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OSCILLATION RESULTS FOR EVEN-ORDER QUASILINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we use the Riccati transformation technique and some inequalities, to establish oscillation theorems for all solutions to even-order quasilinear neutral differential equation

$$\Big(\big[\big(x(t)+p(t)x(\tau(t))\big)^{(n-1)}\big]^{\gamma}\Big)'+q(t)x^{\gamma}\big(\sigma(t)\big)=0,\quad t\ge t_0.$$

Our main results are illustrated with examples.

1. Introduction

Neutral differential equations find numerous applications in natural science and technology; see Hale [1]. Recently, there has been much research activity concerning the oscillation and non-oscillation of solutions of various types of neutral functional differential equations; see for example [2, 3, 4, 6, 7, 11, 12, 14] and the references cited therein.

In this article, we consider the oscillatory behavior of solutions to the even-order neutral differential equation

$$\Big(\big[\big(x(t)+p(t)x(\tau(t))\big)^{(n-1)}\big]^{\gamma}\Big)'+q(t)x^{\gamma}\big(\sigma(t)\big)=0,\quad t\ge t_0. \tag{1.1}$$

We will use the following assumptions:

- (A1) $n\ge 2$ is even and $\gamma\ge 1$ is the ratio of odd positive integers;
- (A2) $p\in C([t_0,\infty),[0,a])$, where a is a constant;
- (A3) $q\in C([t_0,\infty),[0,\infty))$, and q is not eventually zero on any half line $[t_*,\infty)$;
- (A4) $\tau,\sigma\in C([t_0,\infty),\mathbb{R})$, $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma(t)=\infty$, σ^{-1} exists and σ^{-1} is continuously differentiable.

We consider only those solutions x of (1.1) for which $\sup\{|x(t)|:t\ge T\}>0$ for all $T\ge t_0$. We assume that (1.1) possesses such a solution. As usual, a solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[t_0,\infty)$; otherwise, it is called non-oscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

For the oscillation of even-order neutral differential equations, Zafer [5], Karpuz et al. [8], Zhang et al. [10] and Li et al. [13] considered the oscillation of even-order

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neutral equation

$$\big(x(t)+p(t)x(\tau(t))\big)^{(n)}+q(t)x(\sigma(t))=0,\quad t\geq t_0 \tag{1.2}$$

by using the results given in [15]. Meng and Xu [9] studied the oscillation property of the even-order quasi-linear neutral equation

$$\big[r(t)|(z(t))^{(n-1)}|^{\alpha-1}(z(t))^{(n-1)}\big]'+q(t)|x(\sigma(t))|^{\alpha-1}x(\sigma(t))=0,\quad t\geq t_0,$$

with $z(t)=x(t)+p(t)x(\tau(t))$. To the best of our knowledge, there are no results on the oscillation of (1.1) when $p(t)>1$ and $\gamma>1$. The purpose of this paper is to establish some oscillation results for (1.1). The organization of this article is as follows: In Section 2, we give some oscillation criteria for (1.1). In Section 3, we give several examples to illustrate our main results.

Below, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large t.

2. Main results

In this section, we establish some oscillation criteria for (1.1). Let f^{-1} denote the inverse function of f, and for the sake of convenience, we let

$$z(t):=x(t)+p(t)x(\tau(t)),\quad Q(t):=\min\{q(\sigma^{-1}(t)),q(\sigma^{-1}(\tau(t)))\},$$
$$(\rho'(t))_+:=\max\{0,\rho'(t)\}.$$

To prove our main results, we use the following lemmas.

Lemma 2.1 ([2, Lemma 2.2.1]). *Let $u(t)$ be a positive and n-times differentiable function on an interval $[T,\infty)$ with its n-th derivative $u^{(n)}(t)$ non-positive on $[T,\infty)$ and not identically zero on any interval $[T_1,\infty)$, $T_1\geq T$. Then there exists an integer l, $0\leq l\leq n-1$, with $n+l$ odd, such that, for some large $T_2\geq T_1$,*

$$(-1)^{l+j}u^{(j)}(t)>0\quad \text{on } [T_2,\infty)\ (j=l,l+1,\dots,n-1)$$
$$u^{(i)}(t)>0\quad \text{on } [T_2,\infty)\ (i=1,2,\dots,l-1)\ \text{when } l>1.$$

Lemma 2.2 ([2, P. 169]). *Let u be as in Lemma 2.1. If $\lim_{t\to\infty}u(t)\neq 0$, then, for every λ, $0<\lambda<1$, there is $T_\lambda\geq t_0$ such that, for all $t\geq T_\lambda$,*

$$u(t)\geq\frac{\lambda}{(n-1)!}t^{n-1}u^{(n-1)}(t).$$

Lemma 2.3 ([15]). *Let u be as in Lemma 2.1 and $u^{(n-1)}(t)u^{(n)}(t)\leq 0$ for $t\geq t_*$. Then for every constant θ, $0<\theta<1$, there exists a constant $M_\theta>0$ such that*

$$u'(\theta t)\geq M_\theta t^{n-2}u^{(n-1)}(t).$$

Lemma 2.4. *Assume that x is an eventually positive solution of (1.1), and n is even. Then there exists $t_1\geq t_0$ such that, for $t\geq t_1$,*

$$z(t)>0,\quad z'(t)>0,\quad z^{(n-1)}(t)>0,\quad z^{(n)}(t)\leq 0,$$

and $z^{(n)}$ is not identically zero on any interval $[a,\infty)$.

The proof of the above lemma is similar to that of [9, Lemma 2.3], with γ being the ratio of odd integers. We omit it.

Lemma 2.5. *Assume that $\gamma \ge 1$, x_1, $x_2 \in \mathbb{R}$. If $x_1 \ge 0$ and $x_2 \ge 0$, then*
$$x_1{}^\gamma + x_2{}^\gamma \ge \frac{1}{2^{\gamma-1}}(x_1+x_2)^\gamma. \tag{2.1}$$

Proof. (i) Suppose that $x_1 = 0$ or $x_2 = 0$. Then we have (2.1). (ii) Suppose that $x_1 > 0$ and $x_2 > 0$. Define f by $f(x) = x^\gamma$, $x \in (0,\infty)$. Clearly, $f''(x) = \gamma(\gamma-1)x^{\gamma-2} \ge 0$ for $x > 0$. Thus, f is a convex function. By the definition of convex function, for x_1, $x_2 \in (0,\infty)$, we have
$$f\big(\frac{x_1+x_2}{2}\big) \le \frac{f(x_1)+f(x_2)}{2}.$$
That is,
$$x_1{}^\gamma + x_2{}^\gamma \ge \frac{1}{2^{\gamma-1}}(x_1+x_2)^\gamma.$$
This completes the proof. □

First, we establish the following comparison theorems.

Theorem 2.6. *Assume that $(\sigma^{-1}(t))' \ge \sigma_0 > 0$ and $\tau'(t) \ge \tau_0 > 0$. Further, assume that there exists a constant λ, $0 < \lambda < 1$, such that*
$$\big[\frac{y(\sigma^{-1}(t))}{\sigma_0} + \frac{a^\gamma}{\sigma_0\tau_0} y(\sigma^{-1}(\tau(t)))\big]' + \frac{1}{2^{\gamma-1}}\Big(\frac{\lambda}{(n-1)!}t^{n-1}\Big)^\gamma Q(t)y(t) \le 0 \tag{2.2}$$
has no eventually positive solution. Then (1.1) is oscillatory.

Proof. Let x be a non-oscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \ge t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \ge t_1$. Then $z(t) > 0$ for $t \ge t_1$. From (1.1), we obtain
$$\big((z^{(n-1)}(t))^\gamma\big)' = -q(t)x^\gamma(\sigma(t)) \le 0, \quad t \ge t_1.$$

By Lemma 2.4 with n even, there exists $t_2 \ge t_1$ such that $z^{(n)}(t) \le 0$ for $t \ge t_2$. Thus, from Lemma 2.1, there exist $t_3 \ge t_2$ and an odd integer $l \le n-1$ such that, for some large $t_4 \ge t_3$,
$$(-1)^{l+j}z^{(j)}(t) > 0, \quad j = l, l+1, \dots, n-1,\ t \ge t_4 \tag{2.3}$$
and
$$z^{(i)}(t) > 0, \quad i = 1, 2, \dots, l-1,\ t \ge t_4. \tag{2.4}$$
Hence, in view of (2.3) and (2.4), we obtain $z'(t) > 0$ and $z^{(n-1)}(t) > 0$. Therefore, $\lim_{t\to\infty} z(t) \ne 0$. Then, by Lemma 2.2, for every λ, $0 < \lambda < 1$, there exists T_λ such that, for all $t \ge T_\lambda$,
$$z(t) \ge \frac{\lambda}{(n-1)!}t^{n-1}z^{(n-1)}(t). \tag{2.5}$$
It follows from (1.1) that
$$\frac{((z^{(n-1)}(\sigma^{-1}(t)))^\gamma)'}{(\sigma^{-1}(t))'} + q(\sigma^{-1}(t))x^\gamma(t) = 0. \tag{2.6}$$
The above inequality at times $\sigma^{-1}(t)$ and $\sigma^{-1}(\tau(t))$, yields
$$\begin{gathered}\frac{((z^{(n-1)}(\sigma^{-1}(t)))^\gamma)'}{(\sigma^{-1}(t))'} + a^\gamma\frac{((z^{(n-1)}(\sigma^{-1}(\tau(t))))^\gamma)'}{(\sigma^{-1}(\tau(t)))'} \\ + q(\sigma^{-1}(t))x^\gamma(t) + a^\gamma q(\sigma^{-1}(\tau(t)))x^\gamma(\tau(t)) = 0.\end{gathered} \tag{2.7}$$

By (2.1) and the definition of z,

$$\begin{aligned} q(\sigma^{-1}(t))x^\gamma(t)+a^\gamma q(\sigma^{-1}(\tau(t)))x^\gamma(\tau(t)) &\geq Q(t)[x^\gamma(t)+a^\gamma x^\gamma(\tau(t))] \\ &\geq \frac{1}{2^{\gamma-1}}Q(t)[x(t)+ax(\tau(t))]^\gamma \\ &\geq \frac{1}{2^{\gamma-1}}Q(t)z^\gamma(t) \end{aligned} \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$\frac{((z^{(n-1)}(\sigma^{-1}(t)))^\gamma)'}{(\sigma^{-1}(t))'}+a^\gamma\frac{((z^{(n-1)}(\sigma^{-1}(\tau(t))))^\gamma)'}{(\sigma^{-1}(\tau(t)))'}+\frac{1}{2^{\gamma-1}}Q(t)z^\gamma(t)\leq 0. \tag{2.9}$$

From this inequality, $(\sigma^{-1}(t))'\geq\sigma_0>0$ and $\tau'(t)\geq\tau_0>0$, we obtain

$$\frac{((z^{(n-1)}(\sigma^{-1}(t)))^\gamma)'}{\sigma_0}+a^\gamma\frac{((z^{(n-1)}(\sigma^{-1}(\tau(t))))^\gamma)'}{\sigma_0\tau_0}+\frac{1}{2^{\gamma-1}}Q(t)z^\gamma(t)\leq 0. \tag{2.10}$$

Set $y(t)=(z^{(n-1)}(t))^\gamma>0$. From (2.5) and (2.9), we see that y is an eventually positive solution of

$$\Big[\frac{y(\sigma^{-1}(t))}{\sigma_0}+\frac{a^\gamma}{\sigma_0\tau_0}y(\sigma^{-1}(\tau(t)))\Big]'+\frac{1}{2^{\gamma-1}}\Big(\frac{\lambda}{(n-1)!}t^{n-1}\Big)^\gamma Q(t)y(t)\leq 0.$$

The proof is complete. □

Theorem 2.7. *Let τ^{-1} exist. Assume that $\tau(t)\leq t$, $(\sigma^{-1}(t))'\geq\sigma_0>0$ and $\tau'(t)\geq\tau_0>0$. Moreover, assume that there exists a constant λ, $0<\lambda<1$, such that*

$$u'(t)+\frac{1}{2^{\gamma-1}\big(\frac{1}{\sigma_0}+\frac{a^\gamma}{\sigma_0\tau_0}\big)}\Big(\frac{\lambda}{(n-1)!}t^{n-1}\Big)^\gamma Q(t)u(\tau^{-1}(\sigma(t)))\leq 0 \tag{2.11}$$

has no eventually positive solution. Then (1.1) *is oscillatory.*

Proof. Let x be a non-oscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1\geq t_0$ such that $x(t)>0$, $x(\tau(t))>0$ and $x(\sigma(t))>0$ for all $t\geq t_1$. Then $z(t)>0$ for $t\geq t_1$. Proceeding as in the proof of Theorem 2.6, we obtain that $y(t)=(z^{(n-1)}(t))^\gamma>0$ is non-increasing and satisfies inequality (2.2). Define

$$u(t)=\frac{y(\sigma^{-1}(t))}{\sigma_0}+\frac{a^\gamma}{\sigma_0\tau_0}y(\sigma^{-1}(\tau(t))).$$

Then, from $\tau(t)\leq t$, and σ^{-1} begin increasing, we have

$$u(t)\leq\big(\frac{1}{\sigma_0}+\frac{a^\gamma}{\sigma_0\tau_0}\big)y(\sigma^{-1}(\tau(t))).$$

Substituting the above formulas into (2.2), we find u is an eventually positive solution of

$$u'(t)+\frac{1}{2^{\gamma-1}\big(\frac{1}{\sigma_0}+\frac{a^\gamma}{\sigma_0\tau_0}\big)}\Big(\frac{\lambda}{(n-1)!}t^{n-1}\Big)^\gamma Q(t)u(\tau^{-1}(\sigma(t)))\leq 0. \tag{2.12}$$

The proof is complete. □

From Theorem 2.7 and [3, Theorem 2.1.1], we establish the following corollary.

Corollary 2.8. *Let τ^{-1} exist. Assume that $\tau(t) \le t$, $(\sigma^{-1}(t))' \ge \sigma_0 > 0$, $\tau'(t) \ge \tau_0 > 0$, $\tau^{-1}(\sigma(t)) < t$ and*

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^{t} Q(s)(s^{n-1})^\gamma \mathrm{d}s > \frac{2^{\gamma-1}\big(\frac{1}{\sigma_0} + \frac{a^\gamma}{\sigma_0\tau_0}\big)}{e}\big((n-1)!\big)^\gamma. \tag{2.13}$$

Then (1.1) *is oscillatory.*

Proof. From (2.13), one can choose a positive constant $0 < \lambda < 1$ such that

$$\liminf_{t\to\infty} \lambda^\gamma \int_{\tau^{-1}(\sigma(t))}^{t} Q(s)(s^{n-1})^\gamma \mathrm{d}s > \frac{2^{\gamma-1}\big(\frac{1}{\sigma_0} + \frac{a^\gamma}{\sigma_0\tau_0}\big)}{e}((n-1)!)^\gamma.$$

Applying [3, Theorem 2.1.1] to (2.12), with $\tau^{-1}(\sigma(t)) < t$, we complete the proof. □

Theorem 2.9. *Assume that $(\sigma^{-1}(t))' \ge \sigma_0 > 0$, $\tau'(t) \ge \tau_0 > 0$ and $\tau(t) \ge t$. Furthermore, assume that there exists a constant λ, $0 < \lambda < 1$, such that*

$$u'(t) + \frac{1}{2^{\gamma-1}\big(\frac{1}{\sigma_0} + \frac{a^\gamma}{\sigma_0\tau_0}\big)}\Big(\frac{\lambda}{(n-1)!}t^{n-1}\Big)^\gamma Q(t)u(\sigma(t)) \le 0 \tag{2.14}$$

has no eventually positive solution. Then (1.1) *is oscillatory.*

Proof. Let x be a non-oscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \ge t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \ge t_1$. Then $z(t) > 0$ for $t \ge t_1$. Proceeding as in the proof of Theorem 2.6, we obtain that $y(t) = (z^{(n-1)}(t))^\gamma > 0$ is nonincreasing and satisfies inequality (2.2). Define

$$u(t) = \frac{1}{\sigma_0}y(\sigma^{-1}(t)) + \frac{a^\gamma}{\sigma_0\tau_0}y(\sigma^{-1}(\tau(t))).$$

Then, from $\tau(t) \ge t$, we have

$$u(t) \le \Big(\frac{1}{\sigma_0} + \frac{a^\gamma}{\sigma_0\tau_0}\Big)y(\sigma^{-1}(t)).$$

Substituting the above formulas into (2.2), we find u is an eventually positive solution of

$$u'(t) + \frac{1}{2^{\gamma-1}\big(\frac{1}{\sigma_0} + \frac{a^\gamma}{\sigma_0\tau_0}\big)}\Big(\frac{\lambda}{(n-1)!}t^{n-1}\Big)^\gamma Q(t)u(\sigma(t)) \le 0. \tag{2.15}$$

The proof is complete. □

From Theorem 2.9 and [3, Theorem 2.1.1], we establish the following corollary.

Corollary 2.10. *Assume that $(\sigma^{-1}(t))' \ge \sigma_0 > 0$, $\tau'(t) \ge \tau_0 > 0$, $\tau(t) \ge t$, $\sigma(t) < t$ and*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} Q(s)(s^{n-1})^\gamma \mathrm{d}s > \frac{2^{\gamma-1}\big(\frac{1}{\sigma_0} + \frac{a^\gamma}{\sigma_0\tau_0}\big)}{e}\big((n-1)!\big)^\gamma. \tag{2.16}$$

Then (1.1) *is oscillatory.*

Proof. From (2.16), one can choose a positive constant $0 < \lambda < 1$ such that

$$\liminf_{t\to\infty} \lambda^\gamma \int_{\sigma(t)}^{t} Q(s)(s^{n-1})^\gamma \mathrm{d}s > \frac{2^{\gamma-1}\big(\frac{1}{\sigma_0} + \frac{a^\gamma}{\sigma_0\tau_0}\big)}{e}((n-1)!)^\gamma.$$

Applying [3, Theorem 2.1.1] to (2.15), with $\sigma(t) < t$, we complete proof. □

By employing Riccati transformation, we obtain the following criteria.

Theorem 2.11. *Let* $(\sigma^{-1}(t))' \geq \sigma_0 > 0$, $\sigma^{-1}(t) \geq t$, $\sigma^{-1}(\tau(t)) \geq t$ *and* $\tau'(t) \geq \tau_0 > 0$. *Assume that there exists* $\rho \in C^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \Big[\frac{1}{2^{\gamma-1}}\rho(s)Q(s) - \frac{\frac{1}{\sigma_0}+\frac{a^\gamma}{\sigma_0\tau_0}}{(\gamma+1)^{\gamma+1}} \frac{((\rho'(s))_+)^{\gamma+1}}{(\theta M s^{n-2})^\gamma \rho^\gamma(s)}\Big] \mathrm{d}s = \infty \tag{2.17}$$

holds for some constant θ, $0<\theta<1$ *and for all constants* $M>0$. *Then* (1.1) *is oscillatory.*

Proof. Let x be a non-oscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t)>0$, $x(\tau(t))>0$ and $x(\sigma(t))>0$ for all $t\geq t_1$. Then $z(t)>0$ for $t\geq t_1$. Proceeding as in the proof of Theorem 2.6, there exists $t_2 \geq t_1$ such that (2.3), (2.4) and (2.10) hold for $t \geq t_2$.

Using the Riccati transformation

$$\omega(t) = \rho(t)\frac{(z^{(n-1)}(\sigma^{-1}(t)))^\gamma}{z^\gamma(\theta t)}, \quad t\geq t_2. \tag{2.18}$$

Then $\omega(t)>0$ for $t\geq t_2$. Differentiating (2.18), we obtain

$$\begin{aligned} \omega'(t) &= \rho'(t)\frac{(z^{(n-1)}(\sigma^{-1}(t)))^\gamma}{z^\gamma(\theta t)} + \rho(t)\frac{((z^{(n-1)}(\sigma^{-1}(t)))^\gamma)'}{z^\gamma(\theta t)} \\ &\quad - \gamma\theta\rho(t)\frac{(z^{(n-1)}(\sigma^{-1}(t)))^\gamma z'(\theta t)}{z^{\gamma+1}(\theta t)}. \end{aligned} \tag{2.19}$$

By Lemma 2.3 and Lemma 2.4, we have

$$z'(\theta t) \geq M t^{n-2} z^{(n-1)}(t) \geq M t^{n-2} z^{(n-1)}(\sigma^{-1}(t)),$$

for every θ, $0<\theta<1$ and for some $M>0$. Thus, from (2.18) and (2.19), we obtain

$$\omega'(t) \leq \frac{(\rho'(t))_+}{\rho(t)}\omega(t) + \rho(t)\frac{((z^{(n-1)}(\sigma^{-1}(t)))^\gamma)'}{z^\gamma(\theta t)} - \gamma\theta M t^{n-2}\frac{(\omega(t))^{(\gamma+1)/\gamma}}{\rho^{1/\gamma}(t)}. \tag{2.20}$$

Next, define function

$$\psi(t) = \rho(t)\frac{(z^{(n-1)}(\sigma^{-1}(\tau(t))))^\gamma}{z^\gamma(\theta t)}, \quad t\geq t_2. \tag{2.21}$$

Then $\psi(t)>0$ for $t\geq t_2$. Differentiating (2.21), we see that

$$\begin{aligned} \psi'(t) &= \rho'(t)\frac{(z^{(n-1)}(\sigma^{-1}(\tau(t))))^\gamma}{z^\gamma(\theta t)} + \rho(t)\frac{((z^{(n-1)}(\sigma^{-1}(\tau(t))))^\gamma)'}{z^\gamma(\theta t)} \\ &\quad - \gamma\theta\rho(t)\frac{(z^{(n-1)}(\sigma^{-1}(\tau(t))))^\gamma z'(\theta t)}{z^{\gamma+1}(\theta t)}. \end{aligned} \tag{2.22}$$

In view of Lemmas 2.3 and 2.4, we have

$$z'(\theta t) \geq M t^{n-2} z^{(n-1)}(t) \geq M t^{n-2} z^{(n-1)}(\sigma^{-1}(\tau(t))),$$

for every θ, $0<\theta<1$ and for some $M>0$. Hence, by (2.21) and (2.22), we obtain

$$\begin{aligned} \psi'(t) &\leq \frac{(\rho'(t))_+}{\rho(t)}\psi(t) + \rho(t)\frac{((z^{(n-1)}(\sigma^{-1}(\tau(t))))^\gamma)'}{z^\gamma(\theta t)} \\ &\quad - \gamma\theta M t^{n-2}\frac{(\psi(t))^{(\gamma+1)/\gamma}}{\rho^{1/\gamma}(t)}. \end{aligned} \tag{2.23}$$

Therefore, from (2.20) and (2.23) it follows that

$$\begin{aligned} &\frac{\omega'(t)}{\sigma_0}+\frac{a^\gamma}{\sigma_0\tau_0}\psi'(t)\\ &\le \rho(t)\Big[\frac{\frac{((z^{(n-1)}(\sigma^{-1}(t)))^\gamma)'}{\sigma_0}+\frac{a^\gamma}{\sigma_0\tau_0}((z^{(n-1)}(\sigma^{-1}(\tau(t))))^\gamma)'}{z^\gamma(\theta t)}\Big]\\ &\quad+\frac{1}{\sigma_0}\Big[\frac{(\rho'(t))_+}{\rho(t)}\omega(t)-\gamma\theta Mt^{n-2}\frac{(\omega(t))^{(\gamma+1)/\gamma}}{\rho^{1/\gamma}(t)}\Big]\\ &\quad+\frac{a^\gamma}{\sigma_0\tau_0}\Big[\frac{(\rho'(t))_+}{\rho(t)}\psi(t)-\gamma\theta Mt^{n-2}\frac{(\psi(t))^{(\gamma+1)/\gamma}}{\rho^{1/\gamma}(t)}\Big]. \end{aligned} \tag{2.24}$$

Thus, from the above inequality and (2.10), we have

$$\begin{aligned} &\frac{\omega'(t)}{\sigma_0}+\frac{a^\gamma}{\sigma_0\tau_0}\psi'(t)\\ &\le -\frac{1}{2^{\gamma-1}}\rho(t)Q(t)+\frac{1}{\sigma_0}\Big[\frac{(\rho'(t))_+}{\rho(t)}\omega(t)-\gamma\theta Mt^{n-2}\frac{(\omega(t))^{(\gamma+1)/\gamma}}{\rho^{1/\gamma}(t)}\Big]\\ &\quad+\frac{a^\gamma}{\sigma_0\tau_0}\Big[\frac{(\rho'(t))_+}{\rho(t)}\psi(t)-\gamma\theta Mt^{n-2}\frac{(\psi(t))^{(\gamma+1)/\gamma}}{\rho^{1/\gamma}(t)}\Big]. \end{aligned} \tag{2.25}$$

Set

$$A:=\frac{(\rho'(t))_+}{\rho(t)},\quad B:=\frac{\gamma\theta Mt^{n-2}}{\rho^{1/\gamma}(t)},\quad v:=\omega(t),\psi(t).$$

Then, using (2.25) and the inequality

$$Av-Bv^{(\gamma+1)/\gamma}\le\frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}\frac{A^{\gamma+1}}{B^\gamma},\quad B>0, \tag{2.26}$$

we have

$$\frac{\omega'(t)}{\sigma_0}+\frac{a^\gamma}{\sigma_0\tau_0}\psi'(t)\le-\frac{1}{2^{\gamma-1}}\rho(t)Q(t)+\frac{\frac{1}{\sigma_0}+\frac{a^\gamma}{\sigma_0\tau_0}}{(\gamma+1)^{\gamma+1}}\frac{((\rho'(t))_+)^{\gamma+1}}{(\theta Mt^{n-2})^\gamma\rho^\gamma(t)}.$$

Integrating the above inequality from t_2 to t, we obtain

$$\int_{t_2}^t\Big[\frac{1}{2^{\gamma-1}}\rho(s)Q(s)-\frac{\frac{1}{\sigma_0}+\frac{a^\gamma}{\sigma_0\tau_0}}{(\gamma+1)^{\gamma+1}}\frac{((\rho'(s))_+)^{\gamma+1}}{(\theta Ms^{n-2})^\gamma\rho^\gamma(s)}\Big]\mathrm{d}s\le\frac{\omega(t_2)}{\sigma_0}+\frac{a^\gamma}{\sigma_0\tau_0}\psi(t_2),$$

which contradicts (2.17). The proof is complete. □

Remark 2.12. From (2.25), define a Philos-type function $H(t,s)$, and obtain some oscillation criteria for (1.1), the details are left to the reader.

Theorem 2.13. *Let $n=2$, $(\sigma^{-1}(t))'\ge\sigma_0>0$, $\sigma^{-1}(t)\ge t$, $\sigma^{-1}(\tau(t))\ge t$ and $\tau'(t)\ge\tau_0>0$. Assume that there exists $\rho\in C^1([t_0,\infty),(0,\infty))$ such that*

$$\limsup_{t\to\infty}\int_{t_0}^t\Big[\frac{1}{2^{\gamma-1}}\rho(s)Q(s)-\frac{\frac{1}{\sigma_0}+\frac{a^\gamma}{\sigma_0\tau_0}}{(\gamma+1)^{\gamma+1}}\frac{((\rho'(s))_+)^{\gamma+1}}{\rho^\gamma(s)}\Big]\mathrm{d}s=\infty. \tag{2.27}$$

Then (1.1) is oscillatory.

Proof. Define

$$\omega(t)=\rho(t)\frac{(z'(\sigma^{-1}(t)))^\gamma}{z^\gamma(t)},\quad \psi(t)=\rho(t)\frac{(z'(\sigma^{-1}(\tau(t))))^\gamma}{z^\gamma(t)}.$$

The remainder of the proof is similar to that of Theorem 2.11. □

3. Applications

Han et al. [11, 12] considered the oscillation of solutions to the second-order neutral equation

$$(x(t)+p(t)x(\tau(t)))''+q(t)x(\sigma(t))=0, \quad t\geq t_0,$$

where

$$0\leq p(t)\leq p_0<\infty, \quad \tau'(t)\geq \tau_0>0, \quad \tau\circ\sigma=\sigma\circ\tau. \tag{3.1}$$

Li et al. [13] investigated the oscillation of (1.2) when (3.1) holds. It is easy to see that our results weaken the restrictions in [11, 12, 13], since we do not assume $\tau\circ\sigma=\sigma\circ\tau$; instead we assume $\tau^{-1}(\sigma(t))<t$, and bounds on σ', $(\sigma^{-1})'$ and τ^{-1}. Below, we give three examples that illustrate our results.

Example 3.1. Consider the even-order equation

$$\Big(\big[\big(x(t)+ax(t-3)\big)^{(n-1)}\big]^{\gamma}\Big)'+\frac{\beta}{(t^{n-1})^{\gamma}}x^{\gamma}\left(t-6\right)=0, \quad t\geq 1, \tag{3.2}$$

where $\gamma>1$ is the quotient of odd positive integers, $a>0$ and $\beta>0$ are constants. Let $\tau(t)=t-3$, $p(t)=a$, $q(t)=\beta/(t^{n-1})^{\gamma}$ and $\sigma(t)=t-6$. Then $\tau^{-1}(t)=t+3$, $\tau^{-1}(\sigma(t))=t-3$, $\sigma^{-1}(t)=t+6$, $\sigma^{-1}(\tau(t))=t+3$ and $Q(t)=\beta/((t+6)^{n-1})^{\gamma}$. Since

$$\liminf_{t\to\infty}\int_{\tau^{-1}(\sigma(t))}^{t}Q(s)(s^{n-1})^{\gamma}\mathrm{d}s>\frac{\beta}{2^{\gamma(n-1)}}\liminf_{t\to\infty}\int_{t-3}^{t}\mathrm{d}s=\frac{3\beta}{2^{\gamma(n-1)}},$$

by applying Corollary 2.8, Equation (3.2) is oscillatory when

$$\frac{3\beta}{2^{\gamma(n-1)}}\geq\frac{2^{\gamma-1}(1+a^{\gamma})((n-1)!)!}{e}.$$

Example 3.2. Consider the even-order equation

$$\Big(\big[\big(x(t)+ax(t+3)\big)^{(n-1)}\big]^{\gamma}\Big)'+\frac{\beta}{(t^{n-1})^{\gamma}}x^{\gamma}\big(\frac{t}{2}\big)=0, \quad t\geq 1, \tag{3.3}$$

where $\gamma>1$ is the quotient of odd positive integers, $a>0$ and $\beta>0$ are constants. Let $\tau(t)=t+3$, $p(t)=a$, $q(t)=\beta/(t^{n-1})^{\gamma}$ and $\sigma(t)=t/2$. Then $\sigma^{-1}(t)=2t$, $\sigma^{-1}(\tau(t))=2(t+3)$ and $Q(t)=\beta/((2t+6)^{n-1})^{\gamma}$. Since

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t}Q(s)(s^{n-1})^{\gamma}\mathrm{d}s=\infty,$$

by applying Corollary 2.10, Equation (3.3) is oscillatory.

Example 3.3. Consider the even-order equation

$$\Big(\big[\big(x(t)+ax\left(2t\right)\big)^{(n-1)}\big]^{\gamma}\Big)'+\frac{\beta}{t}x^{\gamma}\big(\frac{t}{3}+1\big)=0, \quad t\geq 1, \tag{3.4}$$

where $\gamma>1$ is the quotient of odd positive integers, $a>0$ and $\beta>0$ are constants. Let $\tau(t)=2t$, $p(t)=a$, $q(t)=\beta/t$ and $\sigma(t)=(t/3)+1$. Then $\sigma^{-1}(t)=3(t-1)$, $\sigma^{-1}(\tau(t))=3(2t-1)$ and $Q(t)=\beta/(6t-3)$. Set $\rho(t)=1$. Then, by Theorem 2.11, every solution of (3.4) is oscillatory.

Note that the known results in the literature are not applicable to Equations (3.2), (3.3) and (3.4).

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References

[1] J. K. Hale; *Theory of Functional Differential Equations*, Spring-Verlag, New York, 1977.
[2] R. P. Agarwal, S. R. Grace, D. O'Regan; *Oscillation Theory for Difference and Functional Differential Equations*, Marcel Dekker, Kluwer Academic, Dordrecht, 2000.
[3] G. S. Ladde, V. Lakshmikantham, B. G. Zhang; *Oscillation Theory of Differential Equations with Deviating Arguments*, Marcel Dekker, New York, 1987.
[4] L. Erbe, Q. Kong, B. G. Zhang; *Oscillation Theory for Functional Differential Eqautions*, Marcel Dekker, New York, 1995.
[5] A. Zafer; *Oscillation criteria for even order neutral differential equations*, Appl. Math. Lett. 11 (1998) 21–25.
[6] J. Džurina, D. Hudáková; *Oscillation of second order neutral delay differential equations*, Math. Bohem. 134 (2009) 31–38.
[7] B. Baculíková, J. Džurina; *Oscillation of third-order neutral differential equations*, Math. Comput. Modelling 52 (2010) 215–226.
[8] B. Karpuz, Ö. Ocalan, S. Öztürk; *Comparison theorems on the oscillation and asymptotic behavior of higher-order neutral differential equations*, Glasg. Math. J. 52 (2010) 107–114.
[9] F. W. Meng, R. Xu; *Oscillation criteria for certain even order quasi-linear neutral differential equations with deviating arguments*, Appl. Math. Comput. 190 (2007) 458–464.
[10] Q. X. Zhang, J. R. Yan, L. Gao; *Oscillation behavior of even-order nonlinear neutral differential equations with variable coefficients*, Comput. Math. Appl. 59 (2010) 426–430.
[11] Z. Han, T. Li, S. Sun, W. Chen; *On the oscillation of second-order neutral delay differential equations*, Adv. Differ. Equ. 2010 (2010) 1–8.
[12] Z. Han, T. Li, S. Sun, W. Chen; *Oscillation criteria for second-order nonlinear neutral delay differential equations*, Adv. Differ. Equ. 2010 (2010) 1–23.
[13] T. Li, Z. Han, P. Zhao, S. Sun; *Oscillation of even-order neutral delay differential equations*, Adv. Differ. Equ. 2010 (2010) 1–9.
[14] M. Hasanbulli, Yu. V. Rogovchenko; *Oscillation criteria for second order nonlinear neutral differential equations*, Appl. Math. Comput. 215 (2010) 4392–4399.
[15] Ch. G. Philos; *A new criterion for the oscillatory and asymptotic behavior of delay differential equations*, Bull. Acad. Pol. Sci., Ser. Sci. Math. 39 (1981) 61–64.

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