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EXISTENCE OF NON-NEGATIVE SOLUTIONS FOR PREDATOR-PREY ELLIPTIC SYSTEMS WITH A SIGN-CHANGING NONLINEARITY

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Abstract. By the method of monotone iteration and Schauder fixed point theorem, we prove the existence of non-negative solutions to the system

$$\begin{gathered}-\Delta u = \lambda a(x) f(v) \quad \text{in } \Omega,\\ -\Delta v = \lambda b(x) g(u) \quad \text{in } \Omega,\\ u = v = 0 \quad \text{on } \partial\Omega,\end{gathered}$$

for λ sufficiently small, where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and λ is a positive parameter. In this work, we allow the sign changing nature of a and b with $a(x)b(x) \leq 0, \forall x \in \bar{\Omega}$.

1. Introduction

Let us consider the system

$$\begin{gathered}-\Delta u = \lambda a(x) f(v) \quad \text{in } \Omega,\\ -\Delta v = \lambda b(x) g(u) \quad \text{in } \Omega,\\ u = v = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, λ is a positive parameter. Let $a, b \in L^\infty(\Omega)$ be sign-changing potentials.

In this note, we are interested establishing the existence of a non-negative solution to (1.1) in predator-prey case; i.e., when $ab \leq 0$ a.e. in $\bar{\Omega}$. This case is more delicate as it deals with the predator-prey models arising in the mathematical biology; see [2, 9] and the references cited therein for more details. The present work concludes our earlier study started in [10].

In the recent years, a good amount of research is established for reaction-diffusion systems. Reaction-diffusion systems model many phenomena in biology, ecology, combustion theory, chemical reactions, population dynamics etc. A typical example of these model is

$$\begin{gathered}-\Delta u = f(v) \quad \text{in } \Omega,\\ -\Delta v = g(u) \quad \text{in } \Omega,\\ u = v = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.2}$$

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where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. The existence of the positive solution of (1.2) is established by de Figueiredo et al, [4] in an Orlicz space setting for $N \geq 3$. Hulshof and Vorst [8] established the existence of positive solution to (1.2) by variational technique for $N \geq 1$. Dalmasso [3] established the existence of positive solutions to (1.2) by Schauder fixed point theorem. For the existence and non-existence of positive solutions of (1.2) in ball; we refer the reader to [5] for $N \geq 4$. Recently, by the method of sub and supersolutions and Schauder fixed point theorem, Hai and Shivaji [7] established the existence of a positive solution to the system

$$\begin{gathered} -\Delta u = \lambda f(v) \quad \text{in } \Omega, \\ -\Delta v = \lambda g(u) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

for λ large. So it is natural to explore the existence questions of (1.3) for λ small with sign-changing nonlinearity. In case of a single equation

$$-\Delta u = \lambda a(x) f(u) \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

Hai [6] obtained the existence of a positive solution to (1.4) by Leray-Schauder fixed point theorem without assuming the monotonicity assumption on f, but assumed the continuity of a. Hai obtained an explicit nonnegative lower bound for solutions of (1.4) in product of $\lambda, f(0)$ and $\int_\Omega G(x,y)a^+(y)dy$. In order to obtain nonnegative solutions to the system, which have explicit lower and upper bounds both, the approach of [6] seems difficult due to $ab \leq 0$ a.e. in Ω. Using monotone iterations as in [1], we prove the existence of nonnegative solutions to the system, which have both lower and upper bounds explicitly.

In the present study, we assume the following hypotheses on the nonlinearity and weights:

(H1) $f, g : [0,\infty) \to [0,\infty)$ which are continuous and non-decreasing on $[0,\infty)$.

(H2) There exists $\mu_1 > 0$ such that
$$\int_\Omega G(x,y)a_+(y)dy \geq (1+\mu_1)\int_\Omega G(x,y)a_-(y)dy, \quad \forall x \in \Omega.$$

(H3) There exists $\mu_2 > 0$ such that
$$\int_\Omega G(x,y)b_+(y)dy \geq (1+\mu_2)\int_\Omega G(x,y)b_-(y)dy \quad \forall x \in \Omega,$$

where $G(x,y)$ is the Green's function of $-\Delta$ associated with the Dirichlet boundary condition.

In our earlier study [10], we assumed that $a(x)b(x) \geq 0$ a.e. $x \in \bar{\Omega}$ and proved the existence of a non-negative solution to (1.1). In fact, the main result which is proved in [10] is the following:

> Let $a(x)b(x) \geq 0$ a.e. $x \in \bar{\Omega}$. If $f(0) > 0$, $g(0) > 0$, f and g both are nondecreasing and continuous functions. Suppose (H2), (H3) hold. Then there exists $\lambda^* > 0$ depending on f, g, a, b, μ_i, $i = 1, 2$ such that (1.1) has a non-negative solution for $0 \leq \lambda \leq \lambda^*$.

We point out that the existence of a non-negative solution to (1.1) was left open in case when $a(x)b(x) \leq 0$ a.e. $x \in \bar{\Omega}$, due to the inapplicability of [10, Prop. 2.4].

In this note, we obtain a proposition similar to [10, Prop. 2.4] and by an application of Schauder fixed point theorem, we prove the existence of a non-negative solution to (1.1) in the case $a(x)b(x) \leq 0$ a.e. $x \in \bar{\Omega}$ for λ sufficiently small.

We organize this note as follows: Section 2 deals with some propositions which have been used in the main result. We prove the main theorem in Section 3. In Section 4, we establish the existence of a non-negative solution for a class of $n \times n$ systems with sign-changing nonlinearity. Now we state and prove the main theorem.

Theorem 1.1. *Let $ab \leq 0$ a.e. in $\bar{\Omega}$. If $f(0) > 0$, $g(0) > 0f$, and g both are nondecreasing and continuous functions. Suppose* (H2), (H3) *hold. Then there exists $\lambda^* > 0$ depending on f, g, a, b, μ_i, $i = 1, 2$ such that* (1.1) *has a non-negative solution for $0 \leq \lambda \leq \lambda^*$.*

2. Main Result

Let us assume throughout that $ab \leq 0$ a.e. in $\bar{\Omega}$. Let $A : C(\overline{\Omega}) \times C(\overline{\Omega}) \to C(\overline{\Omega}) \times C(\overline{\Omega})$ be defined by

$$A(u,v)(x) = \Big(\lambda \int_\Omega G(x,y)a(y)f(v(y))dy, \lambda \int_\Omega G(x,y)b(y)g(u(y))dy\Big).$$

Let $\Omega_1 = \{x \in \Omega : a(x) \geq 0, b(x) < 0\}$ and $\Omega_2 = \{x \in \Omega : a(x) < 0, b(x) \geq 0\}$. In this case A can be written as

$$\begin{aligned} A(\phi,\psi)(x) = \Big(&\lambda \int_{\Omega_1} G(x,y)a_+(y)f(\psi(y))dy - \lambda \int_{\Omega_2} G(x,y)a_-(y)f(\psi(y))dy, \\ &\lambda \int_{\Omega_2} G(x,y)b_+(y)g(\phi(y))dy - \lambda \int_{\Omega_1} G(x,y)b_-(y)g(\phi(y))dy\Big). \end{aligned}$$

Also, it can be rewritten as

$$A(\phi,\psi)(x) = \Big(A_1\psi(x) - A_2\psi(x), B_2\phi(x) - B_1\phi(x)\Big),$$

where A_1, B_1 act on $C^+(\overline{\Omega} \cap \Omega_1)$ and A_2, B_2 act on $C^+(\overline{\Omega} \cap \Omega_2)$ and A_i and B_i, $i = 1, 2$ are monotone by (H1).

One can see easily that our problem is exactly to find out the fixed point of the integral operator defined by

$$A(\phi,\psi)(x) = \Big(A_1\psi(x) - A_2\psi(x), B_2\phi(x) - B_1\phi(x)\Big).$$

The difficulty here is that in general, A does not leave invariant the cone of non-negative continuous functions $C^+(\overline{\Omega}) \times C^+(\overline{\Omega})$ invariant. In the ensuing proposition, we construct a closed, convex set $\Gamma \subset C^+(\overline{\Omega}) \times C^+(\overline{\Omega})$, which is left invariant under A.

Indeed, the next proposition is already given in [10], but for the sake of completeness, we repeat the proof here.

Proposition 2.1. *Assume that there exist $\Phi = (\phi^{(1)}, \phi^{(2)})$, $\Psi = (\psi^{(1)}, \psi^{(2)})$ in $C^+(\overline{\Omega}) \times C^+(\overline{\Omega})$ such that $\Phi \leq \Psi$ and*

$$\begin{aligned} (\phi^{(1)}, \phi^{(2)}) &= \Big(A_1\phi^{(2)} - A_2\psi^{(2)}, B_2\phi^{(1)} - B_1\psi^{(1)}\Big), \\ (\psi^{(1)}, \psi^{(2)}) &= \Big(A_1\psi^{(2)} - A_2\phi^{(2)}, B_2\psi^{(1)} - B_1\phi^{(1)}\Big). \end{aligned}$$

Then under the hypothesis (H1),

$$\begin{aligned}\Gamma = \big\{(u,v) \in C(\overline{\Omega}) \times C(\overline{\Omega}) : (0,0) \le (\phi^{(1)}(x), \phi^{(2)}(x))\\ \le (u(x), v(x)) \le (\psi^{(1)}(x), \psi^{(2)}(x)), x \in \overline{\Omega}\big\}\end{aligned}$$

is a closed, convex set and Γ *is invariant under* A.

Proof. It is easy to observe that Γ is a closed, convex set in $C(\overline{\Omega}) \times C(\overline{\Omega})$. Γ is invariant under A for $A(u,v) = (A_1 v - A_2 v, B_2 u - B_1 u)$ where

$$\begin{gathered}A_1 v - A_2 v \le A_1 \psi^{(2)} - A_2 \phi^{(2)} = \psi^{(1)},\\ A_1 v - A_2 v \ge A_1 \phi^{(2)} - A_2 \psi^{(2)} = \phi^{(1)},\\ B_2 u - B_1 u \le B_2 \psi^{(1)} - B_1 \phi^{(1)} = \psi^{(2)},\\ B_2 u - B_1 u \ge B_2 \phi^{(1)} - B_1 \psi^{(1)} = \phi^{(2)}.\end{gathered}$$

This implies

$$(\phi^{(1)}, \phi^{(2)}) \le (A_1 v - A_2 v, B_2 u - B_1 u) \le (\psi^{(1)}, \psi^{(2)}).$$

Therefore, Γ is invariant under A. $\square$

Proposition 2.2. *Let* f *and* g *satisfy* (H1). *Then under the assumptions of Proposition 2.1,* A *has a fixed point on* Γ.

Proof. In view of Proposition 2.1, it is easy to see that A is a completely continuous operator on Γ. By Schauder fixed point theorem A has a fixed point on Γ. $\square$

The fixed point of A is the solution of (1.1). Now we construct such Φ and Ψ (introduced in Proposition 2.1) by the iteration introduced by Cac et al, [1].

Proposition 2.3. *Suppose we have bounded measurable functions* $\Phi_0 = (\phi_0^{(1)}, \phi_0^{(2)})$, $\Psi_0 = (\psi_0^{(1)}, \psi_0^{(2)})$ *on* Ω *such that*

(i) $(0,0) = \mathbf{0} \le \Phi_0 \le \Psi_0$ *on* Ω_1*;* $\mathbf{0} \le \Psi_0 \le \Phi_0$ *on* Ω_2*;*

(ii) $(A_1\psi_0^{(2)} - A_2\psi_0^{(2)}, B_2\phi_0^{(1)} - B_1\phi_0^{(1)}) \le (\psi_0^{(1)}, \psi_0^{(2)})$ *on* Ω_1*;*
$(A_1\psi_0^{(2)} - A_2\psi_0^{(2)}, B_2\phi_0^{(1)} - B_1\phi_0^{(1)}) \le (\phi_0^{(1)}, \phi_0^{(2)})$ *on* Ω_2.

(iii) $(A_1\phi_0^{(2)} - A_2\phi_0^{(2)}, B_2\psi_0^{(1)} - B_1\psi_0^{(1)}) \ge (\phi_0^{(1)}, \phi_0^{(2)})$ *on* Ω_1*;*
$(A_1\phi_0^{(2)} - A_2\phi_0^{(2)}, B_2\psi_0^{(1)} - B_1\psi_0^{(1)}) \ge (\psi_0^{(1)}, \psi_0^{(2)})$ *on* Ω_2.

Define

$$\Phi_1 = \begin{cases} (A_1\phi_0^{(2)} - A_2\phi_0^{(2)}, B_2\psi_0^{(1)} - B_1\psi_0^{(1)}) & \text{on } \Omega_1,\\ (A_1\psi_0^{(2)} - A_2\psi_0^{(2)}, B_2\phi_0^{(1)} - B_1\phi_0^{(1)}) & \text{on } \Omega_2.\end{cases}$$

$$\Psi_1 = \begin{cases} (A_1\psi_0^{(2)} - A_2\psi_0^{(2)}, B_2\phi_0^{(1)} - B_1\phi_0^{(1)}) & \text{on } \Omega_2,\\ (A_1\phi_0^{(2)} - A_2\phi_0^{(2)}, B_2\psi_0^{(1)} - B_1\psi_0^{(1)}) & \text{on } \Omega_2.\end{cases}$$

Then under the hypothesis (H1), Φ_1 *and* Ψ_1 *also satisfy* (i)–(iii).

Proof. Note that (i) implies

$$(A_1\phi_0^{(2)} - A_2\phi_0^{(2)}, B_2\psi_0^{(1)} - B_1\psi_0^{(1)}) \le (A_1\psi_0^{(2)} - A_2\psi_0^{(2)}, B_2\phi_0^{(1)} - B_1\phi_0^{(1)}).$$

This shows that Φ_1 and Ψ_1 satisfy (i). Now we claim that Φ_1 and Ψ_1 satisfy (ii) and (iii). For this,

$$(A_1\psi_1^{(2)} - A_2\psi_1^{(2)}, B_2\phi_1^{(1)} - B_1\phi_1^{(1)})$$

$$\begin{aligned}
&= \Big(A_1(B_2\phi_0^{(1)} - B_1\phi_0^{(1)}) - A_2(B_2\psi_0^{(1)} - B_1\psi_0^{(1)}),\\
&\quad B_2(A_1\psi_0^{(2)} - A_2\psi_0^{(2)}) - B_1(A_1\phi_0^{(2)} - A_2\phi_0^{(2)})\Big)\\
&\le (A_1\psi_0^{(2)} - A_2\psi_0^{(2)}, B_2\phi_0^{(1)} - B_1\phi_0^{(1)}).
\end{aligned}$$

So this implies that Φ_1 and Ψ_1 satisfy (ii). Now

$$\begin{aligned}
&\big(A_1\phi_1^{(2)} - A_2\phi_1^{(2)}, B_2\psi_1^{(1)} - B_1\psi_1^{(1)}\big)\\
&= \Big(A_1(B_2\psi_0^{(1)} - B_1\psi_0^{(1)}) - A_2(B_2\phi_0^{(1)} - B_1\phi_0^{(1)}),\\
&\quad B_2(A_1\phi_0^{(2)} - A_2\phi_0^{(2)}) - B_1(A_1\psi_0^{(2)} - A_2\psi_0^{(2)})\Big)\\
&\ge (A_1\phi_0^{(2)} - A_2\phi_0^{(2)}, B_2\psi_0^{(1)} - B_1\psi_0^{(1)}).
\end{aligned}$$

This implies that Φ_1 and Ψ_1 satisfy (iii). This completes the proof of proposition. $\square$

Proposition 2.4. *Suppose we have bounded measurable functions Φ_0 and Ψ_0 on Ω such that* (i)–(iii) *of Proposition 2.3 hold. Then under the hypothesis* (H1)*, there exist $\Phi, \Psi \in C^+(\overline{\Omega}) \times C^+(\overline{\Omega})$ satisfying the requirement of Proposition 2.1.*

Proof. Let us define

$$\Phi_{n+1} = (\phi_{n+1}^{(1)}, \phi_{n+1}^{(2)}) = \begin{cases} (A_1\phi_n^{(2)} - A_2\phi_n^{(2)}, B_2\psi_n^{(1)} - B_1\psi_n^{(1)}) & \text{on } \Omega_1,\\ (A_1\psi_n^{(2)} - A_2\psi_n^{(2)}, B_2\phi_n^{(1)} - B_1\phi_n^{(1)}) & \text{on } \Omega_2. \end{cases}$$

$$\Psi_{n+1} = (\psi_{n+1}^{(1)}, \psi_{n+1}^{(2)}) = \begin{cases} (A_1\psi_n^{(2)} - A_2\psi_n^{(2)}, B_2\phi_n^{(1)} - B_1\phi_n^{(1)}) & \text{on } \Omega_1,\\ (A_1\phi_n^{(2)} - A_2\phi_n^{(2)}, B_2\psi_n^{(1)} - B_1\psi_n^{(1)}) & \text{on } \Omega_2. \end{cases}$$

By Proposition 2.3 and induction, (Φ_n, Ψ_n) satisfies (i)–(iii) and therefore one can see easily that

$$\begin{aligned}
\mathbf{0} &\le (A_1\phi_n^{(2)} - A_2\phi_n^{(2)}, B_2\psi_n^{(1)} - B_1\psi_n^{(1)})\\
&\le (A_1\phi_{n+1}^{(2)} - A_2\phi_{n+1}^{(2)}, B_2\psi_{n+1}^{(1)} - B_1\psi_{n+1}^{(1)}) \le \dots\\
&\le (A_1\psi_{n+1}^{(2)} - A_2\psi_{n+1}^{(2)}, B_2\phi_{n+1}^{(1)} - B_1\phi_{n+1}^{(1)})\\
&\le (A_1\psi_n^{(2)} - A_2\psi_n^{(2)}, B_2\phi_n^{(1)} - B_1\phi_n^{(1)}) \dots \le (\psi_0^{(1)}, \psi_0^{(2)}) \quad \text{on } \Omega
\end{aligned} \tag{2.1}$$

We can write down the above inequalities as

$$\mathbf{0} \le T(\Phi_n, \Psi_n) \le T(\Phi_{n+1}, \Psi_{n+1}) \le \dots \le T(\Psi_{n+1}, \Phi_{n+1}) \le T(\Psi_n, \Phi_n) \le \dots \le \Psi_0,$$

where

$$\begin{aligned}
T(\Phi_n, \Psi_n) &= (A_1\phi_n^{(2)} - A_2\phi_n^{(2)}, B_2\psi_n^{(1)} - B_1\psi_n^{(1)}),\\
T(\Psi_n, \Phi_n) &= (A_1\psi_n^{(2)} - A_2\psi_n^{(2)}, B_2\phi_n^{(1)} - B_1\phi_n^{(1)}).
\end{aligned}$$

Thus $T(\Phi_n, \Psi_n) \uparrow \Phi$; i.e.,

$$(A_1\phi_n^{(2)} - A_2\phi_n^{(2)}, B_2\psi_n^{(1)} - B_1\psi_n^{(1)}) \to (\phi^{(1)}, \phi^{(2)}) \quad \text{pointwise as } n \to \infty,$$

and $T(\Psi_n, \Phi_n) \downarrow \Psi$; i.e.,

$$(A_1\psi_n^{(2)} - A_2\psi_n^{(2)}, B_2\phi_n^{(1)} - B_1\phi_n^{(1)}) \to (\psi^{(1)}, \psi^{(2)}) \quad \text{pointwise as } n \to \infty.$$

Also, $\Phi \leq \Psi$. From (2.1), $A_1\phi_n^{(2)} - A_2\phi_n^{(2)}$, $B_2\psi_n^{(1)} - B_1\psi_n^{(1)}$, $A_1\psi_n^{(2)} - A_2\psi_n^{(2)}$, $B_2\phi_n^{(1)} - B_1\phi_n^{(1)}$ are bounded functions, $a, b \in L^\infty(\Omega)$ and $G(x,y)$ is integrable. An application of Lebesgue dominated convergence theorem implies

$$\begin{aligned}
A_1(B_2\psi_n^{(1)} - B_1\psi_n^{(1)})(x) &= \int_{\Omega_1} G(x,y)a_+(y)f((B_2\psi_n^{(1)} - B_1\psi_n^{(1)})(y))dy \\
&\to \int_{\Omega_1} G(x,y)a_+(y)f(\phi^{(2)}(y))dy \\
&= A_1\phi^{(2)}(x)
\end{aligned}$$

and similarly, we obtain

$$\begin{gathered}
A_2(B_2\phi_n^{(1)} - B_1\phi_n^{(1)})(x) \to A_2\psi^{(2)}(x), \\
B_1(A_1\psi_n^{(2)} - A_2\psi_n^{(2)})(x) \to B_1\psi^{(1)}(x), \\
B_2(A_1\phi_n^{(2)} - A_2\phi_n^{(2)})(x) \to B_2\phi^{(1)}(x).
\end{gathered}$$

Now using the definition of $T(\Phi_{n+1}, \Psi_{n+1})$,

$$\begin{aligned}
T(\Phi_{n+1}, \Psi_{n+1}) &= (A_1\phi_{n+1}^{(2)} - A_2\phi_{n+1}^{(2)}, B_2\psi_{n+1}^{(1)} - B_1\psi_{n+1}^{(1)}) \\
&= \Big(A_1(B_2\psi_n^{(1)} - B_1\psi_n^{(1)}) - A_2(B_2\phi_n^{(1)} - B_1\phi_n^{(1)}), \\
&\quad B_2(A_1\phi_n^{(2)} - A_2\phi_n^{(2)}) - B_1(A_1\psi_n^{(2)} - A_2\psi_n^{(2)})\Big),
\end{aligned}$$

we obtain

$$(\phi^{(1)}, \phi^{(2)}) = (A_1\phi^{(2)} - A_2\psi^{(2)}, B_2\phi^{(1)} - B_1\psi^{(1)}).$$

By a similar arguments and using the definition of $T(\Psi_{n+1}, \Phi_{n+1})$, we obtain

$$(\psi^{(1)}, \psi^{(2)}) = (A_1\psi^{(2)} - A_2\phi^{(2)}, B_2\psi^{(1)} - B_1\phi^{(1)}).$$

This proves the construction of Φ and Ψ. □

Now we are ready to give the proof of main theorem.

3. Proof of Theorem 1.1

By a simple construction of Φ_0 and Ψ_0, we give some sufficient conditions so that (i)–(iii) of Proposition 2.3 are satisfied. Let

$$\Phi_0 = \begin{cases} (0,0) & \text{on } \Omega_1, \\ (\alpha_1, \alpha_2) & \text{on } \Omega_2; \end{cases} \quad \Psi_0 = \begin{cases} (\alpha_1, \alpha_2) & \text{on } \Omega_1, \\ (0,0) & \text{on } \Omega_2. \end{cases}$$

Then (i) is satisfied if $(\alpha_1, \alpha_2) \geq (0,0)$. Now the condition (ii) is

$$\begin{gathered}
A\Psi_0 = \big(A_1(\alpha_2) - A_2(0), B_2(\alpha_1) - B_1(0)\big) \leq (\alpha_1, \alpha_2) \quad \text{on } \Omega; \\
\text{i.e., } A_1(\alpha_2) - A_2(0) \leq \alpha_1, B_2(\alpha_1) - B_1(0) \leq \alpha_2;
\end{gathered}$$

while (iii) is

$$\begin{gathered}
A\Phi_0 = \big(A_1(0) - A_2(\alpha_2), B_2(0) - B_1(\alpha_1)\big) \geq (0,0) \quad \text{on } \Omega; \\
\text{i.e., } A_1(0) - A_2(\alpha_2) \geq 0, B_2(0) - B_1(\alpha_1) \geq 0.
\end{gathered}$$

Letting

$$z_\pm(x) = \int_\Omega G(x,y)a_\pm(y)dy, \quad Z_\pm(x) = \int_\Omega G(x,y)b_\pm(y)dy,$$

the rest of the proof goes exactly same as the proof of [10, Theorem 1.1]. For the sake of brevity, we omit the details.

4. $n \times n$ SYSTEMS

We consider the existence of a non-negative solution to the $n \times n$ system

$$\begin{gathered} -\Delta u_1 = \lambda a_1(x) f_1(u_2) \quad \text{in } \Omega, \\ -\Delta u_2 = \lambda a_2(x) f_2(u_3) \quad \text{in } \Omega, \\ \dots \\ -\Delta u_{n-1} = \lambda a_{n-1}(x) f_{n-1}(u_n) \quad \text{in } \Omega, \\ -\Delta u_n = \lambda a_n(x) f_n(u_1) \quad \text{in } \Omega, \\ u_1 = u_1 = u_3 = \dots = u_n = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{4.1}$$

where $a_i \in L^\infty(\Omega)$, for all $i = 1, 2, 3, \dots, n$ and $a_i' s$ are sign changing in Ω. Let us consider the case when n is even. Let

$$a_1.a_2 \cdots a_n \le 0 \quad \text{a.e. in } \bar{\Omega}. \tag{4.2}$$

Let us take $\Omega_1 = \{x \in \Omega : a_i(x) \ge 0, \forall \text{i=1, 2, 3,\dots,n-1}, \ a_n(x) < 0\}$ and $\Omega_2 = \{x \in \Omega |\, a_i(x) < 0, \forall\, i = 1, 2, 3, \dots, n-1, \ a_n(x) \ge 0\}$ such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega_1 \cup \Omega_2 = \Omega$. One can take different sets of Ω_1 and Ω_2 but in all sets (4.2) should hold.

We assume the following hypotheses on the nonlinearity and weights:

(H0') $f_i : [0, \infty) \to [0, \infty)$ which are continuous and non-decreasing on $[0, \infty)$, for $i = 1, 2, \dots, n$. There exist $\mu_i > 0$, for $i = 1, 2, \dots, n$ such that

(H1')
$$\int_\Omega G(x, y) a_{1_+}(y) dy \ge (1 + \mu_1) \int_\Omega G(x, y) a_{1_-}(y) dy, \quad \forall\, x \in \Omega.$$

(H2')
$$\int_\Omega G(x, y) a_{2_+}(y) dy \ge (1 + \mu_2) \int_\Omega G(x, y) a_{2_-}(y) dy, \quad \forall\, x \in \Omega.$$

($\dots$) $\dots$

(H(n-1)')
$$\int_\Omega G(x, y) a_{(n-1)_+}(y) dy \ge (1 + \mu_{n-1}) \int_\Omega G(x, y) a_{(n-1)_-}(y) dy, \quad \forall\, x \in \Omega.$$

(Hn')
$$\int_\Omega G(x, y) a_{n_+}(y) dy \ge (1 + \mu_n) \int_\Omega G(x, y) a_{n_-}(y) dy, \quad \forall\, x \in \Omega,$$

where $G(x, y)$ is the Green's function of $-\Delta$ associated with the Dirichlet boundary condition.

One can see easily that the fixed point of the integral operator $A : (C(\overline{\Omega}))^n \to (C(\overline{\Omega}))^n$ defined by

$$\begin{aligned} &A(\phi_1, \phi_2, \dots, \phi_n)(x) \\ &= \Big(\lambda \int_\Omega G(x, y) a_1(y) f_1(\phi_2(y)) dy, \lambda \int_\Omega G(x, y) a_2(y) f_2(\phi_3(y)) dy, \dots, \end{aligned}$$

$$\lambda \int_\Omega G(x,y)a_{n-1}(y)f_{n-1}(\phi_n(y))dy, \lambda \int_\Omega G(x,y)a_n(y)f_n(\phi_1(y))dy\Big)$$

is the solution of (4.1). We note that the operator A can be written as

$$\begin{aligned} &A(\phi_1,\phi_2,\dots,\phi_n)(x)\\ &= \Big(\lambda \int_{\Omega_1} G(x,y)a_{1_+}(y)f(\phi_2(y))dy - \lambda \int_{\Omega_2} G(x,y)a_{1_-}(y)f(\phi_2(y))dy, \dots\\ &\quad \lambda \int_{\Omega_1} G(x,y)a_{n_+}(y)f_n(\phi_1(y))dy - \lambda \int_{\Omega_2} G(x,y)a_{n_-}(y)f_n(\phi_1(y))dy\Big). \end{aligned}$$

For simplicity, A can be rewritten as

$$\begin{aligned} A(\phi_1,\phi_2,\dots,\phi_n)(x) = (&A_1^{(1)}\phi_2(x) - A_2^{(1)}\phi_2(x), A_1^{(2)}\phi_3(x) - A_2^{(2)}\phi_3(x), \dots,\\ &A_2^{(n)}\phi_1(x) - A_1^{(n)}\phi_1(x)), \end{aligned} \tag{4.3}$$

where $A_1^{(i)}$ and $A_2^{(i)}$ act on $C^+(\overline{\Omega} \cap \Omega_1)$ and $C^+(\overline{\Omega} \cap \Omega_2)$, respectively, for $i = 1,2,\dots,n$. By (H'), $A_1^{(i)}$ and $A_2^{(i)}$ are monotone for $i = 1,2,\dots,n$.

Theorem 4.1. *Let $a_1 \cdot a_2 \cdots a_n \le 0$ a.e. in $\bar\Omega$. If $f_i(0) > 0$ and $f_i's$ are non-decreasing, continuous functions for $i = 1,2,\dots,n$. Let* (H1')–(Hn') *hold. Then there exists $\bar\lambda > 0$ depending on f_i, a_i and μ_i, $i = 1,2,3,\dots,n$ such that* (4.1) *has a non-negative solution for* $0 \le \lambda \le \bar\lambda$.

Proof. The proof of this theorem is along the lines of the proof of Theorem 1.1 with an account of propositions similar to Propositions 2.1-2.4. For the sake of brevity, we omit the detailed verification. There exists $\bar\lambda > 0$ depending on f_i, a_i, μ_i, for $i = 1,2,\dots,n$ such that (4.1) has a non-negative solution for $0 \le \lambda \le \bar\lambda$. □

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