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DARBOUX PROBLEM FOR IMPLICIT IMPULSIVE PARTIAL HYPERBOLIC FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this article we investigate the existence and uniqueness of solutions for the initial value problems, for a class of hyperbolic impulsive fractional order differential equations by using some fixed point theorems.

1. Introduction

Fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus since, starting from some speculations of Leibniz (1697) and Euler (1730), it has been developed up to nowadays. The idea of fractional calculus and fractional order differential equations and inclusions has been a subject of interest not only among mathematicians, but also among physicists and engineers. Indeed, we can find numerous applications in rheology, control, porous media, viscoelasticity, electrochemistry, electromagnetism, etc. [14, 16, 20, 21, 23]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Kilbas *et al.* [17], Miller and Ross [22], Podlubny [24], Samko *et al.* [26], the papers of Abbas and Benchohra [2, 3, 4], Abbas *et al.* [1, 5, 6], Belarbi *et al.* [8], Benchohra *et al.* [9, 10, 12], Diethelm [13], Kilbas and Marzan [18], Mainardi [20], Podlubny *et al.* [25], Vityuk and Golushkov [28], Yu and Gao [31], Zhang [32] and the references therein.

The theory of impulsive differential equations have become important in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There has been a significant development in impulse theory in recent years, especially in the area of impulsive differential equations and inclusions with fixed moments; see the monographs of Benchohra *et al.* [11], Lakshmikantham *et al.* [19], the papers of Abbas and Benchohra [3, 4], Abbas *et al.* [1, 5] and the references therein.

The Darboux problem for partial hyperbolic differential equations was studied in the papers of Abbas and Benchohra [2, 3], Abbas *et al.* [7], Vityuk [27], Vityuk and Golushkov [28], Vityuk and Mykhailenko [29, 30] and by other authors.

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In the present article we are concerned with the existence and uniqueness of solutions to fractional order initial-value problem (IVP) for the system

$$\overline{D}_\theta^r u(x,y) = f(x,y,u(x,y),\overline{D}_\theta^r u(x,y)); \quad \text{for } (x,y)\in J,\ x\neq x_k,\ k=1,\dots,m, \tag{1.1}$$

$$u(x_k^+,y) = u(x_k^-,y) + I_k(u(x_k^-,y)); \quad \text{for } y\in[0,b],\ k=1,\dots,m, \tag{1.2}$$

$$\begin{cases} u(x,0)=\varphi(x); \quad x\in[0,a],\\ u(0,y)=\psi(y); \quad y\in[0,b],\\ \qquad \varphi(0)=\psi(0), \end{cases} \tag{1.3}$$

where $J:=[0,a]\times[0,b]$, $a,b>0$, $\theta=(0,0)$, $\overline{D}_\theta^r$ is the mixed regularized derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1]$, $0=x_0<x_1<\dots<x_m<x_{m+1}=a$, $f: J\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$, $I_k:\mathbb{R}^n\to\mathbb{R}^n$, $k=1,\dots,m$, $\varphi:[0,a]\to\mathbb{R}^n$ and $\psi:[0,b]\to\mathbb{R}^n$ are given absolutely continuous functions.

We present two results for the problem (1.1)-(1.3), the first one is based on Banach's contraction principle and the second one on the nonlinear alternative of Leray-Schauder type [15].

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts which are used throughout this paper. By $C(J)$ we denote the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm

$$\|w\|_\infty = \sup_{(x,y)\in J}\|w(x,y)\|,$$

where $\|\cdot\|$ denotes a suitable complete norm on $\mathbb{R}^n$. As usual, by $AC(J)$ we denote the space of absolutely continuous functions from J into $\mathbb{R}^n$ and $L^1(J)$ is the space of Lebegue-integrable functions $w:J\to\mathbb{R}^n$ with the norm

$$\|w\|_1 = \int_0^a\int_0^b \|w(x,y)\|\,dy\,dx.$$

Definition 2.1 ([17, 26])**.** *Let $\alpha\in(0,\infty)$ and $u\in L^1(J)$. The partial Riemann-Liouville integral of order α of $u(x,y)$ with respect to x is defined by the expression*

$$I_{0,x}^\alpha u(x,y) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}u(s,y)ds,$$

for almost all $x\in[0,a]$ and all $y\in[0,b]$, where $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\varsigma)=\int_0^\infty t^{\varsigma-1}e^{-t}\,dt$; $\varsigma>0$.

Analogously, we define the integral

$$I_{0,y}^\alpha u(x,y) = \frac{1}{\Gamma(\alpha)}\int_0^y (y-s)^{\alpha-1}u(x,s)ds,$$

for almost all $x\in[0,a]$ and almost all $y\in[0,b]$.

Definition 2.2 ([17, 26])**.** Let $\alpha\in(0,1]$ and $u\in L^1(J)$. The Riemann-Liouville fractional derivative of order α of $u(x,y)$ with respect to x is defined by

$$(D_{0,x}^\alpha u)(x,y) = \frac{\partial}{\partial x}I_{0,x}^{1-\alpha}u(x,y),$$

for almost all $x\in[0,a]$ and all $y\in[0,b]$.

Analogously, we define the derivative

$$(D^{\alpha}_{0,y}u)(x,y)=\frac{\partial}{\partial y}I^{1-\alpha}_{0,y}u(x,y),$$

for almost all $x\in[0,a]$ and almost all $y\in[0,b]$.

Definition 2.3 ([17, 26])**.** Let $\alpha\in(0,1]$ and $u\in L^1(J)$. The Caputo fractional derivative of order α of $u(x,y)$ with respect to x is defined by the expression

$$^cD^{\alpha}_{0,x}u(x,y)=I^{1-\alpha}_{0,x}\frac{\partial}{\partial x}u(x,y),$$

for almost all $x\in[0,a]$ and all $y\in[0,b]$.

Analogously, we define the derivative

$$^cD^{\alpha}_{0,y}u(x,y)=I^{1-\alpha}_{0,y}\frac{\partial}{\partial y}u(x,y),$$

for almost all $x\in[0,a]$ and almost all $y\in[0,b]$.

Definition 2.4 ([28])**.** *Let $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, $\theta=(0,0)$ and $u\in L^1(J)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by*

$$(I^r_{\theta}u)(x,y)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}u(s,t)\,dt\,ds.$$

In particular,

$$(I^{\theta}_{\theta}u)(x,y)=u(x,y),\ (I^{\sigma}_{\theta}u)(x,y)=\int_0^x\int_0^y u(s,t)\,dt\,ds;$$

for almost all $(x,y)\in J$, where $\sigma=(1,1)$. For instance, $I^r_{\theta}u$ exists for all $r_1,r_2\in(0,\infty)$, when $u\in L^1(J)$. Note also that when $u\in C(J)$, then $(I^r_{\theta}u)\in C(J)$, moreover

$$(I^r_{\theta}u)(x,0)=(I^r_{\theta}u)(0,y)=0;\quad x\in[0,a],\ y\in[0,b].$$

Example 2.5. Let $\lambda,\omega\in(-1,\infty)$ and $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, then

$$I^r_{\theta}x^{\lambda}y^{\omega}=\frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)}x^{\lambda+r_1}y^{\omega+r_2},$$

for almost all $(x,y)\in J$.

By $1-r$ we mean $(1-r_1,1-r_2)\in(0,1]\times(0,1]$. Denote by $D^2_{xy}:=\frac{\partial^2}{\partial x\partial y}$, the mixed second order partial derivative.

Definition 2.6 ([28])**.** Let $r\in(0,1]\times(0,1]$ and $u\in L^1(J)$. The mixed fractional Riemann-Liouville derivative of order r of u is defined by the expression $D^r_{\theta}u(x,y)=(D^2_{xy}I^{1-r}_{\theta}u)(x,y)$ and the Caputo fractional-order derivative of order r of u is defined by the expression $^cD^r_{\theta}u(x,y)=(I^{1-r}_{\theta}D^2_{xy}u)(x,y)$.

The case $\sigma=(1,1)$ is included and we have

$$(D^{\sigma}_{\theta}u)(x,y)=(^cD^{\sigma}_{\theta}u)(x,y)=(D^2_{xy}u)(x,y),$$

for almost all $(x,y)\in J$.

Example 2.7. Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, then

$$D_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} x^{\lambda-r_1} y^{\omega-r_2},$$

for almost all $(x, y) \in J$.

Definition 2.8 ([30])**.** For a function $u : J \to \mathbb{R}^n$, we set

$$q(x, y) = u(x, y) - u(x, 0) - u(0, y) + u(0, 0).$$

By the mixed regularized derivative of order $r = (r_1, r_2) \in (0, 1] \times (0, 1]$ of a function $u(x, y)$, we name the function

$$\overline{D}_\theta^r u(x, y) = D_\theta^r q(x, y).$$

The function

$$\overline{D}_{0,x}^{r_1} u(x, y) = D_{0,x}^{r_1}[u(x, y) - u(0, y)],$$

is called the partial r_1-order regularized derivative of the function $u(x, y) : J \to \mathbb{R}^n$ with respect to the variable x. Analogously, we define the derivative

$$\overline{D}_{0,y}^{r_2} u(x, y) = D_{0,y}^{r_2}[u(x, y) - u(x, 0)].$$

Let $a_1 \in [0, a]$, $z^+ = (a_1, 0) \in J$, $J_z = [a_1, a] \times [0, b]$, $r_1, r_2 > 0$ and $r = (r_1, r_2)$. For $u \in L^1(J_z, \mathbb{R}^n)$, the expression

$$(I_{z^+}^r u)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1^+}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} u(s, t)\, dt\, ds,$$

is called the left-sided mixed Riemann-Liouville integral of order r of u.

Definition 2.9 ([28])**.** For $u \in L^1(J_z, \mathbb{R}^n)$ where $D_{xy}^2 u$ is Lebesque integrable on $[x_k, x_{k+1}] \times [0, b]$, $k = 0, \dots, m$, the Caputo fractional-order derivative of order r of u is defined by the expression $({}^cD_{z^+}^r f)(x, y) = (I_{z^+}^{1-r} D_{xy}^2 f)(x, y)$. The Riemann-Liouville fractional-order derivative of order r of u is defined by $(D_{z^+}^r f)(x, y) = (D_{xy}^2 I_{z^+}^{1-r} f)(x, y)$.

Analogously, we define the derivatives

$$\begin{gathered}
\overline{D}_{z^+}^r u(x, y) = D_{z^+}^r q(x, y), \\
\overline{D}_{a_1,x}^{r_1} u(x, y) = D_{a_1,x}^{r_1}[u(x, y) - u(0, y)], \\
\overline{D}_{a_1,y}^{r_2} u(x, y) = D_{a_1,y}^{r_2}[u(x, y) - u(x, 0)].
\end{gathered}$$

3. Existence of solutions

In what follows set

$$J_k := (x_k, x_{k+1}] \times [0, b].$$

To define the solutions of (1.1)-(1.3), we shall consider the space

$$\begin{aligned}
PC(J) = \big\{ & u : J \to \mathbb{R}^n : u \in C(J_k, \mathbb{R}^n);\ k = 0, 1, \dots, m, \text{ and} \\
& \text{there exist } u(x_k^-, y) \text{ and } u(x_k^+, y);\ k = 1, \dots, m, \\
& \text{with } u(x_k^-, y) = u(x_k, y) \text{ for each } y \in [0, b] \big\}.
\end{aligned}$$

This set is a Banach space with the norm

$$\|u\|_{PC} = \sup_{(x,y)\in J} \|u(x, y)\|.$$

Set

$$J' := J \backslash \{(x_1, y), \dots, (x_m, y),\ y \in [0, b]\}.$$

Definition 3.1. A function $u \in PC(J)$ such that $u, \overline{D}^{r_1}_{x_k,x}u, \overline{D}^{r_2}_{x_k,y}u, \overline{D}^{r}_{z_k^+}u;\ k = 0, \dots, m$, are continuous on J' and $I^{1-r}_{z^+}u \in AC(J')$ is said to be a solution of (1.1)-(1.3) if u satisfies (1.1) on J' and conditions (1.2), (1.3) are satisfied.

For the existence of solutions for (1.1)-(1.3) we need the following lemmas.

Lemma 3.2 ([30]). *Let the function $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous on its variables. Then the problem*

$$\overline{D}^{r}_{\theta}u(x, y) = f(x, y, u(x, y), \overline{D}^{r}_{\theta}u(x, y)); \quad \text{if } (x, y) \in J := [0, a] \times [0, b], \tag{3.1}$$

$$\begin{gathered} u(x, 0) = \varphi(x); \quad x \in [0, a], \\ u(0, y) = \psi(y); \quad y \in [0, b], \\ \varphi(0) = \psi(0), \end{gathered} \tag{3.2}$$

is equivalent to the problem

$$g(x, y) = f(x, y, \mu(x, y) + I^{r}_{\theta}g(x, y), g(x, y)),$$

and if $g \in C(J)$ is the solution of this equation, then $u(x, y) = \mu(x, y) + I^{r}_{\theta}g(x, y)$, where

$$\mu(x, y) = \varphi(x) + \psi(y) - \varphi(0).$$

Lemma 3.3 ([5]). *Let $0 < r_1, r_2 \le 1$ and let $h : J \to \mathbb{R}^n$ be continuous. A function u is a solution of the fractional integral equation*

$$u(x, y) = \begin{cases} \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1-1}(y - t)^{r_2-1}h(s, t)\, dt\, ds; \\ \quad \text{if } (x, y) \in [0, x_1] \times [0, b], \\ \mu(x, y) + \sum_{i=1}^{k}(I_i(u(x_i^-, y)) - I_i(u(x_i^-, 0))) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_0^y (x_i - s)^{r_1-1}(y - t)^{r_2-1}h(s, t)\, dt\, ds \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x - s)^{r_1-1}(y - t)^{r_2-1}h(s, t)\, dt\, ds; \\ \quad \text{if } (x, y) \in (x_k, x_{k+1}] \times [0, b],\ k = 1, \dots, m, \end{cases} \tag{3.3}$$

if and only if u is a solution of the fractional initial-value problem

$$^{c}D^{r}_{z_k^+}u(x, y) = h(x, y), \quad (x, y) \in J',\ k = 1, \dots, m, \tag{3.4}$$

$$u(x_k^+, y) = u(x_k^-, y) + I_k(u(x_k^-, y)), \quad y \in [0, b],\ k = 1, \dots, m. \tag{3.5}$$

By Lemmas 3.2 and 3.3, we conclude the following statement.

Lemma 3.4. *Let the function $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Then problem* (1.1)-(1.3) *is equivalent to the problem*

$$g(x, y) = f(x, y, \xi(x, y), g(x, y)), \tag{3.6}$$

where

$$\xi(x,y)=\begin{cases}\mu(x,y)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds;\\ \ if\ (x,y)\in[0,x_1]\times[0,b],\\ \mu(x,y)+\sum_{i=1}^k(I_i(u(x_i^-,y))-I_i(u(x_i^-,0)))\\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^k\int_{x_{i-1}}^{x_i}\int_0^y(x_i-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds\\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds;\\ \ if\ (x,y)\in(x_k,x_{k+1}]\times[0,b],\ k=1,\dots,m,\end{cases}$$

$$\mu(x,y)=\varphi(x)+\psi(y)-\varphi(0).$$

And if $g\in C(J)$ *is the solution of* (3.6), *then* $u(x,y)=\xi(x,y)$.

Further, we present conditions for the existence and uniqueness of a solution of problem (1.1)-(1.3). We will us the following hypotheses.

- (H1) The function $f:J\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous;
- (H2) For any $u,v,w,z\in\mathbb{R}^n$ and $(x,y)\in J$, there exist constants $l>0$ and $0<l_*<1$ such that
$$\|f(x,y,u,z)-f(x,y,v,w)\|\le l\|u-v\|+l_*\|z-w\|,$$
- (H3) There exists a constant $l^*>0$ such that
$$\|I_k(u)-I_k(\overline{u})\|\le l^*\|u-\overline{u}\|,\quad \text{for } u,\overline{u}\in\mathbb{R}^n,\ k=1,\dots,m.$$

Theorem 3.5. *Assume* (H1)–(H3) *are satisfied. If*

$$2ml^*+\frac{2la^{r_1}b^{r_2}}{(1-l_*)\Gamma(r_1+1)\Gamma(r_2+1)}<1, \tag{3.7}$$

then there exists a unique solution for IVP (1.1)-(1.3) *on* J.

Proof. Transform the problem (1.1)-(1.3) into a fixed point problem. Consider the operator $N:PC(J)\to PC(J)$ defined by

$$\begin{aligned}N(u)(x,y)&=\mu(x,y)+\sum_{0<x_k<x}(I_k(u(x_k^-,y))-I_k(u(x_k^-,0)))\\&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{0<x_k<x}\int_{x_{k-1}}^{x_k}\int_0^y(x_k-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds\\&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds,\end{aligned}\tag{3.8}$$

where $g\in C(J)$ such that

$$g(x,y)=f(x,y,u(x,y),g(x,y)),$$

By Lemma 3.4, the problem of finding the solutions of (1.1)-(1.3) is reduced to finding the solutions of the operator equation $N(u)=u$. Let $v,w\in PC(J)$. Then, for $(x,y)\in J$, we have

$$
\begin{aligned}
&\|N(v)(x,y)-N(w)(x,y)\|\\
&\le \sum_{k=1}^{m}(\|I_k(v(x_k^-,y))-I_k(w(x_k^-,y))\|+\|I_k(v(x_k^-,0))-I_k(w(x_k^-,0))\|)\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y(x_k-s)^{r_1-1}(y-t)^{r_2-1}\|g(s,t)-h(s,t)\|\,dt\,ds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}\|g(s,t)-h(s,t)\|\,dt\,ds,
\end{aligned}
\tag{3.9}
$$

where $g,h\in C(J)$ such that

$$
\begin{gathered}
g(x,y)=f(x,y,v(x,y),g(x,y)),\\
h(x,y)=f(x,y,w(x,y),h(x,y)).
\end{gathered}
$$

By (H2), we obtain

$$
\|g(x,y)-h(x,y)\|\le l\|v(x,y)-w(x,y)\|+l_*\|g(x,y)-h(x,y)\|.
$$

Then

$$
\|g(x,y)-h(x,y)\|\le \frac{l}{1-l_*}\|v(x,y)-w(x,y)\|\le \frac{l}{1-l_*}\|v-w\|_{PC}.
$$

Thus, (H3) and (3.9) imply

$$
\begin{aligned}
&\|N(v)-N(w)\|_{PC}\\
&\le \sum_{k=1}^{m}l^*(\|v(x_k^-,y)-w(x_k^-,y)\|+\|v(x_k^-,0)-w(x_k^-,0)\|)\\
&\quad+\frac{l}{(1-l_*)\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y(x_k-s)^{r_1-1}(y-t)^{r_2-1}\|v-w\|_{PC}\,dt\,ds\\
&\quad+\frac{l}{(1-l_*)\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}\|v-w\|_{PC}\,dt\,ds.
\end{aligned}
$$

However,

$$
\begin{aligned}
&\frac{l}{(1-l_*)\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y(x_k-s)^{r_1-1}(y-t)^{r_2-1}\|v-w\|_{PC}\,dt\,ds\\
&\le \frac{l}{(1-l_*)\Gamma(r_1)\Gamma(r_2)}\frac{b^{r_2}}{r_2}\|v-w\|_{PC}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}(x_k-s)^{r_1-1}ds\\
&\le \frac{l}{(1-l_*)\Gamma(r_1)\Gamma(r_2)}\frac{b^{r_2}}{r_2}\|v-w\|_{PC}\sum_{k=1}^{m}\frac{x_{k-1}^{r_1}}{r_1}\\
&= \frac{l}{(1-l_*)\Gamma(r_1)\Gamma(r_2)}\frac{b^{r_2}}{r_2}\|v-w\|_{PC}\frac{x_{m-1}^{r_1}-x_0^{r_1}}{r_1}\\
&\le \frac{l}{(1-l_*)\Gamma(r_1)\Gamma(r_2)}\frac{b^{r_2}}{r_2}\|v-w\|_{PC}\frac{a^{r_1}}{r_1}\\
&= \frac{la^{r_1}b^{r_2}}{(1-l_*)\Gamma(1+r_1)\Gamma(1+r_2)}\|v-w\|_{PC}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\|N(v)-N(w)\|_{PC}\\
&\le \Big(2ml^* + \frac{la^{r_1}b^{r_2}}{(1-l_*)\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{la^{r_1}b^{r_2}}{(1-l_*)\Gamma(r_1+1)\Gamma(r_2+1)}\Big)\|v-w\|_{PC}\\
&\le \Big(2ml^* + \frac{2la^{r_1}b^{r_2}}{(1-l_*)\Gamma(r_1+1)\Gamma(r_2+1)}\Big)\|v-w\|_{PC}.
\end{aligned}
$$

Hence

$$
\|N(v)-N(w)\|_{PC} \le \Big(2ml^* + \frac{2la^{r_1}b^{r_2}}{(1-l_*)\Gamma(r_1+1)\Gamma(r_2+1)}\Big)\|v-w\|_{PC}.
$$

By (3.7), N is a contraction, and hence N has a unique fixed point by Banach's contraction principle. □

Theorem 3.6 (Nonlinear alternative of Leray-Schauder type [15])**.** *Let X be a Banach space and C a nonempty convex subset of X. Let U a nonempty open subset of C with $0 \in U$ and $T : \overline{U} \to C$ continuous and compact operator. Then either*

(a) *T has fixed points. Or*
(b) *There exist $u \in \partial U$ and $\lambda \in [0,1]$ with $u = \lambda T(u)$.*

For the next theorem, we use the following assumptions:

(H4) There exist $p, q, d \in C(J, \mathbb{R}_+)$ such that
$$
\|f(x,y,u,z)\| \le p(x,y) + q(x,y)\|u\| + d(x,y)\|z\|
$$
for $(x,y) \in J$ and each $u, z \in \mathbb{R}^n$,

(H5) There exists $\psi^* : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that
$$
\|I_k(u)\| \le \psi^*(\|u\|);\ k = 1,\dots,m, \quad \text{for all } u \in \mathbb{R}^n,
$$

(H6) There exists a number $\overline{M} > 0$ such that
$$
\|\mu\|_\infty + 2m\psi^*(\overline{M}) + \frac{2a^{r_1}b^{r_2}\big(p^* + q^*\|\mu\|_\infty + 2mq^*\psi^*(\overline{M})\big)}{\big(1 - d^* - \frac{2q^*a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\big)\Gamma(1+r_1)\Gamma(1+r_2)} < \overline{M},
$$
where $p^* = \sup_{(x,y)\in J} p(x,y)$, $q^* = \sup_{(x,y)\in J} q(x,y)$ and $d^* = \sup_{(x,y)\in J} d(x,y)$.

Theorem 3.7. *Assume* (H1), (H4), (H5), (H6) *hold. If*
$$
d^* + \frac{2q^*a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{3.10}
$$
then (1.1)-(1.3) *has at least one solution on J.*

Proof. Transform problem (1.1)-(1.3) into a fixed point problem. Consider the operator N defined in (3.8). We shall show that the operator N is continuous and compact.

Step 1: N is continuous. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in $PC(J)$. Let $\eta > 0$ be such that $\|u_n\|_{PC} \le \eta$. Then for each $(x,y) \in J$ we have

$$\begin{aligned}
&\|N(u_n)(x,y) - N(u)(x,y)\| \\
&\le \sum_{k=1}^{m}(\|I_k(u_n(x_k^-,y)) - I_k(u(x_k^-,y))\| + \|I_k(u_n(x_k^-,0)) - I_k(u(x_k^-,0))\|) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1}\|g_n(s,t)-g(s,t)\|\,dt\,ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|g_n(s,t)-g(s,t)\|\,dt\,ds,
\end{aligned} \tag{3.11}$$

where $g_n, g \in C(J)$ such that

$$\begin{gathered}
g_n(x,y) = f(x,y,u_n(x,y),g_n(x,y)), \\
g(x,y) = f(x,y,u(x,y),g(x,y)).
\end{gathered}$$

Since $u_n \to u$ as $n \to \infty$ and f is a continuous function, we obtain

$$g_n(x,y) \to g(x,y) \quad \text{as } n \to \infty, \text{ for each } (x,y) \in J.$$

Hence, (3.11) gives

$$\|N(u_n) - N(u)\|_{PC} \le 2ml^*\|u_n - u\|_{PC} + \frac{2a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\|g_n - g\|_\infty \to 0$$

as $n \to \infty$.

Step 2: N maps bounded sets into bounded sets in $PC(J)$. Indeed, it is sufficinet show that for any $\eta^* > 0$, there exists a positive constant M^* such that, for each $u \in B_{\eta^*} = \{u \in PC(J) : \|u\|_{PC} \le \eta^*\}$, we have $\|N(u)\|_{PC} \le M^*$. For $(x,y) \in J$, we have

$$\begin{aligned}
&\|N(u)(x,y)\| \\
&\le \|\mu(x,y)\| + \sum_{k=1}^{m}(\|I_k(u(x_k^-,y))\| + \|I_k(u(x_k^-,0))\|) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1}\|g(s,t)\|\,dt\,ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|g(s,t)\|\,dt\,ds,
\end{aligned} \tag{3.12}$$

where $g \in C(J)$ such that $g(x,y) = f(x,y,u(x,y),g(x,y))$. By (H4), for each $(x,y) \in J$, we have

$$\|g(x,y)\| \le p(x,y) + q(x,y)\|\xi(x,y)\| + d(x,y)\|g(x,y)\|.$$

On the other hand, for each $(x,y) \in J$,

$$\begin{aligned}
\|\xi(x,y)\| &\le \|\mu(x,y)\| + \sum_{k=1}^{m}(\|I_k(u(x_k^-,y))\| + \|I_k(u(x_k^-,0))\|) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1}\|g(s,t)\|\,dt\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^{y}(x-s)^{r_1-1}(y-t)^{r_2-1}\|g(s,t)\|\,dt\,ds\\
&\leq \|\mu\|_\infty+2m\psi^*(\eta^*)+\frac{2a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\|g\|_\infty.
\end{aligned}
$$

Hence, for each $(x,y)\in J$, we have

$$
\|g\|_\infty\leq p^*+q^*\Big(\|\mu\|_\infty+2m\psi^*(\eta^*)+\frac{2a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\|g\|_\infty\Big)+d^*\|g\|_\infty.
$$

Then, by (3.10), we have

$$
\|g\|_\infty\leq\frac{p^*+q^*\big(\|\mu\|_\infty+2m\psi^*(\eta^*)\big)}{1-d^*-\frac{2q^*a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}:=M.
$$

Thus, (3.12) implies

$$
\|N(u)\|_{PC}\leq\|\mu\|_\infty+2m\psi^*(\eta^*)+\frac{2Ma^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}:=M^*.
$$

Step 3: N maps bounded sets into equicontinuous sets in $PC(J)$. Let $(\tau_1,y_1),(\tau_2,y_2)\in J$, $\tau_1<\tau_2$ and $y_1<y_2$, B_{η^*} be a bounded set of $PC(J)$ as in Step 2, and let $u\in B_{\eta^*}$. Then for each $(x,y)\in J$, we have

$$
\begin{aligned}
&\|N(u)(\tau_2,y_2)-N(u)(\tau_1,y_1)\|\\
&\leq\|\mu(\tau_1,y_1)-\mu(\tau_2,y_2)\|+\sum_{k=1}^{m}(\|I_k(u(x_k^-,y_1))-I_k(u(x_k^-,y_2))\|)\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^{y_1}(x_k-s)^{r_1-1}[(y_2-t)^{r_2-1}-(y_1-t)^{r_2-1}]\\
&\quad\times g(s,t)\,dt\,ds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_{y_1}^{y_2}(x_k-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|\,dt\,ds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_0^{y_1}[(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}-(\tau_1-s)^{r_1-1}(y_1-t)^{r_2-1}]\\
&\quad\times g(s,t)\,dt\,ds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|\,dt\,ds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|\,dt\,ds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_0^{y_1}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|\,dt\,ds,
\end{aligned}
$$

where $g\in C(J)$ such that $g(x,y)=f(x,y,u(x,y),g(x,y))$. However, $\|g\|_\infty\leq M$. Thus

$$
\begin{aligned}
&\|N(u)(x_2,y_2)-N(u)(x_1,y_1)\|\\
&\leq\|\mu(\tau_1,y_1)-\mu(\tau_2,y_2)\|+\sum_{k=1}^{m}(\|I_k(u(x_k^-,y_1))-I_k(u(x_k^-,y_2))\|)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{M}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^{y_1}(x_k-s)^{r_1-1}[(y_2-t)^{r_2-1}-(y_1-t)^{r_2-1}]\,dt\,ds\\
&+\frac{M}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_{y_1}^{y_2}(x_k-s)^{r_1-1}(y_2-t)^{r_2-1}\,dt\,ds\\
&+\frac{M}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_0^{y_1}[(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}-(\tau_1-s)^{r_1-1}(y_1-t)^{r_2-1}]\,dt\,ds\\
&+\frac{M}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\,dt\,ds\\
&+\frac{M}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\,dt\,ds\\
&+\frac{M}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_0^{y_1}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\,dt\,ds.
\end{aligned}
$$

As $\tau_1 \to \tau_2$ and $y_1 \to y_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that N is continuous and completely continuous.

Step 4: A priori bounds. We now show there exists an open set $U \subseteq PC(J)$ with $u \neq \lambda N(u)$, for $\lambda \in (0,1)$ and $u \in \partial U$. Let $u \in PC(J)$ and $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus for each $(x,y) \in J$, we have

$$
\begin{aligned}
\|u(x,y)\| &\le \|\lambda\mu(x,y)\| + \sum_{k=1}^{m}\lambda(\|I_k(u(x_k^-,y))\| + \|I_k(u(x_k^-,0))\|)\\
&\quad+\frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^{y}(x_k-s)^{r_1-1}(y-t)^{r_2-1}\|g(s,t)\|\,dt\,ds\\
&\quad+\frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^{y}(x-s)^{r_1-1}(y-t)^{r_2-1}\|g(s,t)\|\,dt\,ds\\
&\le \|\mu\|_\infty + 2m\psi^*(\|u(x,y\|) + \frac{2a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\|g\|_\infty.
\end{aligned}
$$

However,

$$
\|g\|_\infty \le \frac{p^* + q^*\big(\|\mu\|_\infty + 2m\psi^*(\|u\|_{PC})\big)}{1-d^*-\frac{2q^*a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}.
$$

Thus, for each $(x,y) \in J$, we have

$$
\|u\|_{PC} \le \|\mu\|_\infty + 2m\psi^*(\|u\|_{PC}) + \frac{2a^{r_1}b^{r_2}\big(p^*+q^*\|\mu\|_\infty + 2mq^*\psi^*(\|u\|_{PC})\big)}{\big(1-d^*-\frac{2q^*a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\big)\Gamma(1+r_1)\Gamma(1+r_2)}.
$$

Hence

$$
\|u\|_{PC} \le \|\mu\|_\infty + 2mq^*\psi^*(\|u\|_{PC}) + \frac{2a^{r_1}b^{r_2}\big(p^*+q^*\|\mu\|_\infty + 2m\psi^*(\|u\|_{PC})\big)}{\big(1-d^*-\frac{2q^*a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\big)\Gamma(1+r_1)\Gamma(1+r_2)}.
$$

By (H6), there exists $\overline{M}$ such that $\|u\|_{PC} \neq \overline{M}$. Let

$$
U = \{u \in PC(J) : \|u\|_{PC} < \overline{M} + 1\}.
$$

By our choice of U, there is no $u \in \partial U$ such that $u = \lambda N(u)$, for $\lambda \in (0,1)$. As a consequence of Theorem 3.6, we deduce that N has a fixed point u in $\overline{U}$ which is a solution to (1.1)-(1.3). $\square$

4. An Example

As an application of our results we consider the following impulsive implicit partial hyperbolic differential equations

$$\overline{D}_\theta^r u(x,y) = \frac{1}{10e^{x+y+2}(1+|u(x,y)|+|\overline{D}_\theta^r u(x,y)|)}, \quad \text{for } (x,y) \in [0,1]\times[0,1],\ x \neq x_k,\ k=1,\dots,m; \tag{4.1}$$

$$u(x_k^+,y) = u(x_k^-,y) + \frac{1}{6e^{x+y+4}(1+|u(x_k^-,y)|)}; \text{ for } y \in [0,1],\ k=1,\dots,m; \tag{4.2}$$

$$u(x,0) = x, \quad u(0,y) = y^2; \text{for } x,y \in [0,1]. \tag{4.3}$$

Set

$$f(x,y,u,v) = \frac{1}{10e^{x+y+2}(1+|u|+|v|)}, \quad (x,y) \in [0,1]\times[0,1],$$

$$I_k(u(x_k^-,y)) = \frac{1}{6e^{x+y+4}(1+|u(x_k^-,y)|)}, \quad y \in [0,1].$$

Clearly, the function f is continuous. For each $u, v, \overline{u}, \overline{v} \in \mathbb{R}$ and $(x,y) \in [0,1]\times[0,1]$, we have

$$|f(x,y,u,v) - f(x,y,\overline{u},\overline{v})| \le \frac{1}{10e^2}(|u-\overline{u}| + |v-\overline{v}|),$$

$$|I_k(u) - I_k(\overline{u})| \le \frac{1}{6e^4}|u-\overline{u}|.$$

Hence condition (H2) and (H3) are satisfied with $l = l_* = \frac{1}{10e^2}$ and $l^* = \frac{1}{6e^4}$. We shall show that (3.7) holds with $a = b = 1$. Indeed, if we assume, for instance, that the number of impulses $m = 3$, then we have

$$2ml^* + \frac{2la^{r_1}b^{r_2}}{(1-l_*)\Gamma(r_1+1)\Gamma(r_2+1)} = \frac{1}{e^4} + \frac{2}{(10e^2-1)\Gamma(r_1+1)\Gamma(r_2+1)} < 1,$$

which is satisfied for each $(r_1,r_2) \in (0,1]\times(0,1]$. Consequently Theorem 3.5 implies that (4.1)-(4.3) has a unique solution defined on $[0,1]\times[0,1]$.

References

[1] S. Abbas, R. P. Agarwal, M. Benchohra; Darboux problem for impulsive partial hyperbolic differential equations of fractional order with variable times and infinite delay, *Nonlinear Anal. Hybrid Syst.* **4** (2010), 818-829.

[2] S. Abbas, M. Benchohra; Partial hyperbolic differential equations with finite delay involving the Caputo fractional derivative, *Commun. Math. Anal.* **7** (2009), 62-72.

[3] S. Abbas, M. Benchohra; The method of upper and lower solutions for partial hyperbolic fractional order differential inclusions with impulses, *Discuss. Math. Differ. Incl. Control Optim.* **30** (1) (2010), 141-161.

[4] S. Abbas, M. Benchohra; Impulsive partial hyperbolic functional differential equations of fractional order with state-dependent delay, *Frac. Calc. Appl. Anal.* **13** (3) (2010), 225-244.

[5] S. Abbas, M. Benchohra, L. Gorniewicz; Existence theory for impulsive partial hyperbolic functional differential equations involving the Caputo fractional derivative, *Sci. Math. Jpn.*. online e- 2010, 271-282.

[6] S. Abbas, M. Benchohra, J. J. Nieto; Global uniqueness results for fractional order partial hyperbolic functional differential equations, *Adv. Differ. Equations* (2011), Article ID 379876, 25 pages doi:10.1155/2011/379876.
[7] S. Abbas, M. Benchohra, A. N. Vityuk; On fractional order derivatives and Darboux problem for implicit differential equations. (Submitted).
[8] A. Belarbi, M. Benchohra, A. Ouahab; Uniqueness results for fractional functional differential equations with infinite delay in Fréchet spaces, *Appl. Anal.* **85** (2006), 1459-1470.
[9] M. Benchohra, J. R. Graef, S. Hamani; Existence results for boundary value problems of nonlinear fractional differential equations with integral conditions, *Appl. Anal.* **87** (7) (2008), 851-863.
[10] M. Benchohra, S. Hamani, S. K. Ntouyas; Boundary value problems for differential equations with fractional order, *Surv. Math. Appl.* **3** (2008), 1-12.
[11] M. Benchohra, J. Henderson, S. K. Ntouyas; *Impulsive Differential Equations and Inclusions*, Hindawi Publishing Corporation, Vol 2, New York, 2006.
[12] M. Benchohra, J. Henderson, S. K. Ntouyas, A. Ouahab; Existence results for functional differential equations of fractional order, *J. Math. Anal. Appl.* **338** (2008), 1340-1350.
[13] K. Diethelm, N. J. Ford; Analysis of fractional differential equations, *J. Math. Anal. Appl.* **265** (2002), 229-248.
[14] W. G. Glockle, T. F. Nonnenmacher; A fractional calculus approach of selfsimilar protein dynamics, *Biophys. J.* **68** (1995), 46-53.
[15] A. Granas and J. Dugundji; *Fixed Point Theory*, Springer-Verlag, New York, 2003.
[16] R. Hilfer; *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
[17] A. A. Kilbas, Hari M. Srivastava, Juan J. Trujillo; *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.
[18] A. A. Kilbas, S. A. Marzan; Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions, *Differential Equations* **41** (2005), 84-89.
[19] V. Lakshmikantham, D. D. Bainov, P. S. Simeonov; *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.
[20] F. Mainardi; Fractional calculus: Some basic problems in continuum and statistical mechanics, in "Fractals and Fractional Calculus in Continuum Mechanics" (*A. Carpinteri and F. Mainardi, Eds*), pp. 291-348, Springer-Verlag, Wien, 1997.
[21] F. Metzler, W. Schick, H. G. Kilian, T. F. Nonnenmacher; Relaxation in filled polymers: A fractional calculus approach, *J. Chem. Phys.* **103** (1995), 7180-7186.
[22] K. S. Miller, B. Ross; *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.
[23] K. B. Oldham, J. Spanier; *The Fractional Calculus*, Academic Press, New York, London, 1974.
[24] I. Podlubny; *Fractional Differential Equation*, Academic Press, San Diego, 1999.
[25] I. Podlubny, I. Petraš, B. M. Vinagre, P. O'Leary, L. Dorčak; Analogue realizations of fractional-order controllers. fractional order calculus and its applications, *Nonlinear Dynam.* **29** (2002), 281-296.
[26] S. G. Samko, A. A. Kilbas and O. I. Marichev; *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.
[27] A. N. Vityuk; Existence of solutions of partial differential inclusions of fractional order, *Izv. Vyssh. Uchebn. , Ser. Mat.* **8** (1997), 13-19.
[28] A. N. Vityuk, A. V. Golushkov; Existence of solutions of systems of partial differential equations of fractional order, *Nonlinear Oscil.* **7** (3) (2004), 318-325.
[29] A. N. Vityuk, A. V. Mykhailenko; On a class of fractional-order differential equation, *Nonlinear Oscil.* **11** (3), (2008), 307-319.
[30] A. N. Vityuk, A. V. Mykhailenko; The Darboux problem for an implicit fractional-order differential equation, *J. Math. Sci.* **175** (4) (2011), 391-401.
[31] C. Yu and G. Gao; Existence of fractional differential equations, *J. Math. Anal. Appl.* **310** (2005), 26-29.
[32] S. Zhang; Positive solutions for boundary-value problems of nonlinear fractional diffrential equations, *Electron. J. Differential Equations* 2006, No. 36, pp, 1-12.

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