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PERIODIC SOLUTIONS FOR p-LAPLACIAN FUNCTIONAL DIFFERENTIAL EQUATIONS WITH TWO DEVIATING ARGUMENTS

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Abstract. Using the theory of coincidence degree, we prove the existence of periodic solutions for the p-Laplacian functional differential equations with deviating arguments.

1. Introduction

In recent years, the existence of periodic solutions for the Duffing equation, Rayleigh equation and Liénard equation has received a lot of attention; see [1, 2, 4, 5, 6, 7, 8]. For example, Liu [5] studied periodic solutions for the p-Laplacian Liénard equation with a deviating argument. Using Mawhin's continuation theorem, some results on the existence of periodic solution are obtained. But the p-Laplacian Liénard equation with two deviating arguments has been studied far less often.

In this article, we study the existence of periodic solutions for the following Liénard equation with two deviating arguments:

$$(\phi_p(x'(t)))' + f(x(t))x'(t) + g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t))) = e(t), \tag{1.1}$$

where $f, \tau_1, \tau_2, e \in C(\mathbb{R}, \mathbb{R})$; $g_1, g_2 \in C(\mathbb{R}^2, \mathbb{R})$; $\tau_1(t), \tau_2(t), g_1(t, x), g_2(t, x), e(t)$ are periodic functions with period T; $\phi_p(\cdot)$ is the p-Laplacian operator, $1 < p < \infty$. By using the theory of coincidence degree, we obtain some results to guarantee the existence of periodic solutions. Even for $p = 2$, the results in this paper are also new.

In what follows, the L^p-norm in $L^p([0, T], \mathbb{R})$ is defined by

$$\|x\|_p = (\int_0^T |x(t)|^p dt)^{1/p},$$

and the $L^\infty-$norm in $L^\infty([0, T], \mathbb{R})$ is $\|x\|_\infty = \max_{t\in[0,T]} |x(t)|$. Let the Sobolev space $W^{1,p}([0, T], \mathbb{R}]$ be denoted by W.

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Lemma 1.1 ([9]). *Suppose $u \in W$ and $u(0) = u(T) = 0$. Then*
$$\|u\|_\infty \le (T/2)^{1/q}\|u'\|_p.$$

The following Mawhin's continuous theorem is useful in obtaining the existence of T-periodic solutions of (1.1).

Lemma 1.2 ([3]). *Let X and Y be two Banach spaces. Suppose that $L : D(L) \in X \to Y$ is a Fredholm operator with index zero and $N : X \to Y$ is L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X. Moreover, assume that all the following conditions are satisfied:*

1. *$Lx \ne \lambda Nx$, for all $x \in \partial\Omega \cap D(L)$, and all $\lambda \in (0, 1)$;*
2. *$Nx \notin \operatorname{Im} L$, for all $x \in \partial\Omega \cap \ker L$;*
3. *$\deg\{JQN, \Omega \cap \ker L, 0\} \ne 0$, where $J : \operatorname{Im} Q \to \ker L$ is an isomorphism,*

then equation $Lx = Nx$ has a solution on $\Omega \cap D(L)$.

2. Main results

To use coincidence degree theory in the study of T-periodic solutions for (1.1), we rewrite (1.1) in the form
$$\begin{gathered} x'(t) = \phi_q(y(t)) \\ y'(t) = -f(x(t))x'(t) - g_1(t, x(t - \tau_1(t))) - g_2(t, x(t - \tau_2(t))) + e(t). \end{gathered} \tag{2.1}$$

If $z(t) = (x(t), y(t))^T$ is a T-periodic solution of (2.1), then $x(t)$ must be a T-periodic solution of (1.1). Thus, the problem of finding a T-periodic solution for (1.1) reduces to finding one for (2.1).

We set the following notation: $T > 0$ is a constant, $C_T = \{x \in C(\mathbb{R}, \mathbb{R}) : x(t + T) \equiv x(t)\}$ with the norm $\|x\|_\infty = \max_{t\in[0,T]} |x(t)|, X = Z = \{z = (x, y) \in C(\mathbb{R}, \mathbb{R}^2) : z(t) \equiv z(x + T)\}$ with the norm $\|z\| = \max\{\|x\|_\infty, \|y\|_\infty\}$. Clearly, X and Z are Banach spaces. Also let $L : \operatorname{Dom} L \subset X \to Z$ be defined by
$$(Lz)(t) = z'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix},$$
and $N : X \to Z$ defined by
$$(Nz)(t) = \begin{pmatrix} \phi_q(y(t)) \\ -f(x(t))x'(t) - g_1(t, x(t - \tau_1(t))) - g_2(t, x(t - \tau_2(t))) + e(t) \end{pmatrix}$$
It is easy to see that $\ker L = \mathbb{R}^2$, $\operatorname{Im} L = \{z \in Z : \int_0^T z(s)ds = 0\}$. So L is a Fredholm operator with index zero. Let $P : X \to \ker L$ and $Q : Z \to \operatorname{Im} Q$ be defined by
$$Pu = \frac{1}{T}\int_0^T u(s)ds, \quad u \in X; \quad Qv = \frac{1}{T}\int_0^T v(s)ds, \quad v \in Z,$$
and let K_p denote the inverse of $L|_{\ker P\cap \operatorname{Dom} L}$. Obviously, $\ker L = \operatorname{Im} Q = \mathbb{R}^2$ and
$$(K_pz)(t) = \int_0^t z(s)ds - \frac{1}{T}\int_0^T \int_0^t z(s)\, ds\, dt. \tag{2.2}$$
From this equality, one can easily see that N is L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X.

Theorem 2.1. *Suppose that there exist constants $d > 0$ $r_1 \ge 0$ and $r_2 \ge 0$ such that*

(H1) $g_1(t,u)+g_2(t,v)-e(t)>0$ *for all* $t\in\mathbb{R}$, $|\max\{u,v\}|>d$;
(H2) $\lim_{x\to-\infty}\sup_{t\in[0,T]}\frac{|g_1(t,x)|}{|x|^{p-1}}\le r_1$; $\lim_{x\to-\infty}\sup_{t\in[0,T]}\frac{|g_2(t,x))|}{|x|^{p-1}}\le r_2$.

Then (1.1) *has at least one T-periodic solution, if* $4(r_1+r_2)T(T/2)^{p/q}<1$.

Proof. Consider the parametric equation

$$(Lz)(t)=\lambda(Nz)(t),\quad \lambda\in(0,1). \tag{2.3}$$

Let $z(t)=\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}$ be a possible T-periodic solution of (2.3) for some $\lambda\in(0,1)$. One can see $x=x(t)$ is a T-periodic solution of the equation

$$(\phi_p(x'(t)))'+\lambda^{p-1}f(x(t))x'(t)+\lambda^p g_1(t,x(t-\tau_1(t)))+\lambda^p g_2(t,x(t-\tau_2(t)))=\lambda^p e(t). \tag{2.4}$$

Integrating both sides of (2.4) over $[0,T]$, we have

$$\int_0^T[g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))-e(t)]dt=0, \tag{2.5}$$

which implies that there exists $\eta\in[0,T]$ such that

$$g_1(\eta,x(\eta-\tau_1(\eta)))+g_2(\eta,x(\eta-\tau_2(\eta)))-e(\eta)=0.$$

From assumption (H1), we know that there exists $\xi\in\mathbb{R}$ such that $|x(\xi)|\le d$.

Let $\xi=kT+t_0$, where $t_0\in[0,T]$ and k is an integer. Let $\chi(t)=x(t+t_0)-x(t_0)$. Then $\chi(0)=\chi(T)=0$ and $\chi\in W^{1,p}([0,T],\mathbb{R})$. By Lemma 1.1, we have

$$\|x\|_\infty\le\|\chi\|\infty+d\le(\frac{T}{2})^{1/q}\|\chi'\|_p+d=(\frac{T}{2})^{1/q}\|x'\|_p+d.$$

On the other hand, multiplying the two sides of (2.4) by $x(t)$ and integrating them over $[0,T]$, we obtain

$$-\|x'\|_p^p=-\lambda^p\int_o^T x(t)[g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))-e(t)]dt;$$

i.e.,

$$\|x'\|_p^p\le\|x\|_\infty\int_0^T|g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))-e(t)|dt. \tag{2.6}$$

From assumption (H2), there exists a constant $\rho>0$ such that

$$|g_1(t,x)|\le r_1|x|^{p-1},\quad |g_2(t,x)|\le r_2|x|^{p-1},\quad \forall t\in\mathbb{R},\ x<-\rho. \tag{2.7}$$

Let

$$\begin{gathered}
E_1=\{t\in[0,T]:\max\{x(t-\tau_1(t)),x(t-\tau_2(t))\}<-\rho\},\\
E_2=\{t\in[0,T]:\max\{x(t-\tau_1(t)),x(t-\tau_2(t))\}>\rho\},\\
E_3=\{t\in[0,T]:|\max\{x(t-\tau_1(t)),x(t-\tau_2(t))\}|\le\rho\},\\
E_4=\{t\in[0,T]:-\rho\le x(t-\tau_1(t))\le\rho,-\rho\le x(t-\tau_2(t))\le\rho\},\\
E_5=\{t\in[0,T]:x(t-\tau_1(t))<-\rho,-\rho\le x(t-\tau_2(t))\le\rho\},\\
E_6=\{t\in[0,T]:-\rho\le x(t-\tau_1(t))\le\rho,x(t-\tau_2(t))<-\rho\}.
\end{gathered}$$

By (2.4) it is easy to see that

$$\int_0^T[g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))-e(t)]dt=0.$$

Hence

$$\begin{aligned}
&\int_{E_2} |g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))-e(t)|dt\\
&=\int_{E_2} [g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))-e(t)]dt\\
&=-\Big(\int_{E_1}+\int_{E_3}\Big)[g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))-e(t)]dt\\
&\leq \Big(\int_{E_1}+\int_{E_3}\Big)|g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))-e(t)|dt.
\end{aligned}$$

From the above inequality and (2.7) we obtain

$$\begin{aligned}
&\int_0^T |g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))-e(t)|dt\\
&\leq 2\Big(\int_{E_1}+\int_{E_3}\Big)|g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))-e(t)|dt\\
&\leq 2\Big(\int_{E_1}+\int_{E_3}\Big)|g_1(t,x(t-\tau_1(t)))|dt+2\Big(\int_{E_1}+\int_{E_3}\Big)|g_2(t,x(t-\tau_2(t)))|dt\\
&\quad +2\int_0^T|e(t)|dt\\
&\leq 2r_1\int_{E_1}|x(t-\tau_1(t)|^{p-1}dt+2r_2\int_{E_1}|x(t-\tau_2(t)|^{p-1}dt\\
&\quad +2\int_{E_3}(|g_1(t,x(t-\tau_1(t)))|+|g_2(t,x(t-\tau_2(t)))|)dt+2\int_0^T|e(t)|dt\\
&\leq 2(r_1+r_2)T\|x\|_\infty^{p-1}+2\Big(\int_{E_4}+\int_{E_5}+\int_{E_6}\Big)\Big(|g_1(t,x(t-\tau_1(t)))|\\
&\quad +|g_2(t,x(t-\tau_2(t)))|\Big)dt+2\int_0^T|e(t)|dt\\
&\leq 2(r_1+r_2)T\|x\|_\infty^{p-1}+2T(g_{1\rho}+g_{2\rho})+2r_1\int_{E_5}|x(t-\tau_1(t)|^{p-1}dt\\
&\quad +2Tg_{2\rho}+2r_2\int_{E_6}|x(t-\tau_2(t)|^{p-1}dt+2Tg_{1\rho}+2\int_0^T|e(t)|dt\\
&\leq 4(r_1+r_2)T\|x\|_\infty^{p-1}+4T(g_{1\rho}+g_{2\rho})+2\int_0^T|e(t)|dt,
\end{aligned}$$

where

$$g_{1\rho}=\max_{t\in[0,T],|x|\leq\rho}|g_1(t,x(t-\tau_1(t)))|,\quad g_{2\rho}=\max_{t\in[0,T],|x|\leq\rho}|g_2(t,x(t-\tau_2(t)))|.$$

From (2.6) and the above inequality, we have

$$\begin{aligned}
\|x'\|_p^p &\leq \|x\|_\infty\int_0^T|g_1(t,x(t-\tau_1(t)))+g_2(t,x(t-\tau_2(t)))-e(t)|dt\\
&\leq \|x\|_\infty[4(r_1+r_2)T\|x\|_\infty^{p-1}+4T(g_{1\rho}+g_{2\rho})+2\int_0^T|e(t)|dt]
\end{aligned}$$

$$\begin{aligned}&= 4(r_1+r_2)T\Big((\frac{T}{2})^{1/q}\|x'\|_p + d\Big)^p + [4T(g_{1\rho}+g_{2\rho})\\&\quad + 2\int_0^T |e(t)|dt]\Big((\frac{T}{2})^{1/q}\|x'\|_p + d\Big).\end{aligned}$$

Case (1): $\|x'(t)\| = 0$, from (2.6) we see $\|x\|_\infty \le d$.

Case (2): $\|x'(t)\| > 0$, then we know that

$$[(\frac{T}{2})^{1/q}\|x'\|_p + d]^p = (\frac{T}{2})^{p/q}\|x'\|_p^p[1 + (\frac{T}{2})^{-1/q}\frac{d}{\|x'(t)\|_p}]^p.$$

From mathematical analysis, there is a constant $\delta > 0$ such that

$$(1+x)^p < 1 + (1+p)x, \quad \forall x \in [0,\delta]. \tag{2.8}$$

If $(\frac{T}{2})^{-1/q}\frac{d}{\|x'(t)\|_p} > \delta$, then we have $\|x'\|_p < (\frac{T}{2})^{-1/q}\frac{d}{\delta}$.

If $(\frac{T}{2})^{-1/q}\frac{d}{\|x'(t)\|_p} \le \delta$, by (2.8) we know that

$$[(\frac{T}{2})^{1/q}\|x'\|_p + d]^p \le (\frac{T}{2})^{p/q}\|x'(t)\|_p^p + (p+1)(\frac{T}{2})^{p-1)/q} d\|x'(t)\|_p^{p-1}. \tag{2.9}$$

By (2.9), we obtain

$$\begin{aligned}\|x'\|_p^p &\le 4(r_1+r_2)T(\frac{T}{2})^{p/q}\|x'(t)\|_p^p + (p+1)(\frac{T}{2})^{p-1)/q} d\|x'(t)\|_p^{p-1}\\&\quad + (4T(g_{1\rho}+g_{2\rho}) + 2\int_0^T |e(t)|dt)\Big((\frac{T}{2})^{1/q}\|x'\|_p + d\Big).\end{aligned}$$

As $p > 1$, $4(r_1+r_2)T(\frac{T}{2})^{p/q} < 1$, there exists a constant $R_2 > 0$ such that $\|x'\|_p \le R_2$.

Let $R_1 = \max\{(\frac{T}{2})^{-1/q}\frac{d}{\delta}, R_2\}$. Then we have $\|x\|_\infty \le (\frac{T}{2})^{1/q}R_1 := R_0$. By the second equation of (2.1) we obtain

$$y'(t) = -f(x(t))x'(t) - \lambda g_1(t, x(t-\tau_1(t))) - \lambda g_2(t, x(t-\tau_2(t))) + \lambda e(t).$$

Hence

$$\begin{aligned}\int_0^T |y'(t)|dt &\le f_{R_0}\int_0^T |x'(t)|dt + Tg_{1R_0} + Tg_{2R_0} + \int_0^T |e(t)|dt\\&\le f_{R_0}T^{1/q}\|x'\|_p + Tg_{1R_0} + Tg_{2R_0} + \int_0^T |e(t)|dt\\&\le f_{R_0}T^{1/q}R_1 + Tg_{1R_0} + Tg_{2R_0} + \int_0^T |e(t)|dt := R_3,\end{aligned}$$

where

$$f_{R_0} = \max_{|s|\le R_0}|f(s)|, \quad g_{1R_0} = \max_{t\in[0,T], s\le R_0}|g_1(t,s)|, \quad g_{2R_0} = \max_{t\in[0,T], s\le R_0}|g_2(t,s)|.$$

By the first equation of (2.1) we have $\int_0^T \phi_q(y(t))dt = 0$, which implies there exists a constant $t_1 \in [0,T]$ such that $y(t_1) = 0$. So

$$|y(t)| = |\int_{t_1}^t y'(s)ds| \le \int_0^T |y'(s)|ds \le R_3,$$

and $\|y\|_\infty \le R_3$.

Let $R_4 > \max\{R_0, R_3\}$, $\Omega = \{z \in Z : \|z\| < R_4\}$, then $Lz \neq \lambda NZ$, for all $z \in \operatorname{Dom} L \cap \partial\Omega$, $\lambda \in (0,1)$. Since

$$QNz = \frac{1}{T}\int_0^T \begin{pmatrix} \phi_q(y(t)) \\ -f(x(t))x'(t) - g_1(t, x(t-\tau_1(t))) - g_2(t, x(t-\tau_2(t))) + e(t) \end{pmatrix},$$

for any $z \in \ker L \cap \partial\Omega$, if $QNz = 0$, we obtain $y = 0$, $|x| = R_4 > d$. But when $|x| = R_4$, we know that $-g_1(t,x) - g_2(t,x) + e(t) < 0$, which yields a contradiction. So conditions (1) and (2) of Lemma 1.2 is satisfied.

Define the isomorphism $J : \operatorname{Im} Q \to \ker L$ as follows:

$$J(x,y)^T = (-y, x)^T.$$

Let $H(\mu, z) = \mu x + (1-\mu)JQNz, (\mu, z) \in [0,1] \times \Omega$, then we have

$$H(\mu, z) = \begin{pmatrix} \mu x + (1-\mu)\frac{1}{T}\int_0^T [g_1(t,x) + g_2(t,x) - e(t)]dt \\ \mu y + (1-\mu)\phi_q(y) \end{pmatrix},$$

where $(\mu, z) \in [0,1] \times (\ker L \cap \partial\Omega)$. It is obvious that $H(\mu,z) = \mu x + (1-\mu)JQNz \neq 0$ for $(\mu, z) \in [0,1] \times (\ker L \cap \partial\Omega)$. Hence

$$\deg\{JQN, \Omega \cap \ker L, 0\} = \deg\{I, \Omega \cap \ker L, 0\} = 1 \neq 0.$$

So the condition (3) of Lemma 1.2 is satisfied. By applying Lemma 1.2, we conclude that equation $Lz = Nz$ has a solution $z(t) = (x(t), y(t))^T$; i.e., (1.1) has a T-periodic solution $x(t)$. □

Theorem 2.2. *Suppose that there exist constants $d > 0$, $r_1 \geq 0$, and $r_2 \geq 0$ such that* (H1) *holds and*

(H2*) $\lim_{x\to+\infty} \sup_{t\in[0,T]} \frac{|g_1(t,x)|}{|x|^{p-1}} \leq r_1$; $\lim_{x\to+\infty} \sup_{t\in[0,T]} \frac{|g_2(t,x))|}{|x|^{p-1}} \leq r_2$.

Then (1.1) *has at least one T-periodic solution, if $4(r_1 + r_2)T(\frac{T}{2})^{p/q} < 1$.*

Theorem 2.3. *Suppose that $p > 2$ and there exist constants $d > 0$, $b_1 \geq 0$, $b_2 \geq 0$ such that* (H1) *holds and*

(H3) $|g_i(t,u) - g_i(t,v)| \leq b_i|u - v|$ *for all $t, u, v \in \mathbb{R}$, $i = 1, 2$.*

Then (1.1) *has at least one T-periodic solution.*

Proof. By the proof of Theorem 2.1, we have

$$\|x\|_\infty \leq \|\chi\|_\infty + d \leq (\frac{T}{2})^{1/q}\|\chi'\|_p + d = (\frac{T}{2})^{1/q}\|x'\|_p + d,$$

and

$$\|x'\|_p^p \leq \|x\|_\infty \int_0^T |g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t))) - e(t)|dt.$$

From assumption (H3), we have

$$\begin{aligned}
&\int_0^T |g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t))) - e(t)|dt \\
&\leq \int_0^T |g_1(t, x(t-\tau_1(t))) - g_1(t,0)|dt + \int_0^T |g_1(t,0)|dt \\
&\quad + |g_2(t, x(t-\tau_2(t))) - g_1(t,0)|dt + \int_0^T |g_2(t,0)|dt + \int_0^T |e(t)|dt \\
&\leq b_1 \int_0^T |x(t-\tau_1(t))|dt + \int_0^T |g_1(t,0)|dt + b_2 \int_0^T |x(t-\tau_2(t))|dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^T |g_2(t,0)|dt+\int_0^T |e(t)|dt\\
&\le T(b_1+b_2)\|x\|_\infty+Tb
\end{aligned}
$$

where $b=\max\{|g_1(t,0)|+|g_2(t,0)|+|e(t)|\}$. Thus,

$$
\begin{aligned}
\|x'\|_p^p &\le \|x\|_\infty[T(b_1+b_2)\|x\|_\infty+Tb]\\
&\le T(b_1+b_2)[(\frac{T}{2})^{1/q}\|x'\|_p+d]^2+Tb(\frac{T}{2})^{1/q}\|x'\|_p+Tbd.
\end{aligned}
$$

As $p>2$, there exists a constant $R_2>0$ such that $\|x'\|_p\le R_2$. The rest of the proof is same to Theorem 2.1 and is omitted. □

Corollary 2.4. *Suppose that $p=2$ and conditions* (H1), (H3) *hold. Then* (1.1) *has at least one T-periodic solution, if $T(b_1+b_2)(\frac{T}{2})^{\frac{2}{q}}<1$.*

Remark 2.5. If condition (H1) is replaced by

(H1*) $g_1(t,u)+g_2(t,v)-e(t)<0$ for all $t\in\mathbb{R}$, $|\max\{u,v\}|>d$,

then the results in this article still hold.

Example 2.6. Consider the equation

$$
(\phi_3(x'(t)))'+e^{x(t)}x'(t)+g_1(t,x(t-\sin t))+g_2(t,x(t-\cos t))=\frac{1}{\pi}\sin t, \quad (2.10)
$$

where $p=3$, $T=2\pi$, $\tau_1(t)=\sin t$, $\tau_2(t)=\cos t$,

$$
g_1(t,x)=\begin{cases} e^{\sin^2 t}x^3+\frac{1}{\pi}\sin t, & x\ge 0,\\ \frac{x^2}{18e\pi^3}e^{\sin^2 t}+\frac{1}{\pi}\sin t, & x<0,\end{cases}
$$

$$
g_2(t,x)=\begin{cases} e^{\cos^2 t}x^3, & x\ge 0,\\ \frac{x^2}{18e\pi^3}e^{\cos^2 t}, & x<0.\end{cases}
$$

By (2.10), we can get $d=1/10$ (Actually, d can be an arbitrarily small positive), $r_1=r_2=1/(18\pi^3)$, $4(r_1+r_2)T(T/2)^{p/q}<1$ and check that (H1)–(H2) hold. Thus, according to Theorem 2.1, equation (2.10) has at least one 2π-periodic solution.

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