\partial t} v\, dx\Big| \le \Big|\int_{B_R} \Big(\frac{\partial w}{\partial x}\frac{\partial v}{\partial x} + \alpha w \frac{\partial v}{\partial x}\Big)\, dx\Big| + \beta \Big|\int_{B_R} F\big(w, \frac{\partial w}{\partial x}\big)\frac{\partial v}{\partial x}\, dx.\Big|$$

By Hölder inequality,

$$\begin{aligned} \Big|\int_{B_R} \frac{\partial w}{\partial t} v\, dx\Big| &\le \Big(\int_{B_R} \big|\frac{\partial w}{\partial x}\big|^2\, dx\Big)^{1/2} \Big(\int_{B_R} \big|\frac{\partial v}{\partial x}\big|^2\, dx\Big)^{1/2} \\ &\quad + \alpha \Big(\int_{B_R} |w|^2\, dx\Big)^{1/2} \Big(\int_{B_R} \big|\frac{\partial v}{\partial x}\big|^2\, dx\Big)^{1/2} \\ &\quad + \Big(\int_{B_R} \big|F\big(w, \frac{\partial w}{\partial x}\big)\big|^2\, dx\Big)^{1/2} \Big(\int_{B_R} \big|\frac{\partial v}{\partial x}\big|^2\, dx\Big)^{1/2}. \end{aligned}$$

Since $\|v\|_{H_0^1(B_R)} \le 1$, using (H2) and the Poincaré inequality we deduce

$$\Big|\int_{B_R} \frac{\partial w}{\partial t} v\, dx\Big| \le C_{33} \|w(t)\|_{H_0^1(B_R)},$$

where C_{33} is some constant. So

$$\|\frac{\partial w}{\partial t}(t)\|_{H^{-1}(B_R)} \le C_{33} \|w(t)\|_{H_0^1(B_R)}.$$

Therefore,

$$\int_0^T \|\frac{\partial w}{\partial t}(t)\|^2_{H^{-1}(B_R)} dt \le C_{33} \int_0^T \|w(t)\|^2_{H_0^1(B_R)} dt = C_{33} \|w\|^2_{L^2(0,T;H_0^1(B_R))}.$$

Then (3.9) implies

$$\|\frac{\partial w}{\partial t}\|_{L^2(0,T;H^{-1}(B_R))} \le C_{33} \|w_0\|^2_{L^2(B_R)}. \tag{3.10}$$

Take $C = \max\{C_{11}, C_{22}, C_{33}\}$ to obtain the required result in the Theorem. □

Before proving the existence theorem in a ball, we state the following energy estimate from the theory of linear parabolic partial differential equations.

Lemma 3.4 ([5, Theorem 2 page 354]). *Consider the problem*

$$\begin{gathered} \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} - \alpha \frac{\partial w}{\partial x} = f(x,t) \quad \text{in } B_R \times (0,T) \\ w(x,0) = w_0(x) \quad \text{on } B_R \times \{0\} \\ w(x,t) = 0 \quad \text{on } \partial B_R \times [0,T] \end{gathered} \tag{3.11}$$

with $f \in L^2(0,T;L^2(B_R))$ and $w_0 \in L^2(B_R)$.

Then there exists a unique $u \in L^2(0,T;H_0^1(B_R)) \cap C([0,T];L^2(B_R))$ *solution of* (3.11) *that satisfies*

$$\begin{aligned} &\max_{0\le t\le T} \|u(t)\|_{L^2(B_R)} + \|u\|_{L^2(0,T;H_0^1(B_R))} + \|\frac{\partial u}{\partial t}\|_{L^2(0,T;H^{-1}(B_R))} \\ &\le C\Big(\|f\|_{L^2(0,T;L^2(B_R))} + \|w_0\|_{L^2(B_R)}\Big), \end{aligned} \tag{3.12}$$

where C is a positive constant depending only on R and T.

We need another Lemma that follows directly from [5, Theorem 5, page 360].

Lemma 3.5 (Improved regularity)**.** *Consider the problem*

$$\begin{gathered} \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} - \alpha\frac{\partial w}{\partial x} = f(x,t) \quad \text{in } B_R \times (0,T) \\ w(x,0) = w_0(x) \quad \text{on } B_R \times \{0\} \\ w(x,t) = 0 \quad \text{on } \partial B_R \times [0,T] \end{gathered}$$

with $f \in L^2(0,T;L^2(B_R))$ and $w_0 \in H_0^1(B_R)$. Then this problem has a unique weak solution $u \in L^2(0,T;H_0^1(B_R)) \cap C([0,T];L^2(B_R))$, with $\frac{\partial u}{\partial t} \in L^2(0,T;H^{-1}(B_R))$. Moreover,

$$u \in L^2(0,T;H^2(B_R)) \cap L^\infty(0,T;H_0^1(B_R)), \quad \frac{\partial u}{\partial t} \in L^2(0,T;L^2(B_R)).$$

We also have the estimate

$$\begin{aligned} &\operatorname{ess\,sup}_{0\le t\le T} \|u(t)\|_{H_0^1(B_R)} + \|u\|_{L^2(0,T;H^2(B_R))} + \|\frac{\partial u}{\partial t}\|_{L^2(0,T;L^2(B_R))} \\ &\le C''\Big(\|f\|_{L^2(0,T;L^2(B_R))} + \|w_0\|_{H_0^1(B_R)}\Big), \end{aligned} \tag{3.13}$$

where C'' is a positive constant depending only on B_R, T and the coefficients of the operator L.

Next we show the existence of a solution in a ball by using the Schaefer's fixed point Theorem.

Theorem 3.6. *If* (H1)–(H5) *are satisfied, then* (3.1)-(3.3) *has a weak solution* $w \in L^2(0,T;H_0^1(B_R)) \cap C([0,T];L^2(B_R))$.

Proof. Given $w \in L^2(0,T;H_0^1(B_R))$, set $f_w(x,t) := \beta\frac{\partial}{\partial x}F\big(w, \frac{\partial w}{\partial x}\big)$. By (H3), $f_w \in L^2(0,T;L^2(B_R))$. From Lemma 3.4 there exists a unique $v \in L^2(0,T;H_0^1(B_R)) \cap C([0,T];L^2(B_R))$ solution of

$$\begin{gathered} \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} - \alpha\frac{\partial v}{\partial x} = f(x,t) \quad \text{in } B_R \times (0,T) \\ v(x,0) = v_0(x) \quad \text{on } B_R \times \{0\} \\ v(x,t) = 0 \quad \text{on } \partial B_R \times [0,T] \end{gathered} \tag{3.14}$$

Define the mapping

$$A : L^2(0,T;H_0^1(B_R)) \to L^2(0,T;H_0^1(B_R))$$

by $w \mapsto A(w) = v$, where v is derived from w via (3.14).

We now show that the mapping A is continuous and compact. We first prove the continuity. Let $\{w_k\}_k \subset L^2(0,T;H_0^1(B_R))$ be a sequence such that

$$w_k \to w \quad \text{in } L^2(0,T;H_0^1(B_R)). \tag{3.15}$$

By the *improved regularity* (3.13), there exists a constant C'', independent of $\{w_k\}_k$ such that

$$\|v_k\|_{L^2(0,T;H^2(B_R))} \leq C''\Big(\|f_{w_k}\|_{L^2(0,T;L^2(B_R))} + \|w_0\|_{H_0^1(B_R)}\Big), \tag{3.16}$$

for $v_k = A[w_k]$, $k = 1, 2, \dots$. By (H3) as $w_k \to w$ in $L^2(0,T;H_0^1(B_R))$, we must have $f_{w_k}(x,t) \to f_w(x,t)$ in $L^2(0,T;L^2(B_R))$. Therefore $\|f_{w_k}(x,t)\|_{L^2(0,T;L^2(B_R))} \to \|f_w(x,t)\|_{L^2(0,T;L^2(B_R))}$. Then the sequence $\{\|f_{w_k}\|_{L^2(0,T;L^2(B_R))}\}_k$ is bounded and

$$\sup_k \|f_{w_k}\|_{L^2(0,T;L^2(B_R))} \leq C''', \tag{3.17}$$

for a constant C'''. Thus by (3.16) and (3.17) the sequence $\{v_k\}_k$ is bounded uniformly in $L^2(0,T;H^2(B_R))$. Similarly it can be proved that $\{\frac{\partial v_k}{\partial t}\}_k$ is uniformly bounded in $L^2(0,T;H^{-1}(B_R))$. Thus by Rellich's Theorem (see [7]) there exists a subsequence $\{v_{k_j}\}_j \in L^2(0,T;H_0^1(B_R))$ and a function $v \in L^2(0,T;H_0^1(B_R))$ with

$$v_{k_j} \to v \quad \text{in } L^2(0,T;H_0^1(B_R)), \text{ as } j \to \infty. \tag{3.18}$$

Therefore,

$$\int_{B_R} \Big(\frac{\partial v_{k_j}}{\partial t}\phi + \frac{\partial v_{k_j}}{\partial x}\frac{\partial \phi}{\partial x} + \alpha v_{k_j}\frac{\partial \phi}{\partial x}\Big)\,dx = \int_{B_R} f_{w_{k_j}}(x,t)\phi\,dx$$

for each $\phi \in H_0^1(B_R)$. Using (3.15) and (3.18) we see that

$$\int_{B_R} \Big(\frac{\partial v}{\partial t}\phi + \frac{\partial v}{\partial x}\frac{\partial \phi}{\partial x} + \alpha v\frac{\partial \phi}{\partial x}\Big)\,dx = \int_{B_R} f_w(x,t)\phi\,dx.$$

Thus $v = A[w]$. Therefore

$$A[w_k] \to A[w] \text{ in } L^2(0,T;H_0^1(B_R)).$$

The compactness result follows from similar arguments.

To apply Schaefer's fixed point Theorem in $L^2(0,T;H_0^1(B_R))$ we need to show that the set $\{w \in L^2(0,T;H_0^1(B_R)) : w = \lambda A[w] \text{ for some } 0 \leq \lambda \leq 1\}$ is bounded. This follows directly from the a priori estimate (Theorem 3.3) with $\lambda = 1$. $\square$

Remark 3.7. Theorem 3.6 shows that $w = \frac{\partial u}{\partial x} \in L^2(0,T;H_0^1(B_R))$ solves problem (3.1)-(3.3); so $u \in L^2(0,T;H_0^1(B_R) \cap H^2(B_R))$ and is a strong solution of problem (1.4)-(1.5) in the bounded domain $B_R \times [0,T]$ with zero Dirichlet condition on the lateral boundary of the domain.

4. Construction of the solution in the whole domain

The next step is to construct a solution of (3.1)-(3.3) in the whole real line. To do that, we approximate the real line by

$$\mathbb{R} = \cup_{N\in\mathbb{N}} B_N = \lim_{N\to\infty} B_N$$

where $B_N = \{x \in \mathbb{R} : |x| < N\}$. We also approximate w_0 by a sequence of bounded function w_{0N} defined in B_N such that $|w_{0N}| \leq |w_0|$ and $w_{0N} \to w_0$ in $L^2_{\rm loc}(\mathbb{R})$. For $N \in \mathbb{N}$, there exists $w_N \in L^2(0,T;H_0^1(B_N)) \cap C([0,T];L^2(B_N))$ with $\frac{\partial w_N}{\partial t} \in L^2(0,T;H^{-1}(B_N))$, weak solution of

$$-\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + \alpha\frac{\partial w}{\partial x} = \beta\frac{\partial}{\partial x}F\big(w, \frac{\partial w}{\partial x}\big) \quad (x,t) \in B_N \times (0,T) \tag{4.1}$$

$$w(x,0) = w_{0N}(x) \quad x \in B_N \tag{4.2}$$

$$w(x,t)=0, \quad (x,t)\in\partial B_N\times[0,T]. \tag{4.3}$$

For any given $\rho>0$, the following sequences are bounded uniformly for $N>2\rho$:

$$\begin{gathered}\{w_N\}_N \quad \text{in } L^2(0,T;H_0^1(B_\rho)),\\ \{\frac{\partial w_N}{\partial t}\}_N \quad \text{in } L^2(0,T;H^{-1}(B_\rho))\end{gathered}$$

Since these spaces are compactly embedded in $L^2((B_\rho\times(0,T))$ then the sequence $\{w_N\}_N$ is relatively compact in $L^2((B_\rho\times(0,T))$. By a standard diagonal process we may select a subsequence also denoted $\{w_N\}_N$ so that

$$\begin{gathered}w_N\to w \quad \text{a.e. in } L^2(0,T;L^2_{\rm loc}(\mathbb{R})),\\ w_N\to w \quad \text{weakly in } L^2(0,T;H^1_{\rm loc}(\mathbb{R})).\end{gathered}$$

Since F is continuous, passing to the limit in (3.4) yields that w is a weak solution of (3.1)-(3.3) in $\mathbb{R}$.

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