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POSITIVE SOLUTIONS FOR SYSTEMS OF THIRD-ORDER GENERALIZED STURM-LIOUVILLE BOUNDARY-VALUE PROBLEMS WITH (p,q)-LAPLACIAN

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Abstract. In this work, we use the Leggett-Williams fixed point theorem to prove the existence of at least three positive solutions to a system of third-order ordinary differential equations with (p,q)-Laplacian

1. Introduction

In this article, we prove the existence of at least three positive solutions to a boundary-value problem with the (p,q)-Laplacian:

$$
\begin{gathered}
(\phi_p(u''(t)))' + a_1(t) f_1(t,u(t),v(t)) = 0 \quad 0 \le t \le 1, \\
(\phi_q(v''(t)))' + a_2(t) f_2(t,u(t),v(t)) = 0 \quad 0 \le t \le 1, \\
\alpha_1 u(0) - \beta_1 u'(0) = \mu_{11} u(\xi_1), \quad \gamma_1 u(1) + \delta_1 u'(1) = \mu_{12} u(\eta_1), \quad u''(0) = 0, \\
\alpha_2 v(0) - \beta_2 v'(0) = \mu_{21} v(\xi_2), \quad \gamma_2 v(1) + \delta_2 v'(1) = \mu_{22} u(\eta_2), \quad v''(0) = 0,
\end{gathered}
\tag{1.1}
$$

where $\phi_p(s) = |s|^{p-2}s$ and $\phi_q(s) = |s|^{q-2}s$ are p,q-Laplacian operators; $p > 1$, $q > 1$, $0 < \xi_i < 1$, $0 < \eta_i < 1$, for $i = 1, 2$.

Il'in and Moiseev [4] studied the existence of solutions for a linear multi-point boundary-value problem. Gupta [3] studied certain three-point boundary-value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multi-point boundary-value problems have been studied by several authors because multi-point boundary-value problems describe many phenomena of applied mathematics and physics (see [2, 5, 12, 11]). There is much current interest in questions of positive solutions of boundary-value problems for ordinary differential equations, on may see [1, 8, 9, 10, 13, 14, 15] and references therein. Motivated by the works [7, 16], in this paper we will show the existence of three positive solutions for the problem (1.1).

The basic space used in this paper is a real Banach space $E = (C[0,1],\mathbb{R}) \times (C[0,1],\mathbb{R})$ with the norm $\|(u,v)\| := \|u\| + \|v\|$, where $\|u\| = \max_{t\in[0,1]} |u(t)|$. For convenience, we make the following assumptions:

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(H1) $\alpha_i \geq 0$, $\beta_i \geq 0$, $\gamma_i \geq 0$, $\delta_i \geq 0$, $\rho_i = \alpha_i\gamma_i + \beta_i\gamma_i + \alpha_i\delta_i > 0$, $\rho_i - \mu_{i2}\psi(\eta_i) > 0$, $\rho_i - \mu_{i1}\varphi(\xi_i) > 0$, $\mu_{i1}, \mu_{i2} > 0$, $\Delta_i < 0$, for $i = 1, 2$, and $\sigma \in (0, 1/2)$,

$$\Delta_i = \begin{vmatrix} -\mu_{i1}\psi(\xi_i) & \rho - \mu_{i1}\varphi(\xi_i) \\ \rho - \mu_{i2}\psi(\eta_i) & -\mu_{i2}\varphi(\eta_i) \end{vmatrix}, \ i = 1, 2,$$

where

$$\psi_i(t) = \beta_i + \alpha_i t, \quad \varphi_i(t) = \gamma_i + \delta_i - \gamma_i t, \quad t \in [0, 1], \quad i = 1, 2, \tag{1.2}$$

are linearly independent solutions of the equation $x''(t) = 0$, $t \in [0, 1]$. Obviously, ψ_i is non-decreasing on $[0, 1]$ and φ_i is non-increasing on $[0, 1]$.

(H2) $f_i \in C([0, 1] \times [0, +\infty) \times [0, +\infty), [0, +\infty))$, and $a_i : (0, 1) \to [0, +\infty)$ is continuous and $a_i(t) \not\equiv 0$, for $i = 1, 2$ on any subinterval of $(0, 1)$, and

$$0 < \int_0^1 a_i(s)ds < +\infty.$$

For the convenience of the reader, we present here the Leggett-Williams fixed point theorem [6].

Given a cone K in a real Banach space E, a map α is said to be a nonnegative continuous concave (resp. convex) functional on K provided that $\alpha : K \to [0. +\infty)$ is continuous and

$$\alpha(tx + (1 - t)y) \geq t\alpha(x) + (1 - t)\alpha(y),$$
$$(resp.\alpha(tx + (1 - t)y) \leq t\alpha(x) + (1 - t)\alpha(y)),$$

for all $x, y \in K$ and $t \in [0, 1]$.

Let $0 < a < b$ be given and let α be a nonnegative continuous concave functional on K. Define the convex sets P_r and $P(\alpha, a, b)$ by

$$P_r = \{x \in K | \|x\| < r\},$$
$$P(\alpha, a, b) = \{x \in K | a \leq \alpha(x), \|x\| \leq b\}.$$

Theorem 1.1 (Leggett-Williams fixed point theorem)**.** . *Let $A : \overline{P_c} \to \overline{P_c}$ be a completely continuous operator and let α be a nonnegative continuous concave functional on K such that $\alpha(x) \leq \|x\|$ for all $x \in \overline{P_c}$. Suppose there exist $0 < a < b < d \leq c$ such that*

(A1) *$x \in P(\alpha, b, d) | \alpha(x) > b \neq \emptyset$, and $\alpha(Ax) > b$ for $x \in P(\alpha, b, d)$;*
(A2) *$\|Ax\| < a$ for $\|x\| \leq a$; and*
(A3) *$\alpha(Ax) > b$ for $x \in P(\alpha, b, c)$ with $\|Ax\| > d$.*

Then A has at least three fixed points x_1, x_2, and x_3 and such that $\|x_1\| < a, b < \alpha(x_2)$ and $\|x_3\| > a$, with $\alpha(x_3) < b$.

Inspired and motivated by the works mentioned above, in this work we will consider the existence of positive solutions to (1.1). We shall first give a new form of the solution, and then determine the properties of the Green's function for associated linear boundary-value problems; finally, by employing the Leggett-Williams fixed point theorem, some sufficient conditions guaranteeing the existence of three positive solutions. The rest of the article is organized as follows: in Section 2, we present some preliminaries that will be used in Section 3. The main results and proofs will be given in Section 3. Finally, in Section 4, an example are given to demonstrate the application of our main result.

2. Preliminaries

In this section, we present some notations and preliminary lemmas that will be used in the proof of the main result.

Definition 2.1. Let X be a real Banach space. A non-empty closed convex set $P \subset X$ is called a cone of X if it satisfies the following conditions:

(1) $x \in P, \mu \geq 0$ implies $\mu x \in P$;
(2) $x \in P, -x \in P$ implies $x = 0$.

Let $y_1(t) = -\phi_p(u''(t))$, $y_2(t) = -\phi_q(v''(t))$, then for $i = 1, 2$, the following two boundary-value problems

$$\begin{gathered}(\phi_p(u''(t)))' + a_1(t)f_1(t, u(t), v(t)) = 0, \quad 0 \leq t \leq 1, \\ u''(0) = 0,\end{gathered}$$

and

$$\begin{gathered}(\phi_q(v''(t)))' + a_2(t)f_2(t, u(t), v(t)) = 0, \quad 0 \leq t \leq 1, \\ v''(0) = 0,\end{gathered}$$

are turned into the following two boundary-value problems

$$\begin{gathered}y_i'(t) - a_i(t)f_i(t, u(t), v(t)) = 0, \quad 0 \leq t \leq 1, \\ y_i(0) = 0,\end{gathered} \qquad i = 1, 2. \tag{2.1}$$

Lemma 2.2. *Problem* (2.1) *has a unique solution*

$$y_i(t) = \int_0^t a_i(s)f_i(s, u(s), v(s))ds, \quad i = 1, 2. \tag{2.2}$$

Lemma 2.3. *If* (H1) *holds, then for $y_i(t) \in C([0,1])$, the following two boundary-value problems*

$$\begin{gathered}u''(t) + \phi_p^{-1}(y_1(t)) = 0, \quad 0 \leq t \leq 1, \\ \alpha_1 u(0) - \beta_1 u'(0) = \mu_{11}u(\xi_1), \\ \gamma_1 u(1) + \delta_1 u'(1) = \mu_{12}u(\eta_1),\end{gathered} \tag{2.3}$$

and

$$\begin{gathered}v''(t) + \phi_q^{-1}(y_2(t)) = 0, \quad 0 \leq t \leq 1, \\ \alpha_2 v(0) - \beta_2 v'(0) = \mu_{21}v(\xi_2), \\ \gamma_2 v(1) + \delta_2 v'(1) = \mu_{22}v(\eta_2),\end{gathered} \tag{2.4}$$

have a unique solutions

$$u(t) = \int_0^1 G_1(t, s)\phi_p^{-1}(y_1(s))ds + A_1(\phi_p^{-1}(y_1))\psi_1(t) + B(\phi_p^{-1}(y_1))\varphi_1(t), \tag{2.5}$$

$$v(t) = \int_0^1 G_2(t, s)\phi_q^{-1}(y_2(s))ds + A_2(\phi_q^{-1}(y_2))\psi_2(t) + B_2(\phi_q^{-1}(y_2))\varphi_2(t), \tag{2.6}$$

where

$$G_i(t, s) = \frac{1}{\rho_i}\begin{cases}\varphi_i(t)\psi_i(s), & s \leq t, \\ \varphi_i(s)\psi_i(t), & t \leq s,\end{cases} \quad i = 1, 2, \tag{2.7}$$

$$A_1(\phi_p^{-1}(y_1)) = \frac{1}{\Delta_1}\begin{vmatrix}\mu_{11}\int_0^1 G_1(\xi_1, s)\phi_p^{-1}(y_1)ds & \rho_1 - \mu_{11}\varphi_1(\xi_1) \\ \mu_{12}\int_0^1 G_1(\eta_1, s)\phi_p^{-1}(y_1)ds & -\mu_{12}\varphi_1(\eta_1)\end{vmatrix}, \tag{2.8}$$

$$B_1(\phi_p^{-1}(y_1)) = \frac{1}{\Delta_1}\begin{vmatrix} -\mu_{11}\psi_1(\xi_1) & \mu_{11}\int_0^1 G_1(\xi_1,s)\phi_p^{-1}(y_1)ds \\ \rho_1-\mu_{12}\psi_1(\eta_1) & \mu_{12}\int_0^1 G_1(\eta_1,s)\phi_p^{-1}(y_1)ds \end{vmatrix}, \tag{2.9}$$

$$A_2(\phi_q^{-1}(y_2)) = \frac{1}{\Delta_2}\begin{vmatrix} \mu_{21}\int_0^1 G_2(\xi_2,s)\phi_q^{-1}(y_2)ds & \rho_2-\mu_{21}\varphi_2(\xi_2) \\ \mu_{22}\int_0^1 G_2(\eta_2,s)\phi_q^{-1}(y_2)ds & -\mu_{22}\varphi_2(\eta_2) \end{vmatrix}, \tag{2.10}$$

and

$$B_2(\phi_q^{-1}(y_2)) = \frac{1}{\Delta_2}\begin{vmatrix} -\mu_{21}\psi_2(\xi_2) & \mu_{21}\int_0^1 G_2(\xi_2,s)\phi_q^{-1}(y_2)ds \\ \rho_2-\mu_{22}\psi_2(\eta_2) & \mu_{22}\int_0^1 G_2(\eta_2,s)\phi_q^{-1}(y_2)ds \end{vmatrix}. \tag{2.11}$$

The proof of the above theorem follows by routine calculations. We omit it.

Remark 2.4. For a fixed integrable function y, it is obvious that $A(\phi_p^{-1}(y))$ and $B(\phi_q^{-1}(y))$ are constant.

For convenience, let

$$\Lambda_0 = \min\Big\{\frac{\varphi_1(1-\sigma)}{\varphi_1(\sigma)}, \frac{\psi_1(\sigma)}{\psi_1(1)}, \frac{\varphi_2(1-\sigma)}{\varphi_2(\sigma)}, \frac{\psi_2(\sigma)}{\psi_2(1)}\Big\},$$
$$\Lambda_1 = \max\Big\{1, \|\psi_1\|, \|\varphi_1\|, \|\psi_2\|, \|\varphi_2\|\Big\},$$
$$\Lambda_2 = \min\Big\{\min_{t\in[\sigma,1-\sigma]}\varphi_1(t), \min_{t\in[\sigma,1-\sigma]}\psi_1(t), \min_{t\in[\sigma,1-\sigma]}\varphi_2(t), \min_{t\in[\sigma,1-\sigma]}\psi_2(t), 1\Big\},$$
$$\lambda = \min\Big\{\Lambda_0, \frac{\Lambda_2}{\Lambda_1}\Big\}.$$

If (H1) and (H2) hold, then from Lemmas 2.2 and 2.3, we know that $(u(t),v(t))$ is a solution of (1.1) if and only if

$$u(t) = \int_0^1 G_1(t,s)\phi_p^{-1}(W_1(s))ds + A_1(\phi_p^{-1}(W_1(s)))\psi_1(t) + B_1(\phi_p^{-1}(W_1(s)))\varphi_1(t),$$
$$v(t) = \int_0^1 G_2(t,s)\phi_q^{-1}(W_2(s))ds + A_2(\phi_q^{-1}(W_2(s)))\psi_2(t) + B_2(\phi_q^{-1}(W_2(s)))\varphi_2(t),$$

where $0 \le t \le 1$, and $W_i(s) = \int_0^s a_i(\tau)f_i(\tau,u(\tau),v(\tau))d\tau$, for $i = 1,2$.

We need some properties of the functions $G_i, i = 1,2$, in order to discuss the existence of positive solutions. For the Green's functions $G_i(t,s)$, we have the following two Lemmas [7].

Lemma 2.5. *If* (H1) *and* (H2) *hold, then*

$$0 \le G_i(t,s) \le G_i(s,s), \quad t,s \in [0,1], \tag{2.12}$$
$$G_i(t,s) \ge \Lambda_0 G_i(s,s), \quad t \in [\sigma,1-\sigma], s \in [0,1], \tag{2.13}$$

for $i = 1,2$.

Denote

$$K = \big\{(u,v) \in E : u(t) \ge 0, v(t) \ge 0, \min_{t\in[\sigma,1-\sigma]}(u(t)+v(t)) \ge \lambda\|(u,v)\|\big\}.$$

It is obvious that K is cone. Define the operator $T : E \to E$ by

$$T(u,v)(t) = (T_1(u,v)(t), T_2(u,v)(t)), \quad \forall t \in (0,1), \tag{2.14}$$

where

$$
\begin{aligned}
T_1(u,v)(t) &= \int_0^1 G_1(t,s)\phi_p^{-1}(W_1(s))ds + A_1(\phi_p^{-1}(W_1(s)))\psi_1(t) \\
&\quad + B_1(\phi_p^{-1}(W_1(s)))\varphi_1(t), \quad 0 \le t \le 1, \\
T_2(u,v)(t) &= \int_0^1 G_2(t,s)\phi_q^{-1}(W_2(s))ds + A_2(\phi_q^{-1}(W_2(s)))\psi_2(t) \\
&\quad + B_2(\phi_q^{-1}(W_2(s)))\varphi_2(t), \quad 0 \le t \le 1,
\end{aligned}
\tag{2.15}
$$

where $W_i(s) = \int_0^s a_i(\tau) f_i(\tau, u(\tau), v(\tau))d\tau$, for $i = 1, 2$. Evidently, $(u(t), v(t))$ is a solution of (1.1) if and only if $(u(t), v(t))$ is a fixed point of operator T.

Lemma 2.6. *If* (H1) *and* (H2) *hold, then the operator defined in* (2.14) *satisfies* $T(K) \subseteq K$.

Proof. For $(u,v) \in K$, then from properties of $G_1(t,s)$ and $G_2(t,s), T_1(u,v)(t) \ge 0$, $T_2(u,v)(t) \ge 0$, $t \in [0,1]$, and it follows form (2.15) that

$$
\begin{aligned}
&T_1(u,v)(t) \\
&= \int_0^1 G_1(t,s)\phi_p^{-1}(W_1(s))ds + A_1(\phi_p^{-1}(W_1(s)))\psi_1(t) + B_1(\phi_p^{-1}(W_1(s)))\varphi_1(t) \\
&\le \int_0^1 G_1(s,s)\phi_p^{-1}(W_1(s))ds + \Lambda_1[A_1(\phi_p^{-1}(W_1(s))) + B_1(\phi_p^{-1}(W_1(s)))].
\end{aligned}
$$

Thus,

$$
\|T_1(u,v)\| \le \int_0^1 G_1(s,s)\phi_p^{-1}(W_1(s))ds + \Lambda_1[A_1(\phi_p^{-1}(W_1(s))) + B_1(\phi_p^{-1}(W_1(s)))].
$$

On the other hand, for $t \in [\sigma, 1-\sigma]$, we have

$$
\begin{aligned}
&\min_{t\in[\sigma,1-\sigma]} T_1(u,v)(t) \\
&= \min_{t\in[\sigma,1-\sigma]} \Big[\int_0^1 G_1(t,s)\phi_p^{-1}(W_1(s))ds + A_1(\phi_p^{-1}(W_1))\psi_1(t) + B_1(\phi_p^{-1}(W_1))\varphi_1(t)\Big] \\
&\ge \Lambda_0 \int_0^1 G_1(s,s)\phi_p^{-1}(W_1(s))ds + A_1(\phi_p^{-1}(W_1))\psi_1(t) + B_1(\phi_p^{-1}(W_1))\varphi_1(t) \\
&\ge \Lambda_0 \int_0^1 G_1(s,s)\phi_p^{-1}(W_1(s))ds + \frac{\Lambda_2}{\Lambda_1}.\Lambda_1.[A_1(\phi_p^{-1}(W_1)) + B_1(\phi_p^{-1}(W_1))] \\
&\ge \lambda\big[\int_0^1 G_1(s,s)\phi_p^{-1}(W_1(s))ds + \Lambda_1.[A_1(\phi_p^{-1}(W_1)) + B_1(\phi_p^{-1}(W_1))]\big] \\
&\ge \lambda\|T_1(u,v)\|.
\end{aligned}
$$

In this way, for any $(u,v) \in K$, we have

$$
\min_{t\in[\sigma,1-\sigma]} T_2(u,v)(t) \ge \lambda\|T_2(u,v)\|.
$$

Therefore,

$$
\begin{aligned}
\min_{t\in[\sigma,1-\sigma]} \big(T_1(u,v)(t), T_2(u,v)(t)\big) &\ge \lambda\|T_1(u,v)\| + \lambda\|T_2(u,v)\| \\
&= \lambda\|(T_1(u,v), T_2(u,v))\|.
\end{aligned}
$$

From the above, we obtain $T(K) \subseteq K$. This completes the proof. □

3. Main results

In this section, we discuss the existence of positive solutions of (1.1). We define the nonnegative continuous concave functional on K by

$$\alpha(u,v)=\min_{\sigma\le t\le 1-\sigma}(u(t)+v(t)).$$

It is obvious that, for each $(u,v)\in K$, $\alpha(u,v)\le\|(u,v)\|$.

In this section, for convenience, we denote

$$\widetilde{A_i}=\frac{1}{\Delta_i}\begin{vmatrix}\mu_{i1} & \rho_i-\mu_{i1}\varphi_i(\xi_i)\\ \mu_{i2} & -\mu_{i2}\varphi_i(\eta_i)\end{vmatrix},\quad \widetilde{B_0}=\frac{1}{\Delta_i}\begin{vmatrix}-\mu_{i1}\psi_i(\xi_i) & \mu_{i1}\\ \rho_i-\mu_{i2}\psi_i(\eta_i) & \mu_{i2}\end{vmatrix},$$

$$M_i=\max_{0\le t\le 1}\int_0^1 G_i(t,s)ds,\quad m_i=\min_{\sigma\le t\le 1-\sigma}\int_\sigma^{1-\sigma} G_i(t,s)ds,\quad i=1,2.$$

Also we use the following assumptions: There exist nonnegative numbers a,b,c such that $0<a<b\le\min\{\lambda,\frac{m_1}{p_1M_1},\frac{m_2}{p_2M_2}\}c$, and $f_i(t,u,v)$ satisfy the following conditions:

(H3) $f_1(t,u,v)<\frac{1}{\int_0^1 a_1(t)dt}\phi_p\big(\frac{c}{p_1M_1[1+\Lambda_1\widetilde{A_1}+\Lambda_1\widetilde{B_1}]}\big)$, and

$$f_2(t,u,v)<\frac{1}{\int_0^1 a_2(t)dt}\phi_q\big(\frac{c}{p_2M_2[1+\Lambda_1\widetilde{A_2}+\Lambda_1\widetilde{B_2}]}\big),$$

for any $t\in[0,1],u+v\in[0,c]$;

(H4) $f_1(t,u,v)<\frac{1}{\int_0^1 a_1(t)dt}\phi_p\big(\frac{a}{p_1M_1[1+\Lambda_1\widetilde{A_1}+\Lambda_1\widetilde{B_1}]}\big)$, and

$$f_2(t,u,v)<\frac{1}{\int_0^1 a_2(t)dt}\phi_q\big(\frac{a}{p_2M_2[1+\Lambda_1\widetilde{A_2}+\Lambda_1\widetilde{B_2}]}\big),$$

for any $\forall t\in[0,1],u+v\in[0,a]$;

(H5) $f_1(t,u,v)>\frac{1}{\int_\sigma^{1-\sigma} a_1(t)dt}\phi_p\big(\frac{b}{m_1[1+\Lambda_2\widetilde{A_1}+\Lambda_2\widetilde{B_1}]}\big)$, or

$$f_2(t,u,v)>\frac{1}{\int_\sigma^{1-\sigma} a_2(t)dt}\phi_q\big(\frac{b}{m_2[1+\Lambda_2\widetilde{A_2}+\Lambda_2\widetilde{B_2}]}\big)$$

for any $t\in[0,1],u+v\in[b,\frac{b}{\lambda}]$, where $\frac{1}{p_1}+\frac{1}{p_2}\le 1$.

Theorem 3.1. *Under assumptions* (H1)–(H5)*, Problem* (1.1) *has at least three positive solutions* $(u_1,v_1),(u_2,v_2),(u_3,v_3)$ *such that* $\|(u_1,v_1)\|<a$, $b<\alpha((u_2,v_2))$, *and* $\|(u_3,v_3)\|>a$, *with* $\alpha((u_3,v_3))<b$.

Proof. First, we show that $T:\overline{P_c}\to\overline{P_c}$ is a completely continuous operator. If $(u,v)\in\overline{P_c}$, by condition (H3), we have

$$\begin{aligned}&A_1(\phi_p^{-1}(y))\\&\le\frac{1}{\Delta_1}\begin{vmatrix}\mu_{11}\int_0^1 G_1(\xi_1,s)\phi_p^{-1}(\int_0^s a_1(\tau)f_1(\tau,u(\tau),v(\tau))d\tau)ds & \rho_1-\mu_{11}\varphi_1(\xi_1)\\ \mu_{12}\int_0^1 G_1(\xi_1,s)\phi_p^{-1}(\int_0^s a_1(\tau)f_1(\tau,u(\tau),v(\tau))d\tau)ds & -\mu_{12}\varphi_1(\eta_1)\end{vmatrix}\\&\le\frac{c}{p_1[1+\Lambda_1\widetilde{A_1}+\Lambda_1\widetilde{B_1}]}\widetilde{A_1},\end{aligned}$$

and

$$B_1(\phi_p^{-1}(y))$$

$$
\begin{aligned}
&\le \frac{1}{\Delta_1}\begin{vmatrix} -\mu_{11}\psi(\xi_1) & \mu_{11}\int_0^1 G_1(\xi_1,s)\phi_p^{-1}(\int_0^s a_1(\tau)f_1(\tau,u(\tau),v(\tau))d\tau)ds \\ \rho_1-\mu_{12}\psi_1(\eta_1) & \mu_{12}\int_0^1 G_1(\xi_1,s)\phi_p^{-1}(\int_0^s a_1(\tau)f_1(\tau,u(\tau),v(\tau))d\tau)ds \end{vmatrix}\\
&\le \frac{c}{p_1[1+\Lambda_1\widetilde{A_1}+\Lambda_1\widetilde{B_1}]}\widetilde{B_1}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|T_1(u,v)\|\\
&= \max_{0\le t\le 1}|T_1(u,v)(t)|\\
&= \max_{0\le t\le 1}\Big(\int_0^1 G_1(t,s)\phi_p^{-1}(W_1(s))ds + A_1(\phi_p^{-1}(W_1))\psi_1(t)+B_1(\phi_p^{-1}(W_1))\varphi_1(t)\Big)\\
&\le \frac{c}{p_1[1+\Lambda_1\widetilde{A_1}+\Lambda_1\widetilde{B_1}]}+\frac{c}{p_1[1+\Lambda_1\widetilde{A_1}+\Lambda_1\widetilde{B_1}]}\widetilde{A_1}\varphi_1(t)\\
&\quad+\frac{c}{p_1[1+\Lambda_1\widetilde{A_1}+\Lambda_1\widetilde{B_1}]}\widetilde{B_1}\varphi_1(t)\\
&\le \frac{c}{p_1[1+\Lambda_1\widetilde{A_1}+\Lambda_1\widetilde{B_1}]}[1+\Lambda_1\widetilde{A_1}+\Lambda_1\widetilde{B_1}]=\frac{c}{p_1}.
\end{aligned}
$$

In the same way, for any $(u,v)\in\overline{P_c}$, we have

$$
\|T_2(u,v)\|\le\frac{c}{p_2}.
$$

thus

$$
\|T(u,v)\|=\|T_1(u,v)\|+\|T_2(u,v)\|\le\frac{c}{p_1}+\frac{c}{p_2}\le c.
$$

Therefore, $\|T(u,v)\|\le c$, that is, $T:\overline{P_c}\to\overline{P_c}$. The operator T is completely continuous by an application of the Ascoli-Arzela theorem.

In the same way, the condition (H4) implies that the condition (A2) of Theorem 1.1 is satisfied. We now show that condition (A1) of Theorem 1.1 is satisfied. Clearly, $\{(u,v)\in P(\alpha,b,\frac{b}{\lambda})|\alpha(u,v)>b\}\neq\emptyset$. If $(u,v)\in P(\alpha,b,\frac{b}{\lambda})$, then $b\le u(s)+v(s)\le\frac{b}{\lambda}$, $s\in[\sigma,1-\sigma]$. By condition (H5), we obtain

$$
\begin{aligned}
&A_1(\phi_p^{-1}(y))\\
&\ge\frac{1}{\Delta_1}\begin{vmatrix}\mu_{11}\int_\sigma^{1-\sigma}G_1(\xi_1,s)\phi_p^{-1}(\int_\sigma^{1-\sigma}a_1(\tau)f_1(\tau,u(\tau),v(\tau))d\tau)ds & \rho_1-\mu_{11}\varphi_1(\xi_1)\\ \mu_{12}\int_\sigma^{1-\sigma}G_1(\xi_1,s)\phi_p^{-1}(\int_\sigma^{1-\sigma}a_1(\tau)f_1(\tau,u(\tau),v(\tau))d\tau)ds & -\mu_{12}\varphi_1(\eta_1)\end{vmatrix}\\
&\ge\frac{b}{[1+\Lambda_2\widetilde{A_1}+\Lambda_2\widetilde{B_1}]}\widetilde{A_1},
\end{aligned}
$$

and

$$
\begin{aligned}
&B(\phi_p^{-1}(y))\\
&\ge\frac{1}{\Delta_1}\begin{vmatrix}-\mu_{11}\psi_1(\xi_1) & \mu_{11}\int_\sigma^{1-\sigma}G_1(\xi_1,s)\phi_p^{-1}(\int_\sigma^{1-\sigma}a_1(\tau)f_1(\tau,u(\tau),v(\tau)d\tau)ds\\ \rho_1-\mu_{12}\psi_1(\eta_1) & \mu_{12}\int_\sigma^{1-\sigma}G_1(\xi_1,s)\phi_p^{-1}(\int_\sigma^{1-\sigma}a_1(\tau)f_1(\tau,u(\tau),v(\tau))d\tau)ds\end{vmatrix}\\
&\ge\frac{b}{[1+\Lambda_2\widetilde{A_1}+\Lambda_2\widetilde{B_1}]}\widetilde{B_1}.
\end{aligned}
$$

Thus,

$\alpha(T(u,v)(t))$

$$
\begin{aligned}
&= \min_{\sigma\le t\le 1-\sigma}(T_1(u,v)(t)+T_2(u,v)(t))\\
&\ge \min_{\sigma\le t\le 1-\sigma}\Big(\int_0^1 G_1(t,s)\phi_p^{-1}(W_1(s))ds + A_1(\phi_p^{-1}(W_1))\psi_1(t)+B_1(\phi_p^{-1}(W_1))\varphi_1(t)\Big)\\
&\quad + \min_{\sigma\le t\le 1-\sigma}\Big(\int_0^1 G_2(t,s)\phi_q^{-1}(W_2(s))ds + A_2(\phi_q^{-1}(W_2))\psi_2(t)\\
&\quad + B_2(\phi_q^{-1}(W_2))\varphi_2(t)\Big)\\
&\ge \frac{b}{1+\Lambda_2\widetilde{A_1}+\Lambda_2\widetilde{B_1}} + \frac{b}{1+\Lambda_1\widetilde{A_1}+\Lambda_2\widetilde{B_1}}\widetilde{A_1}\psi_1(t) + \frac{b}{1+\Lambda_2\widetilde{A_1}+\Lambda_2\widetilde{B_1}}\widetilde{B_1}\varphi_1(t)\\
&\ge \frac{b}{1+\Lambda_2\widetilde{A_1}+\Lambda_2\widetilde{B_1}}[1+\Lambda_2\widetilde{A_1}+\Lambda_2\widetilde{B_1}] = b
\end{aligned}
$$

Therefore, condition (A1) of Theorem 1.1 is satisfied.

Finally, we show that the condition (A3) of Theorem 1.1 is also satisfied. If $(u,v)\in P(\alpha,b,c)$, and $\|T(u,v)\|>b/\lambda$, then

$$\alpha(T(u,v)(t)) = \min_{\sigma\le t\le 1-\sigma} T(u,v)(t) \ge \lambda\|T(u,v)\| > b.$$

Therefore, the condition (A3) of Theorem 1.1 is also satisfied.

By Theorem 1.1, there exist three positive solutions $(u_1,v_1),(u_2,v_2),(u_3,v_3)$ such that $\|(u_1,v_1)\|<a, b<\alpha((u_2,v_2))$, and $\|(u_3,v_3)\|>a$, with $\alpha((u_3,v_3))<b$. we have the conclusion. □

4. Application

As an example, we consider the boundary-value problem

$$
\begin{gathered}
(\phi_p(u''(t)))' + a_1(t)f_1(t,u(t),v(t)) = 0 \quad 0\le t\le 1,\\
(\phi_q(v''(t)))' + a_2(t)f_2(t,u(t),v(t)) = 0 \quad 0\le t\le 1,\\
u(0)-u'(0)=u(\frac14),\quad u(1)+u'(1)=\frac12 u(\frac12),\quad u''(0)=0,\\
v(0)-v'(0)=v(\frac14),\quad v(1)+v'(1)=\frac12 v(\frac12),\quad v''(0)=0,
\end{gathered}
\tag{4.1}
$$

where $a_i(t)=1$, $\alpha_i=\beta_i=\gamma_i=\delta_i=1$, for $i=1,2$, and

$$
\begin{aligned}
&f_1(t,u,v)=f_2(t,u,v)\\
&=\begin{cases}
\frac{t}{1000}+\frac{u+v}{1000}, & t\in[0,1],\ 0\le u+v\le 1,\\
\frac{t}{1000}+2((u+v)^2-(u+v))+\frac{1}{1000}, & t\in[0,1],\ 1<u+v<2,\\
\frac{t}{1000}+2\big[\log_2(u+v)+\frac{u+v}{2}\big]+\frac{1}{1000}, & t\in[0,1],\ 2\le u+v\le 4,\\
\frac{t}{1000}+4\sqrt{u+v}+\frac{1}{1000}, & t\in[0,1],\ 4<u+v<+\infty,
\end{cases}
\end{aligned}
$$

Choose $\sigma=\frac14$, $p=q=3$, $p_1=p_2=2$. Then by direct calculations, we obtain

$$
\begin{gathered}
\rho_i = 3\psi_i(t)=1+t, \varphi_i(t)=2-t, t\in[0,1],\\
\Delta_i = \begin{vmatrix} -\mu_{i1}\psi_i(\xi_i) & \rho_i-\mu_{i1}\varphi_1(\xi_1)\\ \rho_i-\mu_{i2}\psi_i(\eta_i) & -\mu_{i2}\varphi_i(\eta_i)\end{vmatrix} = -\frac{15}{8},\quad \widetilde{A_i}=\frac{11}{15},\quad \widetilde{B_i}=\frac{23}{8},\\
M_i = \max_{0\le t\le 1}\int_0^1 G_i(t,s)ds = \frac43,\quad m_i=\min_{\sigma\le t\le 1-\sigma}\int_\sigma^{1-\sigma} G_i(t,s)ds = \frac{25}{96},
\end{gathered}
$$

$$\int_0^1 a_i(t)dt = 1, \quad \int_\sigma^{1-\sigma} a_i(t)dt = \frac{1}{2} \quad i = 1,2,$$
$$\Lambda_0 = \min\Big\{\frac{\varphi_1(1-\sigma)}{\varphi_1(\sigma)}, \frac{\psi_2(\sigma)}{\psi_2(1)}, \frac{\varphi_2(1-\sigma)}{\varphi_2(\sigma)}, \frac{\psi_2(\sigma)}{\psi_2(1)}\Big\} = \frac{5}{7},$$
$$\Lambda_1 = \max\Big\{1, \|\psi_1\|, \|\varphi_1\|, \|\psi_2\|, \|\varphi_2\|\Big\} = 2,$$
$$\Lambda_2 = \min\Big\{\min_{t\in[\sigma,1-\sigma]} \varphi_1(t), \min_{t\in[\sigma,1-\sigma]} \psi_1(t), \min_{t\in[\sigma,1-\sigma]} \varphi_2(t), \min_{t\in[\sigma,1-\sigma]} \psi_2(t)1\Big\} = 1,$$
$$\lambda = \min\Big\{\Lambda_0, \frac{\Lambda_2}{\Lambda_1}\Big\} = \frac{1}{2},$$
$$\frac{1}{p_iM_i[1+\Lambda_1\widetilde{A_i}+\Lambda_1\widetilde{B_i}]} = \frac{45}{986}, \quad \frac{1}{m_i[1+\Lambda_2\widetilde{A_i}+\Lambda_2\widetilde{B_i}]} = \frac{2304}{2765}, \quad i = 1,2.$$

So we choose $a = 1$, $b = 2$, $c = 200$, Then, by Theorem 3.1, system (4.1) has at least three positive solutions (u_1, v_1), (u_2, v_2), (u_3, v_3) such that $\|(u_1, v_1)\| < a, b < \alpha((u_2, v_2))$, and $\|(u_3, v_3)\| > a$, with $\alpha((u_3, v_3)) < b$.

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