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OTT-SUDAN-OSTROVSKIY EQUATIONS ON A RIGHT HALF-LINE

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Abstract. We consider initial-boundary value problems for the Ott-Sudan-Ostrovskiy equation on a right half-line. We show the the existence of solutions, global in time, and study their asymptotic behavior for large time.

1. Introduction

This article is devoted to the study of the initial-boundary value problem for the Ott-Sudan-Ostrovskiy equation on the right half-line,

$$\begin{gathered} u_t + uu_x + \alpha u_{xxx} + \int_0^{+\infty} \frac{\operatorname{sign}(x-y)u_y(y,t)}{\sqrt{|x-y|}} dy = 0, \quad t > 0,\ x > 0, \\ u(x,0) = u_0(x), \quad x > 0, \\ u(0,t) = 0, \quad t > 0, \end{gathered} \tag{1.1}$$

where $\alpha > 0$.

The Ott-Sudan-Ostrovskiy equation is a simple universal model equation which appears as the first approximation in the description of the ion-acoustic waves in plasma [16, 19]. We study traditionally important questions in the theory of nonlinear partial differential equations, such as global in time existence of solutions to the initial-boundary value problem (1.1)) and the asymptotic behavior of solutions for large time.

Many publications have dealt with asymptotic representations of solutions to the Cauchy problem for nonlinear evolution equations in the previous twenty years. While not attempting to provide a complete review of these publications, we do list some known results: [2, 3, 5, 6, 8, 10], where, in particular, the optimal time decay estimates and asymptotic formulas of solutions to different nonlinear local and nonlocal dissipative equations were obtained. In the case of dispersive equations some progress in the asymptotic methods was achieved due to the discovery of the Inverse Scattering Transform method (see books [1, 17]). Some other functional analytic methods were applied for the study of the large time asymptotic behavior of solutions to dispersive equations in [4, 9, 14, 18].

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The theory of the initial-boundary value problems on a half-line for the nonlinear non local equations is relatively young and traditional questions of a general theory are far from their conclusion ([15, 20]). There are many open natural questions which we need to study. First of them is how many boundary data should be posed in the initial-boundary value problems for it's correct solvability. Another difficulty of the non local equation on a half-line is due to the influence of the boundary data. A description of the large time asymptotic behavior of solutions requires new approach and some reorientation of the points of view comparing with the Cauchy problem. For example, comparing with the corresponding case of the Cauchy problem, solutions can obtain rapid oscillations, can converge to a self-similar profile, can grow or decay faster, and so on. So every type of the nonlinearity and boundary data should be studied individually. For the general theory of nonlinear pseudodifferential equations on a half-line we refer to the book [11]. This book is the first attempt to develop systematically a general theory of the initial-boundary value problems for nonlinear evolution equations on a half-line, where pseudodifferential operator K on a half-line was introduced by virtue of the inverse Laplace transformation of the product of the symbol $K(p) = O(p^{\beta})$ which is analytic in the right complex half-plane, and the Laplace transform of the derivative $\partial_x^{[\beta]} u$. The main difficulty in the boundary value problem (1.1) is that the operator

$$Ku = \alpha u_{xxx} + \int_0^{+\infty} \frac{\operatorname{sign}(x-y) u_y(y,t)}{\sqrt{|x-y|}} dy$$

in equation (1.1) has a symbol $K(p) = \alpha p^3 + |p|^{1/2}$, which is non analytic. Thus we can not use methods of the book [11] directly. Also the order of the first term of the symbol $K(p)$ is critical, since the number of the boundary data depends also on the sign of α (see [11]). In paper [13] the initial-boundary value problem for the nonlinear nonlocal Ott-Sudan-Ostrovskiy type on the left half-line ($\alpha < 0$) was studied. It was proved that because of the negative sign of the term u_{xxx} two boundary conditions have to be imposed at $x = 0$. In the present paper we develop the theory of the Ott-Sudan-Ostrovskiy equation (1.1) considering the case of the right half-line and $\alpha > 0$. We will show below that only one boundary value is necessary and sufficient to pose in the problem (1.1) for its solvability and uniqueness. Our approach here is based on the L^p estimates of the Green function. For constructing the Green operator in the present paper we follow the idea of paper [12], reducing the linear problem (1.1) to the corresponding Riemann problem. The Laplace transform requires the boundary data $u(0,t)$, $u_x(0,t)$, $u_{xx}(0,t)$ and so $u_x(0,t)$ and $u_{xx}(0,t)$ should be determined by the given data. To achieve this we need to solve the system of nonlinear singular integro-differential equations with Hilbert kernel. We believe that the method developed in this paper could be applicable to a wide class of dissipative nonlinear non local equations.

To state precisely the results of the present paper we give some notation. We denote $\langle t\rangle = 1 + t$, $\{t\} = t/\langle t\rangle$. Direct Laplace transformation $\mathcal{L}_{x\to\xi}$ is

$$\widehat{u}(\xi) \equiv \mathcal{L}_{x\to\xi} u = \int_0^{+\infty} e^{-\xi x} u(x)\, dx$$

and the inverse Laplace transformation $\mathcal{L}^{-1}_{\xi\to x}$ is defined by

$$u(x) \equiv \mathcal{L}^{-1}_{\xi\to x}\widehat{u} = (2\pi i)^{-1}\int_{-i\infty}^{i\infty} e^{\xi x} u(\xi)d\xi.$$

Weighted Lebesgue space $L^{q,a}(\mathbb{R}^+) = \{\varphi \in \mathcal{S}'; \|\varphi\|_{L^{q,a}} < \infty\}$, where

$$\|\varphi\|_{L^{q,a}} = \Big(\int_0^{+\infty}(1+x)^{aq}|\varphi(x)|^q dx\Big)^{1/q}$$

for $a > 0$, $1 \le q < \infty$ and

$$\|\varphi\|_{L^\infty} = \operatorname{ess\,sup}_{x\in\mathbb{R}^+}|\varphi(x)|$$

By $\Phi^{\pm}$ we denote a left and right limiting values of sectionally analytic function Φ given by integral of Cauchy type

$$\Phi(z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\phi(q)}{q-z}dq$$

All the integrals are understood in the sense of the Cauchy principal value. We introduce $\Lambda(s) \in L^\infty(\mathbb{R}^+)$ by formula

$$\begin{aligned}\Lambda(s) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{ps-|p|^{1/2}}dp - \big(\frac{1}{2\pi i}\big)^3\int_{-i\infty}^{i\infty}d\xi e^{\xi}\int_0^{+\infty} e^{-ps} \\ &\quad\times \frac{\sqrt{p}(p^2+|\xi|^4)^{1/8}}{(\sqrt{ip}+\xi)(\sqrt{-ip}+\xi)}dp\int_0^\infty\frac{1}{q+p}\frac{1}{(q^2+|\xi|^4)^{1/8}}dq.\end{aligned} \tag{1.2}$$

We define the linear operator

$$f(\phi) = \int_0^{+\infty}\phi(y)dy. \tag{1.3}$$

Now we state our main result.

Theorem 1.1. *Suppose that for small $a > 0$ the initial data*

$$u_0 \in Z = L^1(\mathbb{R}^+) \cap L^{1,a}(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$$

are such that $\|u_0\|_Z \le \varepsilon$ is sufficiently small. Then there exists a unique global solution

$$u \in C([0,\infty); L^1(\mathbb{R}^+) \cap L^{1,a}(\mathbb{R}^+)) \cap C((0,\infty); L^\infty(\mathbb{R}^+))$$

to the initial-boundary value problem (1.1). *Moreover*

$$u = A\Lambda(xt^{-2})t^{-2} + O(t^{-2-\gamma}), \tag{1.4}$$

as $t \to \infty$ in L^∞, where $\gamma \in (0, \min(1,a))$ and

$$A = f(u_0) - \int_0^{+\infty} f(u_x u))d\tau.$$

2. Preliminaries

We consider the following linear initial-boundary value problem on half-line

$$\begin{gathered} u_t+\alpha u_{xxx}+\int_0^{+\infty}\frac{\operatorname{sign}(x-y)u_y(y,t)}{\sqrt{|x-y|}}dy=0,\quad t>0,\ x>0,\\ u(x,0)=u_0(x),\quad x>0,\\ u(0,t)=0,\quad t>0. \end{gathered}\tag{2.1}$$

Denote

$$K(q)=\alpha q^3+|q|^{1/2},\quad K_1(q)=\alpha q^3+q^{1/2},\quad K_1(k(\xi))=-\xi,\tag{2.2}$$

where $\operatorname{Re}k(\xi)>0$ for $\operatorname{Re}\xi>0$. We define

$$\mathcal{G}(t)\phi=\int_0^{+\infty}G(x,y,t)\phi(y)dy,\tag{2.3}$$

where

$$\begin{aligned} &G(x,y,t)\\ &=\big(\frac{1}{2\pi i}\big)^2\int_{-i\infty}^{i\infty}d\xi e^{\xi t}\int_{-i\infty}^{i\infty}dpe^{px}\frac{Y^-(p,\xi)}{K_1(p)+\xi}(p-k_1)(p-k_2)I^-(p,\xi,y) \end{aligned}\tag{2.4}$$

for $x>0$, $y>0$, $t>0$. Here and below

$$Y^{\pm}=e^{\Gamma^{\pm}},\tag{2.5}$$

$\Gamma^+(p,\xi)$ and $\Gamma^-(p,\xi)$ are a left and right limiting values of sectionally analytic function

$$\Gamma(z,\xi)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\ln\frac{K(q)+\xi}{K_1(q)+\xi}dq,\tag{2.6}$$

and

$$I(z,\xi,y)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{-qy}}{q-z}\frac{1}{(q-k_2)(q-k_1)}\frac{1}{Y^+(q,\xi)}dq.\tag{2.7}$$

All the integrals are understood in the sense of the principal values.

Proposition 2.1. *Let the initial data u_0 be in L^1. Then there exists a unique solution $u(x,t)$ of the initial-boundary value problem* (2.1)*, which has integral representation*

$$u(x,t)=\mathcal{G}(t)u_0.\tag{2.8}$$

Proof. To derive an integral representation for the solutions of (2.1) we suppose that there exists a solution $u(x,t)$ of problem (2.1), which is continued by zero outside of $x>0$:

$$u(x,t)=0\quad\text{for all }x<0.$$

Let $\phi(p)$ be a function of the complex variable $\operatorname{Re}p=0$, which obeys the Hölder condition for all finite p and tends to 0 as $p\to\pm i\infty$. We define the operator

$$\mathbb{P}\phi(z)=-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{p-z}\phi(p)dp.$$

Note that $\mathbb{P}\phi(z)=F(z)$ constitutes a function analytic n the left and right semi-planes. Here and below these functions will be denoted $F^+(z)$ and $F^-(z)$, respectively. These functions have the limiting values $F^+(p)$ and $F^-(p)$ at all points of imaginary axis $\operatorname{Re}p=0$, on approaching the contour from the left and from the right, respectively. These limiting values are expressed by Sokhotzki-Plemelj formula

$$F^{+}(p)-F^{-}(p)=\phi(p). \tag{2.9}$$

We have for the Laplace transform

$$\mathcal{L}_{x\to p}\Big\{\int_0^{+\infty}\frac{\operatorname{sign}(x-y)u_y(y,t)}{\sqrt{|x-y|}}dy\Big\}=\mathbb{P}\Big\{|p|^{1/2}(\mathcal{L}_{x\to p}\{u\}-\frac{u(0,t)}{p})\Big\}.$$

Since $\mathcal{L}_{x\to q}\{u\}$ is analytic for all $\operatorname{Re} q>0$ we have

$$\widehat{u}(q,t)=\mathcal{L}_{x\to q}\{u\}=\mathbb{P}\{\widehat{u}(p,t)\} \tag{2.10}$$

and

$$\mathcal{L}_{x\to q}\{\alpha u_{xxx}\}=\mathbb{P}\Big\{\alpha p^3\Big(\widehat{u}(p,t)-\sum_{j=1}^{3}\frac{\partial_x^{j-1}u(0,t)}{p^j}\Big)\Big\}.$$

Therefore, applying the Laplace transform to (2.1) with respect to space and time variables we obtain for $\operatorname{Re} p=0$, $t>0$

$$\begin{gathered}\widehat{\widehat{u}}(p,\xi)=\frac{1}{K(p)+\xi}\big(\Xi(p,\xi)+\widehat{\Phi}(p,\xi)\big)\\ \Xi(p,\xi)=\widehat{u}_0(p)+\alpha p\widehat{u}_x(0,\xi)+\alpha\widehat{u}_{xx}(0,\xi).\end{gathered} \tag{2.11}$$

with some function $\widehat{\Phi}(p,\xi)=O(p^{-1/2})$ such that for all $\operatorname{Re} z>0$

$$\mathbb{P}^{-}\{\widehat{\Phi}\}(z,\xi)=0 \tag{2.12}$$

Here the functions $\widehat{\widehat{u}}(p,\xi)$, $\widehat{\Phi}(p,\xi)$, $\widehat{u}_x(0,\xi)$ and $\widehat{u}_{xx}(0,\xi)$ are the Laplace transforms for $\widehat{u}(p,t)$, $\Phi(p,t)$, $u_x(0,t)$ and $u_{xx}(0,t)$ with respect to time, respectively. We will find the function $\widehat{\Phi}(p,\xi)$ using the analytic properties of function $\widehat{\widehat{u}}$ in the right-half complex planes $\operatorname{Re} p>0$ and $\operatorname{Re}\xi>0$. For $\operatorname{Re} p=0$, we have

$$\widehat{\widehat{u}}(p,\xi)=-\frac{1}{\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-p}\widehat{\widehat{u}}(q,\xi)dq. \tag{2.13}$$

As in [12] we perform the condition (2.13) in the form of nonhomogeneous Riemann problem to find

$$\widehat{\Phi}(p,\xi)=-Y^{+}(p,\xi)U^{+}(p,\xi),$$

where

$$U(z,\xi)=\mathbb{P}\Big\{\frac{1}{Y^{+}(p,\xi)}\frac{K_1(p)-K(p)}{(K(p)+\xi)(K_1(p)+\xi)}\Xi(p,\xi)\Big\}. \tag{2.14}$$

We now return to solution $u(x,t)$ of the problem (2.1). From (2.11) with the help of the integral representation (2.14) and Sokhotzki-Plemelj formula (2.9) we have for Laplace transform of solution of the problem (2.1)

$$\widehat{\widehat{u}}(p,\xi)=\frac{1}{K_1(p)+\xi}(\widehat{u}_0(p)+\alpha(p\widehat{u}_x(0,\xi)+\widehat{u}_{xx}(0,\xi))-Y^{-}U^{-}). \tag{2.15}$$

There exist two roots $k_j(\xi)$ of equation $K_1(z)=-\xi$ such that $\operatorname{Re} k_j(\xi)>0$ for all $\operatorname{Re}\xi>0$. Therefore in the expression for the function $\widehat{\widehat{u}}$ the factor $1/(K_1(z)+\xi)$ has two poles in the point $z=k_j(\xi)$, $j=1,2$, $\operatorname{Re} z>0$. However the function $\widehat{\widehat{u}}$ is the limiting value of an analytic function in $\operatorname{Re} z>0$. Thus in general case the problem (2.1) is insolvable. It is soluble only when the functions $U^{-}(z,\xi)$ satisfies additional conditions. For analyticity of $\widehat{u}(z,\xi)$ in points $z = k_j(\xi)$ it is necessary that

$$\mathrm{Res}_{z=k_j(\xi)}\{\widehat{u}_0(z) + \alpha(z\widehat{u}_x(0,\xi) + \widehat{u}_{xx}(0,\xi)) - Y^-U^-\} = 0, \quad j = 1,2. \tag{2.16}$$

Therefore, we obtain for the Laplace transforms of boundary data $u_x(0,t)$, $u_{xx}(0,t)$ the following system

$$B_1(k_j(\xi),\xi)\widehat{u}_x(0,\xi) + B_2(k_j(\xi),\xi)\widehat{u}_{xx}(0,\xi) = \frac{1}{\alpha}\{\mathcal{I}_3\widehat{u}_0\}(k_j), \quad j = 1,2, \tag{2.17}$$

where

$$B_1(k_j(\xi),\xi) = k_j(\xi) - Y^-(k_j(\xi),\xi)\lim_{y\to 0}\partial_y\Upsilon_1(k_j(\xi),\xi,y), \tag{2.18}$$

$$B_2(k_j(\xi),\xi) = 1 - Y^-(k_j(\xi),\xi)\lim_{y\to 0}\Upsilon_1(k_j(\xi),\xi,y), \tag{2.19}$$

$$\{\mathcal{I}_3\widehat{u}_0\}(k_j) = \int_0^{+\infty} dy u_0(y)\Big(Y^-(k_j(\xi),\xi)\Upsilon_1(k_j(\xi),\xi,y) - e^{-k_j(\xi)y}\Big), \tag{2.20}$$

where

$$\Upsilon_1(z,\xi,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{1}{Y^+(q,\xi)}\frac{K_1(q)-K(q)}{K_1(q)+\xi}e^{-qy}dq. \tag{2.21}$$

To solve this system firstly we consider the sectionally analytic function $\Upsilon_1(z,\xi,y)$ given by Cauchy type integral. Since, by (2.9),

$$\frac{1}{Y^+(q,\xi)}\frac{K(q)+\xi}{K_1(q)+\xi} = \frac{1}{Y^-(q,\xi)}, \tag{2.22}$$

we have

$$\Upsilon_1(z,\xi,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\Big(\frac{1}{Y^+(q,\xi)} - \frac{1}{Y^-(q,\xi)}\Big)e^{-qy}dq \tag{2.23}$$

Observe that the function $1/\big(Y^-(q,\xi)\big)$ is analytic for all $\mathrm{Re}\, q > 0$. Therefore, by the Cauchy Theorem for $y > 0$, we find

$$-\lim_{z\to p,\ \mathrm{Re}\, z>0}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{1}{Y^-(q,\xi)}e^{-qy}dq = \frac{1}{Y^-(p,\xi)}e^{-py}. \tag{2.24}$$

Thus from (2.23) and (2.24), we obtain the relation

$$\Upsilon_1^-(p,\xi,y) = \Psi^-(p,\xi,y) + \frac{1}{Y^-(p,\xi)}e^{-py}, \tag{2.25}$$

where

$$\Psi(z,\xi,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{1}{Y^+(q,\xi)}e^{-qy}dq.$$

Therefore,

$$B_1(k_j(\xi),\xi) = Y^-(k_j(\xi),\xi)\partial_y\Psi(k_j,\xi,0), \quad j = 1,2,$$
$$B_2(k_j(\xi),\xi) = -Y^-(k_j(\xi),\xi)\Psi(k_j,\xi,0), \quad j = 1,2$$

and

$$\begin{aligned}\{\mathcal{I}_3\widehat{u}_0\}(k_j) &= \int_0^{+\infty} dy u_0(y)\big[Y^-(k_j(\xi),\xi)\Upsilon_1^-(k_j,\xi,y) - e^{-k_j(\xi)y}\big] \\ &= Y^-(k_j(\xi),\xi)\int_0^{+\infty} dy u_0(y)\Psi(k_j(\xi),\xi,y), \quad j = 1,2.\end{aligned}$$

Here and below

$$\partial_y\Psi(z,\xi,0)=\lim_{y\to 0}\partial_y\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{-zy}}{q-z}\frac{1}{Y^+(q,\xi)}dq,$$
$$\Psi(z,\xi,0)=\lim_{y\to 0}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{-zy}}{q-z}\frac{1}{Y^+(q,\xi)}dq,$$

Substituting these formulas in (2.17) we obtain

$$\begin{aligned}&\widehat{u}_x(0,\xi)\\&=\frac{1}{\alpha}\int_0^{+\infty}dyu_0(y)\frac{\Psi(k_1,\xi,0)\Psi(k_2(\xi),\xi,y)-\Psi(k_2,\xi,0)\Psi(k_1(\xi),\xi,y)}{\partial_y\Psi(k_2,\xi,0)-\partial_y\Psi(k_1,\xi,0)}\end{aligned}\tag{2.26}$$

and

$$\begin{aligned}&\widehat{u}_{xx}(0,\xi)\\&=\frac{1}{\alpha}\int_0^{+\infty}dyu_0(y)\frac{\partial_y\Psi(k_1,\xi,0)\Psi(k_2(\xi),\xi,y)-\partial_y\Psi(k_2,\xi,0)\Psi(k_1(\xi),\xi,y)}{\partial_y\Psi(k_2,\xi,0)-\partial_y\Psi(k_1,\xi,0)}.\end{aligned}\tag{2.27}$$

Now we return to formula (2.15). In accordance with (2.26)–(2.27), the function $\widehat{\widehat{u}}(p,\xi)$ constitutes the limiting value of an analytic function in $\operatorname{Re} z>0$ and, as a consequence, its inverse Laplace transform vanish for all $x<0$.

Under assumption $u(0,t)=0$, via definition (2.21) the integral representation (2.14) for the function $U^-(p,\xi)$, in $\operatorname{Re} z>0$, takes form

$$\begin{aligned}&U^-(p,\xi)\\&=\int_0^{+\infty}dyu_0(y)\Upsilon_1^-(p,\xi,y)-\alpha\widehat{u}_x(0,\xi)\partial_y\Upsilon_1^-(p,\xi,0)+\alpha\Upsilon_1^-(p,\xi,0)\widehat{u}_{xx}(0,\xi),\end{aligned}$$

where the function $\Upsilon_1(z,\xi,y)$ is defined by (2.23). Using (2.25) from (2.15) we obtain

$$\begin{aligned}\widehat{\widehat{u}}=&\int_0^{\infty}u_0(y)\frac{Y^-(p,\xi)}{K_1(p)+\xi}\Psi^-(p,\xi,y)dy\\&+\alpha\widehat{u}_x(0,\xi)\partial_y\Psi^-(p,\xi,0)-\alpha\Psi^-(p,\xi,0)\widehat{u}_{xx}(0,\xi)\end{aligned}\tag{2.28}$$

where $\widehat{u}_x(0,\xi)$ and $\widehat{u}_{xx}(0,\xi)$ are defined by (2.26)–(2.27)

$$\Psi(z,\xi,y)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{-qy}}{q-z}\frac{1}{Y^+(q,\xi)}dq.$$

Applying to (2.28) the inverse Laplace transform with respect to time, and the inverse Fourier transform with respect to space variables, we obtain

$$u(x,t)=\mathcal{G}(t)u_0=\int_0^{\infty}G(x,y,t)u_0(y)dy,$$

where for $x>0$, $y>0$, $t>0$, the $G(x,y,t)$ was defined by

$$\begin{aligned}&G(x,y,t)\\&=\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\xi e^{\xi t}\int_{-i\infty}^{i\infty}dpe^{px}\frac{Y^-}{K_1(p)+\xi}\Big(\Psi^-(p,\xi,y)\\&\quad+\frac{(\partial_y\Psi^-(p,\xi,0)-\widetilde{\Psi}(k_2))\Psi(k_1,\xi,y)-(\partial_y\Psi^-(p,\xi,0)-\widetilde{\Psi}(k_1))\Psi(k_2,\xi,y)}{\partial_y\Psi(k_2,\xi,0)-\partial_y\Psi(k_1,\xi,0)}\Big),\end{aligned}\tag{2.29}$$

where

$$\widetilde{\Psi}(k_j) = \Psi(k_j,\xi,0)\partial_y\Psi(k_j,\xi,0), \quad j=1,2,$$
$$\Psi(z,\xi,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{-qy}}{q-z}e^{-\Gamma^+(q,\xi)}dq,$$
$$\Gamma(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\ln\frac{K(q)+\xi}{K_1(q)+\xi}dq.$$

For subsequent considerations it is required to investigate the behavior of the function $\Gamma(z,\xi)$. Set

$$\phi(p,\xi) = \ln\frac{K(p)+\xi}{K_1(p)+\xi} \neq 0, \quad \operatorname{Re} p = 0, \quad \operatorname{Re}\xi \geq 0.$$

Observe that the function $\phi(p,\xi)$ satisifes the Hölder condition for all finite p and tends to a definite limit $\phi(\infty,\xi)$ as $p \to \pm i\infty$,

$$\phi(\infty,\xi) = 0.$$

It can be easily obtained that for large p and some fixed ξ,

$$|\phi(p,\xi) - \phi(\infty,\xi)| \leq C\frac{\xi}{\langle|p|\rangle^3}. \tag{2.30}$$

Therefore,

$$|e^{\Gamma^{\pm}(z,\xi)}| \leq C \tag{2.31}$$

for all $\operatorname{Re}\xi \geq 0$. Moreover we have

$$\frac{1}{q-z} = -\frac{1}{z} - \frac{q}{z^2} + \frac{q^2}{z^2(q-z)}.$$

and therefore

$$\Gamma(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\ln(\frac{K(q)+\xi}{K_1(q)+\xi})dq = A_1(\xi)\frac{1}{z} + O(|z|^{-2}), \tag{2.32}$$

where

$$A_1(\xi) = \int_{-i\infty}^{i\infty}\ln\big\{\frac{K(q)+\xi}{K_1(q)+\xi}\big\}dq. \tag{2.33}$$

In view of (2.30), for large $|q|$, the integrand in (2.32) is of order $|q|^{-3}$; whence the corresponding integrals with infinite limits are convergent, in accordance with well-known criterion for convergence improper integrals.

Using the above estimate, for $|z| > 1$, we obtain

$$e^{-\Gamma^+(z,\xi)} - 1 - A_1(\xi)\frac{1}{z} = e^{-A_1(\xi)/z + O(|z|^{-2})} - 1 + A_1(\xi)\frac{1}{z} = O(|z|^{-2}).$$

In view of this fact and the Cauchy Theorem, we obtain

$$\begin{aligned}\Psi^-(p,\xi,0) &= \frac{1}{2\pi i}\lim_{y\to 0}\lim_{z\to p,\,\operatorname{Re} z>0}\int_{-i\infty}^{i\infty}\frac{e^{-qy}}{q-z}\big(\frac{1}{Y^+(q,\xi)} - 1\big)dq \\ &\quad + \frac{1}{2\pi i}\lim_{y\to 0}\lim_{z\to p,\,\operatorname{Re} z>0}\int_{-i\infty}^{i\infty}\frac{e^{-qy}}{q-z}dq \\ &= -1\end{aligned}$$

and $\Psi(k_j,\xi,0)=-1$ for $j=1,2$. In the same way

$$\begin{aligned}\partial_y\Psi^-(p,\xi,0)&=\frac{1}{2\pi i}\lim_{z\to p,\,\mathrm{Re}\,z>0}\int_{-i\infty}^{i\infty}\frac{q}{q-z}\big(\frac{1}{Y^+(q,\xi)}-1+A_1(\xi)\frac{1}{q}\big)dq\\ &\quad-\lim_{y\to 0}\partial_y\lim_{z\to p,\mathrm{Re}\,z>0}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{-qy}}{q-z}\big(\frac{A_1(\xi)}{q}-1\big)dq\\ &=-A_1(\xi)+p.\end{aligned}$$

and

$$\partial_y\Psi(k_j,\xi,0)=-A_1(\xi)+k_j, j=1,2.$$

Thus after some calculation we attain

$$\begin{aligned}&[\partial_y\Psi(k_2,\xi,0)-\partial_y\Psi(k_1,\xi,0)]^{-1}\\ &\times(\partial_y\Psi^-(p,\xi,0)-\Psi(k_2,\xi,0)\partial_y\Psi(k_2,\xi,0))\Psi(k_1,\xi,y)\\ &-(\partial_y\Psi^-(p,\xi,0)-\Psi(k_1,\xi,0)\partial_y\Psi(k_1,\xi,0))\Psi(k_2,\xi,y))\\ &=\frac{(p-k_2)\Psi(k_1,\xi,y)-(p-k_1)\Psi(k_2,\xi,y)}{k_2(\xi)-k_1(\xi)}.\end{aligned}$$

Substituting the above relation in (2.29), we obtain (2.4). The proof is complete. □

Now we collect some preliminary estimates of the Green operator $\mathcal{G}(t)$ defined in (2.3).

Lemma 2.2. *The following estimates are true, provided that the right-hand sides are finite*

$$\|\partial_x^n\mathcal{G}(t)\phi\|_{L^{s,\mu}}\le C\widetilde{t}^{-(\frac{1}{r}-\frac{1+\mu}{s})-n}\|\phi(\cdot)\|_{L^r}+\widetilde{t}^{-(\frac{1}{r}-\frac{1}{s})-n}\|\phi(\cdot)\|_{L^{r,\mu}},\tag{2.34}$$

$$\|(\mathcal{G}(t)\phi-t^{-2}\Lambda(xt^{-2})f(\phi))\|_{L^\infty}\le Ct^{-2-2\mu}\|(\cdot)^\mu\phi\|_{L^1},\tag{2.35}$$

where $\widetilde{t}=\{t\}^{1/3}\langle t\rangle^2$*, small* $\mu>0$*,* $1\le r\le s\le\infty$*,* $n=0,1$*,* $\Lambda(s)$ *is given by* (1.2) *and* $f(\phi)$ *is given by* (1.3).

Proof. By Sokhotzki-Plemelj formula we have

$$I^-(p,\xi,y)=I^+(p,\xi,y)-\frac{e^{-py}}{Y^+(p,\xi)}.$$

Inserting this expression in (2.4), we have

$$G(x,y,t)=J_1(x,y,t)+J_2(x,y,t),\tag{2.36}$$

where

$$J_1(x,y,t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{px-K(p)t}e^{-py}dp\tag{2.37}$$

and

$$\begin{aligned}&J_2(x,y,t)\\ &=-(\frac{1}{2\pi i})^2\int_{-i\infty}^{i\infty}d\xi e^{\xi t}\int_{-i\infty}^{i\infty}e^{px}\frac{Y^+(p,\xi)(p-k_1(\xi))(p-k_2(\xi))}{K(p)+\xi}I^+(p,\xi,y)dp.\end{aligned}\tag{2.38}$$

Denote

$$\mathcal{J}_j(x,t)\phi=\int_0^{+\infty}J_j(x,y,t)\phi(y)dy$$

for $x > 0$. From [10] we have

$$\|\partial_x^n \mathcal{J}_1(t)\phi\|_{L^{s,\mu}} \leq C\widetilde{t}^{-(\frac{1}{r}-\frac{1+\mu}{s})-n}\|\phi(\cdot)\|_{L^r} + \widetilde{t}^{-(\frac{1}{r}-\frac{1}{s})-2n}\|\phi(\cdot)\|_{L^{r,\mu}}$$

for $n = 0, 1, 2,\ s \geq r,\ \mu \in (0, 1)$. Let the contours $\mathcal{C}_i$ be defined as

$$\mathcal{C}_i = \{p \in (\infty e^{-i(\frac{\pi}{2}-(-1)^i\varepsilon_i)}, 0) \cup (0, \infty e^{i(\frac{\pi}{2}-(-1)^i\varepsilon_i)})\}, i = 1, 2, 3, \tag{2.39}$$

where $\varepsilon_j > 0$ are small enough, can be chosen such that all functions under integration are analytic and $\operatorname{Re} k_j(\xi) > 0,\ j = 1, 2$ for $\xi \in \mathcal{C}_3$. In particular, for example, $K(p) + \xi \neq 0$ outside the origin for all $p \in \mathcal{C}_1$ and $\xi \in \mathcal{C}_3$.

Firstly we estimate $\mathcal{J}_1(t)\phi$. Let $x - y > 0$. Let $x - y < 0$. We can rewrite (2.37) in the form

$$J_1(x, y, t) = \frac{1}{2\pi i}\int_{\mathcal{C}_1} e^{\xi t} d\xi \frac{1}{2\pi i}\int_{\mathcal{C}_2} e^{-p(y-x)}\frac{1}{K(p) + \xi}dp \tag{2.40}$$

By the choice of contour $\mathcal{C}_2$ we have $|e^{-p(y-x)}| \leq Ce^{-C|p||x-y|}$. Therefore, using obvious estimates:

$$\|e^{-|p||\cdot|}\|_{L^{s,\mu}} \leq C|p|^{-\frac{1+s\mu}{s}}, \tag{2.41}$$

$$\|\int_0^{+\infty} e^{-C|p||x-y|}\phi(y)dy\|_{L^{s,\mu}} \leq C|p|^{-1+\frac{1}{r}-\frac{1+s\mu}{s}}\|\phi\|_{L^r} + C|p|^{-1+\frac{1}{r}-\frac{1}{s}}\|\phi\|_{L^{r,\mu}}$$

for all $s, r \geq 1$, for $p \in \mathcal{C}_2$, we obtain

$$\begin{aligned}\|\partial_x^n \mathcal{J}_1(t)\phi\|_{L^{s,\mu}} &\leq C\|\phi\|_{L^r}\int_{\mathcal{C}_1} e^{-C|\xi|t}d\xi \int_{\mathcal{C}_2}\frac{1}{|K(p) + \xi|}|p|^{-1+n+\frac{1}{r}-\frac{1+s\mu}{s}}dp \\ &\quad + C\|\phi\|_{L^{r,\mu}}\int_{\mathcal{C}_1} e^{-C|\xi|t}d\xi \int_{\mathcal{C}_2}\frac{1}{|K(p) + \xi|}|p|^{-1+\frac{1}{r}+n-\frac{1}{s}}dp \\ &\leq C\widetilde{t}^{-(\frac{1}{r}-\frac{1+\mu}{s})-n}\|\phi(\cdot)\|_{L^r} + \widetilde{t}^{-(\frac{1}{r}-\frac{1}{s})-n}\|\phi(\cdot)\|_{L^{r,\mu}}\end{aligned} \tag{2.42}$$

for $n = 0, 1,\ r > s,\ \mu \in (0, 1)$. In the case of $x - y > 0$ we use the fact that for $p \in \mathcal{C}_3$, $\operatorname{Re} K(p) > 0$ and

$$\operatorname{Re} K(p) = O(\{p\}^{1/2}\langle p\rangle^3).$$

Therefore, changing in formula (2.37) contour of integration by $\mathcal{C}_3$ from the book [10] we obtain estimate (2.42).

Now we estimate $\mathcal{J}_2(t)\phi$. For any analytic in the left half-complex plane function $\phi^+(p)$ we have

$$\begin{aligned}&\int_{-i\infty}^{i\infty}\frac{1}{K(p) + \xi}\phi^+(p)dp \\ &= (e^{i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}})\int_0^{+\infty}\phi^+(-p)\frac{\sqrt{p}}{(\sqrt{ip} - p^3 + \xi)(\sqrt{-ip} - p^3 + \xi)}\end{aligned}$$

where $\operatorname{Re}\xi = 0$. Here

$$K(p) = \begin{cases}\alpha p^3 + (-ip)^{1/2}, & \operatorname{Im} p > 0;\\ \alpha p^3 + (ip)^{1/2}, & \operatorname{Im} p < 0.\end{cases}$$

Using the above relation, for $J_2(x,y,t)$ we obtain

$$\begin{aligned} J_2(x,y,t) = -(\frac{1}{2\pi i})^2 i\sqrt{2}\int_{\mathcal{C}_1} d\xi e^{\xi t}\int_0^\infty e^{-px} Z^+(p,\xi,y) \\ \times \frac{\sqrt{p}}{(\sqrt{ip}-p^3+\xi)(\sqrt{-ip}-p^3+\xi)}dp. \end{aligned} \tag{2.43}$$

where

$$Z^+(p,\xi,y) = Y^+(-p,\xi)(p+k_1(\xi))(p+k_2(\xi))I^+(-p,\xi,y).$$

To estimate $J_2(x,y,t)$ in the case $t<1$ we rewrite I^+ in the form

$$\begin{aligned} &I^+(p,\xi,y) \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}_2}\frac{e^{-qy}}{(q-p)(q-k_1(\xi))(q-k_2(\xi))}(\frac{1}{Y^-(q,\xi)}\frac{K_1(q)+\xi}{K(q)+\xi}-\frac{1}{Y^-(q,\xi)})dq \\ &\quad+\frac{1}{2\pi i}\int_{\mathcal{C}_2}\frac{e^{-qy}}{(q-p)(q-k_1(\xi))(q-k_2(\xi))}\frac{1}{Y^-(q,\xi)}dq \end{aligned}$$

Therefore, using analytic properties of integrand function in the second term by the Cauchy Theorem, we obtain

$$\begin{aligned} I^+(p,\xi,y) &= \frac{1}{(k_2(\xi)-k_1(\xi))}\sum_{j=1}^{2}(-1)^j\frac{e^{-k_j(\xi)y}}{(p-k_j(\xi))}\frac{1}{Y^-(k_j,\xi)} \\ &\quad+\frac{1}{2\pi i}\int_{\mathcal{C}_2}\frac{e^{-qy}}{(q-p)(q-k_1(\xi))(q-k_2(\xi))}\frac{1}{Y^-(q,\xi)}\frac{K_1(q)-K(q)}{K(q)+\xi}dq \end{aligned}$$

After this observation from the integral representation (2.37) by Holder inequality we have arrived at the following estimate for $r\geq 1$, $s\geq 1$, $l^{-1}=1-r^{-1}$, small $\mu\geq 0$, $n=0,1$, $t>1$,

$$\begin{aligned} &\|\partial_x^n\int_0^{+\infty} J_2(\cdot,y,t)\phi(y)dy\|_{L^{s,\mu}} \\ &\leq C\|\phi\|_{L^r}\int e^{-|\xi|t}d\xi\int_0^\infty dp|p|^{n-\frac{1+\mu}{s}+\frac12}\frac{|p+|\xi|^{\frac13}|^2}{|-p^3+\xi|^2} \\ &\quad\times\Big(\int_{\mathcal{C}_2}dq\frac{1}{|q-p||q|^{\frac12-\frac1r}}\frac{1}{|-q^3+\xi|}+\frac{1}{\langle|\xi|\rangle^{\frac13+1-\frac1r}}\Big) \\ &\leq C\|\phi\|_{L^r}t^{-(\frac1r-\frac{1+\mu}{s}+n)/3}. \end{aligned} \tag{2.44}$$

Here we used (2.31) and the estimate $k_j(\xi)=O(\langle\xi\rangle^{1/3})$. Therefore according to (2.42) and (2.44), for $t<1$, we obtain the estimate (2.34).

Now we prove the asymptotic formula (2.35). For large $t>1$ and $|\xi|>1$, the integrand $e^{\xi t}$ decays as e^{-Ct}. However, in the neighborhood of $\xi=0$, the integrand $e^{\xi t}$ changes relatively slowly. By this reason we split the integrals (2.43) into integrals over sections $|p|<1$, $|\xi|<1$ and over the rest of the ranges of integration. In the neighborhood of $p=0$ we have

$$K(p)+\xi=\sqrt{|p|}+\xi+O(p^{7/2}),\quad K_1(p)+\xi=\sqrt{p}+\xi+O(p^{7/2}).$$

Also for small ξ by construction $k_j(\xi) = \alpha^{-2/5} e^{i(-1)^j 2\pi/5} + O(\xi)$. To separate the principal part of the expansion of

$$\Gamma^+(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z} \ln\frac{K(q)+\xi}{K_1(q)+\xi} dq$$

near points $p=0$, $\xi=0$ we introduce the new function $\widetilde{\Gamma}^+(z,\xi)$ by the relation

$$\begin{aligned}
\widetilde{\Gamma}^{\pm}(z,\xi) &= \frac{1}{2\pi i}\Big\{\int_{i|\xi|^2}^{i} + \int_{-i}^{-i|\xi|^2}\Big\}\frac{1}{q-z}\ln\frac{\sqrt{|q|}}{\sqrt{q}} dq \\
&= \frac{1}{2\pi i}(-i\frac{\pi}{4})\int_{i|\xi|^2}^{i}\big(\frac{1}{q-z}+\frac{1}{q+z}\big)dq \\
&= \frac{1}{8}\ln\frac{z^2+|\xi|^4}{z^2+1}
\end{aligned} \tag{2.45}$$

It can be proved by a direct calculation that for small $|z|<1$, $|\xi|<1$,

$$\widetilde{\Gamma}^+(z,\xi) - \Gamma^+(z,\xi) = O(\xi).$$

Also via (2.45), the functions

$$w^{\pm}(z,\xi) = e^{\widetilde{\Gamma}^{\pm}(z,\xi)} = (\frac{z^2+|\xi|^4}{z^2+1})^{1/8}$$

are analytic in $z \in C$ except for $z \in [-i,-i|\xi|^2] \cup [i|\xi|^2, i]$ and therefore

$$w^+(z,\xi) = w^-(z,\xi)$$

for all $z \notin [-i,-i|\xi|^2] \cup [i|\xi|^2, i]$. Via the last comments we obtain for small $p>0$, $\xi \in \mathcal{C}_1$

$$I^+(p,\xi,y) = \frac{1}{2\pi i}\frac{1}{k_1(\xi)k_2(\xi)}\int_{-i\infty}^{i\infty}\frac{1}{q-p}\frac{1}{(q^2+|\xi|^4)^{1/8}}\frac{K(q)}{K_1(q)}dq(1+O(y^\mu p^\mu)),$$

where $\mu \in (0,\frac{1}{4})$.

On the basis last relations for points of the contours close the points $p=0$ and $\xi=0$ integrand of $\mathcal{M}_1(x,y,t)$ has the following representation

$$-\big(\frac{1}{2\pi i}\big)^2 i\sqrt{2}e^{\xi t}e^{-px}\frac{\sqrt{p}(p^2+|\xi|^4)^{1/8}}{(\sqrt{ip}-p^3+\xi)(\sqrt{-ip}-p^3+\xi)}\widetilde{I}^+(p,\xi,y).$$

where

$$\widetilde{I}^+(p,\xi,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-p}\frac{1}{(q^2+|\xi|^4)^{1/8}}\frac{K(q)}{K_1(q)}dq.$$

On integrating this function by extending the limits to $-i\infty$ and $+i\infty$ we obtain the contributions to $\mathcal{M}_1(x,y,t)$ from the neighborhoods of $\xi=0$ and $p=0$

$$-t^{-2}(\frac{1}{2\pi i})^2\int_{\mathcal{C}_1}d\xi e^{\xi}\int_0^{+\infty}e^{-pxt^{-2}}\frac{\sqrt{p}(p^2+|\xi|^4)^{1/8}}{(\sqrt{ip}+\xi)(\sqrt{-ip}+\xi)}dp \tag{2.46}$$

$$\times\int_0^{\infty}\frac{1}{q+p}\frac{1}{(q^2+|\xi|^4)^{1/8}}dq. \tag{2.47}$$

To estimate the contribution of $R(x,y,t)$ to $G(x,y,t)$ over the rest of the range of integration we use standard method of "partition of unity" and Watson Lemma. Due to smoothing properties of integrand functions by integrating by part we obtain

$$R(x,y,t) = y^\mu O(t^{-2-2\mu}) \tag{2.48}$$

For the function $J_1(x,y,t)$ defined by (2.37), we have

$$J_1(x,y,t) = \frac{1}{2\pi i} t^{-2} \int_{-i\infty}^{i\infty} e^{pxt^{-2} - |p|^{1/2}} dp + y^{\mu} O(t^{-2-2\mu}). \tag{2.49}$$

Thus from (2.46), (2.48) and (2.49), via (2.36) we obtain

$$\|\mathcal{G}\phi - t^{-2}\Lambda(xt^{-2}) f(\phi)\|_{L^\infty} \le Ct^{-2-2\mu} \|\langle x\rangle^{\mu}\phi\|_{L^1}.$$

By the same argument, it is easy to prove (2.34) for $t > 1$. The proof is complete. □

Theorem 2.3. *Let the initial data u_0 belong to $L^1(\mathbb{R}^+) \cap L^{1,a}(\mathbb{R}^+)$, with small $a \ge 0$. Then for some $T > 0$ there exists a unique solution*

$$u \in C([0,T]; L^1(\mathbb{R}^+) \cap L^{1,a}(\mathbb{R}^+)) \cap C((0,T]; H^1_\infty(\mathbb{R}^+))$$

to the initial-boundary value problem (1.1)

3. Proof of Theorem 1.1

By Proposition 2.1 we rewrite the initial-boundary value problem (1.1) as the integral equation

$$u(t) = \mathcal{G}(t)u_0 - \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))d\tau \tag{3.1}$$

where $\mathcal{G}$ is the Green operator of the linear problem (2.1). We choose the space

$$Z = \{\phi \in L^1(\mathbb{R}^+) \cap L^{1,a}(\mathbb{R}^+)\}$$

with $a > 0$ is small and the space

$$X = \{\phi \in C([0,\infty); Z) \cap C((0,\infty); H^1_\infty(\mathbb{R}^+)) : \|\phi\|_X < \infty\},$$

where

$$\|\phi\|_X = \sup_{t\ge 0} \Big(\{t\}^{-a/3} \langle t\rangle^{-2a} \|\phi(t)\|_{L^{1,a}} + \|\phi(t)\|_{L^1} + \sum_{n=0}^{1} \{t\}^{\frac{n+1}{3}} \langle t\rangle^{2(n+1)} \|\phi(t)\|_{L^\infty} \Big)$$

shows the optimal time decay properties of the solution. We apply the contraction mapping principle in a ball $X_\rho = \{\phi \in X : \|\phi\|_X \le \rho\}$ in the space X of radius

$$\rho = \frac{1}{2C}\|u_0\|_Z > 0.$$

For $v \in X_\rho$ we define the mapping $\mathcal{M}(v)$ by

$$\mathcal{M}(v) = \mathcal{G}(t)u_0 - \int_0^t e^{-\tau}\mathcal{G}(t-\tau)\mathcal{N}(v(\tau))d\tau. \tag{3.2}$$

We first prove that $\|\mathcal{M}(v)\|_X \le \rho$, where $\rho > 0$ is sufficiently small. By Lemma 2.2, we obtain

$$\|\mathcal{G}(t)\phi\|_{L^1} \le C\{t\}^{-\frac{n}{3}} \langle t\rangle^{-2n} \|\phi\|_{L^1},$$
$$\|\mathcal{G}(t)\phi\|_{L^{1,a}} \le C\{t\}^{\frac{a}{3}} \langle t\rangle^{2a} \|\phi\|_{L^1} + \|\phi\|_{L^{1,a}},$$
$$\|\partial_x^n \mathcal{G}(t)\phi\|_{L^\infty} \le C\{t\}^{-(n+1)/3} \langle t\rangle^{-2(n+1)} \|\phi\|_{L^1}$$

for all $t \ge 0$. Therefore,

$$\|\mathcal{G}u_0\|_X \le C\|u_0\|_Z. \tag{3.3}$$

Also since $v \in X_\rho$, we obtain

$$\|\mathcal{N}(v(\tau))\|_{L^{1,a}} \le C\|v\|_{L^{1,a}}\|v_x\|_{L^\infty} \le C\{\tau\}^{-(2-a)/3}\langle\tau\rangle^{2a-4}\|v\|_X^2,$$
$$\|\mathcal{N}(v(\tau))\|_{L^1} \le C\|v\|_{L^1}\|v_x\|_{L^\infty} \le C\{\tau\}^{-2/3}\langle\tau\rangle^{-3}\|v\|_X^2,$$

for all $\tau > 0$, and

$$\|\mathcal{N}(v(\tau))\|_{L^\infty} \le C\|v\|_{L^\infty}\|v_x\|_{L^\infty} \le C\langle\tau\rangle^{-6}\ \|v\|_X^2.$$

for all $\tau > 1$. Now by Lemma 2.2 we obtain

$$\begin{aligned}
&\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v(\tau))d\tau\|_{L^{1,a}}\\
&\le \int_0^t \{t-\tau\}^{\frac{a}{3}}\langle t-\tau\rangle^{2a}\|\mathcal{N}(v(\tau))\|_{L^1}d\tau + \int_0^t \|\mathcal{N}(v(\tau))\|_{L^{1,a}}d\tau\\
&\le C\|v\|_X^2\Big[\int_0^t \{t-\tau\}^{\frac{a}{3}}\langle t-\tau\rangle^{2a}\{\tau\}^{-2/3}\langle\tau\rangle^{-4}\ d\tau + \int_0^t \{\tau\}^{-\frac{2-a}{3}}\langle\tau\rangle^{2a-3}\ d\tau\Big]\\
&\le C\langle t\rangle^{2a}\{t\}^{\frac{a}{3}}\|v\|_X^2
\end{aligned}$$

for all $t \ge 0$. In the same manner by Lemma 2.2 we have

$$\begin{aligned}
\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v(\tau))d\tau\|_{L^1} &\le \int_0^t \|\mathcal{N}(v(\tau))\|_{L^1}d\tau\\
&\le C\|v\|_X^2\int_0^t \{\tau\}^{-2/3}\langle\tau\rangle^{-3}d\tau\\
&\le C\|v\|_X^2
\end{aligned}$$

for all $t \ge 0$. Also in view of Lemma 2.2, we find

$$\begin{aligned}
&\|\partial_x^n\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v(\tau))d\tau\|_{L^\infty}\\
&\le \int_0^t \{t-\tau\}^{-(n+1)/3}\langle t-\tau\rangle^{-2(n+1)}\|\mathcal{N}(v(\tau))\|_{L^1}d\tau d\tau\\
&\le C\|v\|_X^2\Big(\int_0^t \{t-\tau\}^{-(n+1)/3}\langle t-\tau\rangle^{-2(n+1)}\{\tau\}^{-2/3}\langle\tau\rangle^{-3}d\tau\Big)\\
&\le C\{t\}^{-(n+1)/3}\langle t\rangle^{-2(n+1)}\|v\|_X^2
\end{aligned}$$

for $t \ge 0$. Thus we obtain

$$\|\int_0^t e^{-\tau}\mathcal{G}(t-\tau)\mathcal{N}(v(\tau))d\tau\|_X \le C\|v\|_X^2,$$

hence in view of (3.2) and (3.3),

$$\begin{aligned}
\|\mathcal{M}(v)\|_X &\le \|\mathcal{G}u_0\|_X + \|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v(\tau))d\tau\|_X\\
&\le C\|u_0\|_Z + C\|v\|_X^2\\
&\le \frac{\rho}{2} + C\rho^2 < \rho.
\end{aligned}$$

since $\rho > 0$ is sufficiently small. Hence the mapping $\mathcal{M}$ transforms a ball X_ρ into itself. In the same manner we estimate the difference

$$\|\mathcal{M}(w) - \mathcal{M}(v)\|_X \le \frac{1}{2}\|w - v\|_X,$$

which shows that $\mathcal{M}$ is a contraction mapping. Therefore, there exists a unique solution $v \in C([0,\infty); L^1(\mathbb{R}^+) \cap L^{1,a}(\mathbb{R}^+)) \cap C((0,\infty); H^1_\infty)$ to the initial-boundary value problem (1.1). Now we can prove the asymptotic formula

$$v(x,t) = A_1 \Lambda(xt^{-2}) t^{-2} + O(t^{-2-\gamma}), \tag{3.4}$$

where

$$A_1 = f(u_0) - \int_0^{+\infty} e^{-\tau} f(\mathcal{N}(v)) d\tau.$$

Denote $G_0(t) = t^{-2}\ \Lambda(xt^{-2})$. From Lemma 2.2, we have

$$t^{2+\gamma} \|\mathcal{G}(t)\phi - G_0(t) f(\phi)\|_{L^\infty} \le C\|\phi\|_Z \tag{3.5}$$

for all $t > 1$. Also in view the definition of the norm X we have

$$|f(\mathcal{N}(v(\tau)))| \le \|\mathcal{N}(v(\tau))\|_{L^1} \le C\{\tau\}^{-2/3} \langle \tau\rangle^{-4} \|v\|_X^2.$$

By a direct calculation, for some small $\gamma_1 > 0$, $\gamma > 0$, we have

$$\begin{aligned}
&\| \int_0^{t/2} |G_0(t-\tau) - G_0(t)| f(\mathcal{N}(v(\tau))) d\tau \|_{L^\infty} \\
&\le \langle t\rangle^{-1} C \|v\|_X^2 \int_0^{t/2} \|(G_0(t-\tau) + G_0(t))\|_{L^\infty} \{\tau\}^{-\gamma_1} \langle\tau\rangle^{-\gamma_2} d\tau \\
&\le C\langle t\rangle^{-2} \int_0^{t/2} \{\tau\}^{-\gamma_1} \langle\tau\rangle^{-\gamma_2} d\tau \le C\langle t\rangle^{-\gamma-2}
\end{aligned} \tag{3.6}$$

and in the same way,

$$\|\langle t\rangle^{\gamma} G_0(t) \int_{t/2}^{\infty} f(\mathcal{N}(v(\tau))) d\tau \|_{L^\infty} \le C\|v\|_X^2 \tag{3.7}$$

Also we have

$$\begin{aligned}
&\| \int_0^{t/2} (\mathcal{G}(t-\tau)\mathcal{N}(v(\tau)) - G_0(t-\tau) f(\mathcal{N}(v(\tau)))) d\tau \|_{L^\infty} \\
&+ \| \int_{t/2}^{t} \mathcal{G}(t-\tau)\mathcal{N}(v(\tau)) d\tau \|_{L^\infty} \\
&\le C \int_0^{t/2} (t-\tau)^{-2} \|\mathcal{N}(v(\tau))\|_{L^1} d\tau + Ce^{-\frac{t}{2}} \int_{t/2}^{t} \|\mathcal{N}(v(\tau))\|_{L^1} d\tau \\
&\le Ct^{-2-\gamma} \|v\|_X^2
\end{aligned} \tag{3.8}$$

for all $t > 1$. By (3.1), we obtain

$$\begin{aligned}
&\langle t\rangle^{\gamma+2} \|(v(t) - AG_0(t))\|_X \\
&\le \|(\mathcal{G}(t)u_0 - G_0(t) f(u_0)\|_{L^\infty} \\
&\quad + \langle t\rangle^{\gamma+2} \| \int_0^{\frac{t}{2}} (\mathcal{G}(t-\tau)\mathcal{N}(v(\tau)) - G_0(t-\tau) f(\mathcal{N}(v(\tau)))) d\tau \|_{L^\infty} \\
&\quad + \langle t\rangle^{\gamma+2} \| \int_{t/2}^{t} \mathcal{G}(t-\tau)\mathcal{N}(v(\tau)) d\tau \|_{L^\infty} + \langle t\rangle^{\gamma+2} \|G_0(t) \int_{t/2}^{\infty} f(\mathcal{N}(v(\tau))) d\tau \|_{L^\infty} \\
&\quad + \langle t\rangle^{\gamma+2} \| \int_0^{t/2} (G_0(t-\tau) - G_0(t)) f(\mathcal{N}(v(\tau))) d\tau \|_{L^\infty}.
\end{aligned} \tag{3.9}$$

All the summands in the right-hand side of above inequality are estimated by $C\|u_0\|_Z + C\|v\|_X^2$ via estimates (3.6)–(3.8). Thus by (3.9) the asymptotic formula (3.4) is valid, which completes the proof.

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