Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 138, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

SUBHARMONIC SOLUTIONS FOR NON-AUTONOMOUS SECOND-ORDER SUBLINEAR HAMILTONIAN SYSTEMS WITH p-LAPLACIAN

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Abstract. In this article, we study the existence of subharmonic solutions to the non-autonomous second-order sublinear Hamiltonian systems with p-Laplacian. Introducing some kinds of control functions, infinitely many subharmonic solutions are obtained by using the minimax methods in critical point theory. We point out that our results are new even in the case $p = 2$.

1. Introduction and main results

Consider the second-order system

$$\frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)) + \nabla F(t, u(t)) = 0 \quad \text{a.e. } t \in \mathbb{R}. \tag{1.1}$$

where $p > 1$, $F : \mathbb{R} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is T-periodic ($T > 0$) in its first variable for all $x \in \mathbb{R}^{\mathbb{N}}$, and satisfies the assumption

- (A1) $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^{\mathbb{N}}$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+), b \in L^1(0, T; \mathbb{R}^+)$ such that
$$|F(t, x)| + |\nabla F(t, x)| \le a(|x|)b(t)$$
for all $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0, T]$.

A solution is called subharmonic solution if it is kT-periodic solution for some positive integer k (see for example [10]).

Recently, considerable attention has been paid to subharmonic solutions of second-order Hamiltonian systems with p-Laplacian; see [2, 7, 11, 18, 20, 21, 23]. When $p = 2$, Equation (1.1) reduces to the second-order non-autonomous Hamiltonian system

$$\ddot{u}(t) + \nabla F(t, u(t)) = 0 \quad \text{a.e. } t \in \mathbb{R}. \tag{1.2}$$

Using the variational methods, many existence results are obtained under suitable conditions, we refer the reader to [1, 3, 4, 5, 6, 8, 9, 10, 12, 13, 14, 15, 16, 17, 19, 22] and the reference therein. In particular, in [6], Tang and Wu have proved the following result.

2000 *Mathematics Subject Classification.* 34B15, 34K13, 58E30.
Key words and phrases. Control functions; subharmonic solutions; p-Laplacian systems; minimax methods.

Submitted August 1, 2011. Published October 20, 2011.

Theorem 1.1 ([6]). *Suppose that F satisfies assumption* (A1) *and the following conditions:*

- (S1) *There exist $f, g \in L^1(0, T; \mathbb{R}^+)$ and $\alpha \in [0, 1)$ such that*
$$|\nabla F(t, x)| \le f(t)|x|^\alpha + g(t)$$
for all $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0, T]$;
- (S2) *There exists $\gamma \in L^1(0, T; \mathbb{R})$ such that*
$$F(t, x) \ge \gamma(t)$$
for all $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0, T]$;
- (S3) *There exists a subset E of $[0, T]$ with* $\operatorname{meas}(E) > 0$ *such that*
$$\frac{1}{|x|^{2\alpha}} F(t, x) \to +\infty \quad \text{as } |x| \to +\infty$$
for a.e. $t \in E$.

Then problem (1.2) *has a kT-periodic solution u_k for every positive integer k, and* $\max_{0 \le t \le kT} |u(t)| \to +\infty$ *as $k \to +\infty$.*

Subsequently, Pasca and Tang in [7] dealt with the second order differential inclusions systems with p-Laplacian. They generalized Theorem 1.1 in a more general sense. Note that in [6, 7], it is usually assumed that (S1) holds, for p-Laplacian systems, $\alpha \in [0, p - 1)$. This means that nonlinearity $\nabla F(t, x)$ is sublinear.

Recently, the author and Zhang [22], introduced a control function $h(t)$, consider the case in which nonlinearity $\nabla F(t, x)$ is only weak sublinear: It is assumed that there exists a positive function $h \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfied the following restrictions

- (i) $h(s) \le h(t)$ for all $s \le t$, $s, t \in \mathbb{R}^+$;
- (ii) $h(s + t) \le C^*(h(s) + h(t))$ for all $s, t \in \mathbb{R}^+$;
- (iii) $0 < h(t) \le K_1 t^\alpha + K_2$ for all $t \in \mathbb{R}^+$;
- (iv) $h(t) \to +\infty$ as $t \to +\infty$.

Here C^*, K_1, K_2 are positive constants, $\alpha \in [0, 1)$, and $h(t)$ need just to satisfy conditions (i)-(iii) if $\alpha = 0$. Moreover, conditions
$$|\nabla F(t, x)| \le f(t)h(|x|) + g(t),$$
$$\frac{1}{h^2(|x|)} \int_0^T F(t, x)dx \to \pm\infty \quad \text{as } |x| \to +\infty$$
are posed. Under these assumptions, they show that second-order system
$$\begin{gathered} \ddot{u}(t) = \nabla F(t, u(t)) \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{gathered} \tag{1.3}$$
has a T-periodic solution. In addition, if the nonlinearity $\nabla F(t, x)$ grows slightly slower than $|x|^{p-1}$ at infinity, such as
$$\nabla F(t, x) = \frac{t|x|^{p-1}}{\ln(e + |x|^2)}, \tag{1.4}$$
solutions are saddle points of problem
$$\begin{gathered} \frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)) = \nabla F(t, u(t)) \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{gathered} \tag{1.5}$$
which have been obtained in [23] by minimax methods.

In the present article, we will focus on the subharmonic solutions for (1.1) by replacing in assumptions (S1) and (S3) the term $|x|$ with more general control functions $h(|x|)$. Here, we emphasize that our results are still new when $p = 2$.

We will establish our main results:

Theorem 1.2. *Suppose that F satisfies assumption* (A1) *and the following conditions:*

- (H1) *There exist constants $C^* > 0$, $K_1 > 0$, $K_2 > 0$, $\alpha \in [0, p-1)$ and a positive function $h \in C(\mathbb{R}^+, \mathbb{R}^+)$ with the properties* (i)–(iv)*. Moreover, there exist $f, g \in L^1(0, T; \mathbb{R}^+)$ such that*
$$|\nabla F(t,x)| \le f(t)h(|x|) + g(t)$$
for all $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0,T]$;
- (H2) *There exists $\gamma \in L^1(0,T;\mathbb{R})$ such that*
$$F(t,x) \ge \gamma(t)$$
for all $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0,T]$;
- (H3) *There exist a positive function $h \in C(\mathbb{R}^+, \mathbb{R}^+)$ which satisfies the conditions* (i)–(iv)*, and a subset E of $[0,T]$ with* $\mathrm{meas}(E) > 0$ *such that*
$$\frac{1}{h^q(|x|)} F(t,x) \to +\infty \quad \text{as } |x| \to +\infty$$
for a.e. $t \in E$, here $q := \frac{p}{p-1}$.

Then (1.1) *has kT-periodic solution $u_k \in W^{1,p}_{kT}$ for every positive integer k such that $\|u_k\|_\infty \to +\infty$ as $k \to +\infty$, where*
$$W^{1,p}_{kT} := \{u : [0,kT] \to \mathbb{R}^{\mathbb{N}} | \ u \text{ is absolutely continuous}, \\ u(0) = u(kT), \dot{u} \in L^p(0,kT;\mathbb{R}^{\mathbb{N}})\}$$
is a Banach space with the norm
$$\|u\| := \Big(\int_0^{kT} |u(t)|^p dt + \int_0^{kT} |\dot{u}(t)|^p dt \Big)^{1/p}$$
and $\|u_k\|_\infty := \max_{0 \le t \le kT} |u(t)|$ for $u \in W^{1,p}_{kT}$.

Remark 1.3. Theorem 1.2 generalizes Theorem 1.1. In fact, when $p = 2$, Theorem 1.1 is a special case of Theorem 1.2 with control function $h(t) = t^\alpha$, $\alpha \in [0, p-1)$, $t \in \mathbb{R}^+$. Furthermore, there are functions $F(t,x)$ satisfying Theorem 1.2 and not satisfying Theorem 1.1 and earlier results in the references [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23], even for $p = 2$.

Example 1.4. Consider the function
$$F(t,x) = \sin[(1+|x|^2)^{1/2} \ln^{1/2}(e + |x|^2)] + |\sin \omega t| \ln^{\frac{3}{2}}(e + |x|^2)$$
for all $x \in \mathbb{R}^{\mathbb{N}}$ and $t \in \mathbb{R}$, where $\omega = 2\pi/T$. It is apparent that
$$|\nabla F(t,x)| \le \ln^{1/2}(e + |x|^2) + 10,$$
which implies that $F(t,x)$ is not bounded. Moreover, one has
$$\frac{1}{|x|^{2\alpha}} F(t,x) \to 0 \quad \text{as } |x| \to +\infty$$

for any $\alpha \in (0,1)$ and $t \in \mathbb{R}$. Hence, this example can not be solved by Theorem 1.1 and the results in [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23].

On the other hand, put $h(t) = \ln^{1/2}(e+t^2)$, it is not difficult to see that the properties (i)-(iv) of $h(t)$ are all satisfied. By simple computation, (H1)-(H3) remains true. Therefore, $F(t,x)$ satisfies all the conditions of Theorem 1.2, then problem (1.1) with $p = 2$ has kT-periodic solution $u_k \in W^{1,p}_{kT}$ for every positive integer k such that $\|u_k\|_\infty \to +\infty$ as $k \to +\infty$.

Theorem 1.5. *Suppose that F satisfies assumption* (A1), (H2) *and the following conditions:*

- (H4) *There exists constant $C^* > 0$ and a positive function $h^* \in C(\mathbb{R}^+, \mathbb{R}^+)$ with the properties:*
  - (i*) $h^*(s) \le h^*(t)$ *for all* $s \le t$, $s,t \in \mathbb{R}^+$;
  - (ii*) $h^*(s+t) \le C^*(h^*(s) + h^*(t))$ *for all* $s,t \in \mathbb{R}^+$;
  - (ii*) $th^*(t) - pH^*(t) \to -\infty$ *as* $t \to +\infty$, *where* $H^*(t) = \int_0^t h^*(s)ds$;
  - (iv*) $H^*(t)/t^p \to 0$ *as* $t \to +\infty$.

  Moreover, there exist $f, g \in L^1(0,T;\mathbb{R}^+)$ such that
  $$|\nabla F(t,x)| \le f(t)h^*(|x|) + g(t)$$
  for all $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0,T]$;
- (H5) *There exist a positive function $h^* \in C(\mathbb{R}^+, \mathbb{R}^+)$ which satisfies the conditions* (i*)–(iv*), *and a subset E of $[0,T]$ with* $\operatorname{meas}(E) > 0$ *such that*
  $$\frac{1}{H^*(|x|)} F(t,x) > 0 \quad \text{as } |x| \to +\infty$$
  for a.e. $t \in E$.

Then (1.1) *has kT-periodic solution $u_k \in W^{1,p}_{kT}$ for every positive integer k such that $\|u_k\|_\infty \to +\infty$ as $k \to +\infty$.*

Remark 1.6. (1) In contrast to the result in Theorem 1.2, if $\nabla F(t,x)$ grows faster at infinity, with the rate like $\frac{|x|^{p-1}}{\ln(e+|x|^2)}$, from the proof we see that, the approach of Theorem 1.2 can not be repeated unless $f(t)$ satisfies certain restrictions, and α has a wider range, say, $\alpha \in [0, p-1]$. Meanwhile Theorem 1.5 needs only $f(t)$ belonging to $L^1(0,T;\mathbb{R}^+)$.

(2) Comparing with [23], we emphasize that Theorem 1.5 can not only treat the case like (1.4), but also cases like (S1)-(S3). Details for this assertion can be found in Example 1.7 below, also in the example in Section 4. Furthermore, our methods here are simpler and more direct than those in [23].

(3) We must point out that assumption (H4) leads to $H^*(t) \to +\infty$ as $t \to +\infty$ (for details see Lemma 2.2), then (H5) is stronger than (H3) (or (S3)) with $\alpha = 0$. Therefore, Theorem 1.5 is a new result, and do not cover Theorem 1.1.

(4) There are functions $F(t,x)$ satisfying Theorem 1.5 and not satisfying Theorem 1.1, Theorem 1.2, or the assumptions in [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23].

Example 1.7. Consider the function
$$F(t,x) = |\sin \omega t||f(t)|\frac{|x|^p}{\ln(e+|x|^2)},$$

where $f(t) \in L^1(\mathbb{R}, \mathbb{R}^+)$. Clearly, for all $x \in \mathbb{R}^{\mathbb{N}}$ and $t \in \mathbb{R}$, one has

$$|\nabla F(t,x)| \leq (2+p)|f(t)| \frac{|x|^{p-1}}{\ln(e+|x|^2)},$$

which implies that (H1) does not hold for any $\alpha \in [0, p-1)$. Moreover, as mentioned before $f(t)$ only belongs to $L^1(\mathbb{R}, \mathbb{R}^+)$ and no other further requirements on $f(t)$ are posed, then the approach of Theorem 1.2 can not be applied. This is the key feature that Theorem 1.5 is different from Theorem 1.2. Thus, this example can not be solved by earlier results even if $p = 2$.

On the other hand, take $h^*(t) = \frac{t^{p-1}}{\ln(e+t^2)}$, $H^*(t) = \int_0^t \frac{s^{p-1}}{\ln(e+s^2)} ds$, $s, t \in \mathbb{R}^+$, then we can find that conditions (H2), (H4) and (H5) are all satisfied, by Theorem 1.5, problem (1.1) has kT-periodic solution $u_k \in W^{1,p}_{kT}$ for every positive integer k such that $\|u_k\|_\infty \to +\infty$ as $k \to +\infty$.

Remark 1.8. Without loss of generality, we may assume that functions b in assumption (A1), f, g in (H1), (H4) and γ in (H2) are T-periodic. Then assumptions (A1), (H1), (H2), (H4) hold for all $t \in R$ by the T-periodicity of $F(t,x)$ in the first variable.

The remainder of this article is organized as follows. In Section 2 we give some notations and the estimates of control functions $h^*(t)$ and $H^*(t)$. Section 3 are devoted to the proofs of main theorems. Finally, we will give a new example to illustrate our results in Section 4.

2. Preliminaries

Let k be a positive integer. For convenience, in the following we will denote various positive constants as $C_i, i = 0, 1, 2, \cdots$. For $u \in W^{1,p}_{kT}$, let $\bar{u} := \frac{1}{kT}\int_0^{kT} u(t)dt$ and $\tilde{u}(t) := u(t) - \bar{u}$, then one has: Sobolev's inequality

$$\|\tilde{u}\|_\infty \leq C_0 \int_0^{kT} |\dot{u}(t)|^p dt$$

and Wirtinger's inequality

$$\int_0^{kT} |\tilde{u}(t)|^p dt \leq C_0 \int_0^{kT} |\dot{u}(t)|^p dt.$$

It follows from assumption (A1) that functional φ_k on $W^{1,p}_{kT}$ give by

$$\varphi_k(u) = \frac{1}{p}\int_0^{kT} |\dot{u}(t)|^p dt - \int_0^{kT} F(t, u(t))dt$$

is continuously differentiable on $W^{1,p}_{kT}$ (see [10]). Moreover, one has

$$(\varphi_k'(u), v) = \int_0^{kT} (|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t))dt - \int_0^{kT} (\nabla F(t, u(t)), v(t))dt$$

for all $u, v \in W^{1,p}_{kT}$. It is well known that the solutions to problem (1.1) correspond to the critical points of the functional φ_k.

To proof of our main theorems, we need the following auxiliary results.

Lemma 2.1 ([4]). *Suppose that F satisfies assumption (A1), and E is a measurable subset of $[0,T]$. Assume that*

$$F(t,x)\to+\infty \quad \text{as } |x|\to\infty$$

for a.e. $t\in E$. Then for every $\delta>0$, there exists subset E_δ of E with $\operatorname{meas}(E\backslash E_\delta)<\delta$ such that

$$F(t,x)\to+\infty \quad \text{as } |x|\to+\infty$$

uniformly for all $t\in E_\delta$.

Lemma 2.2. *Suppose that there exists a positive function h^* which satisfies the conditions (i*), (iii*), (iv*), then we have the following estimates:*

(a) $0<h^*(t)\le \epsilon t^{p-1}+C_1$ *for any* $\epsilon>0$, $t\in\mathbb{R}^+$,
(b) $h^{*q}(t)/H^*(t)\to 0$ *as* $t\to+\infty$,
(c) $H^*(t)\to+\infty$ *as* $t\to+\infty$.

Proof. It follows from (iv*) that, for any $\epsilon>0$, there exists $M_1>0$ such that

$$H^*(t)\le\varepsilon t^p \quad \forall t\ge M_1.$$

Observe that by (iii*), there exists $M_2>0$ such that

$$th^*(t)-pH^*(t)\le 0 \quad \forall t\ge M_2,$$

which implies that

$$h^*(t)\le\frac{pH^*(t)}{t}\le p\epsilon t^{p-1} \quad \forall t\ge M,$$

where $M:=\max\{M_1,M_2\}$. Hence we obtain

$$h^*(t)\le p\epsilon t^{p-1}+h^*(M)$$

for all $t>0$ by (i*) of (H4). Obviously, $h^*(t)$ satisfies (a) due to the definition of $h^*(t)$ and the above inequality.

Next, we turn to (b). Recalling property (iv*) and the fact $q=\frac{p}{p-1}$, we obtain

$$\begin{aligned}0<\frac{h^{*q}(t)}{H^*(t)}&=\frac{h^{*q}(t)}{H^{*q}(t)}\cdot H^{*q-1}(t)\le\big(\frac{p}{t}\big)^q\cdot H^{*q-1}(t)\\&=p^q\cdot\frac{H^{*q-1}(t)}{t^q}=p^q\big(\frac{H^*(t)}{t^p}\big)^{1/(p-1)}\to 0 \quad \text{as } t\to+\infty.\end{aligned}$$

Therefore, estimate (b) holds.

Finally, we show that (c) is also true. By (iii*), one arrives at, for every $L>0$, there exists $M_3>0$ such that

$$th^*(t)-pH^*(t)\le -L \quad \forall t\ge M_3.$$

So, one has

$$\theta th^*(\theta t)-pH^*(\theta t)\le -L$$

for all $|\theta t|\ge M_3$. Then we have

$$\frac{d}{d\theta}\big[\frac{H^*(\theta t)}{\theta^p}\big]=\frac{\theta t\cdot h^*(\theta t)-pH^*(\theta t)}{\theta^{p+1}}\le-\frac{L}{\theta^{p+1}}=\frac{d}{d\theta}\big(\frac{L}{p\theta^p}\big).$$

Let $\theta>1$, integrating both sides of the above inequality from 1 to θ, we obtain

$$\frac{H^*(\theta t)}{\theta^p}-H^*(t)\le\frac{L}{p\theta^p}-\frac{L}{p}=\frac{L}{p}\big(\frac{1}{\theta^p-1}\big).$$

Let $\theta \to +\infty$ in the above inequality, and by (iv*), one has

$$H^*(t) \geq \frac{L}{p}$$

for all $t \geq M_3$. By the arbitrariness of L, we have $H^*(t) \to +\infty$ as $t \to +\infty$. This completes the proof. $\square$

3. Proof of main results

In this section, firstly, we discuss the (PS) condition.

Lemma 3.1. *Assume that F satisfies assumptions* (A1), (H1)–(H3). *Then φ_k satisfies the* (PS) *condition, that is, $\{u_n\}$ has a convergent subsequence whenever it satisfies $\varphi_k'(u_n) \to 0$ as $n \to +\infty$ and $\{\varphi_k(u_n)\}$ is bounded.*

Proof. It follows from (H1) and Sobolev's inequality that

$$\begin{aligned}
&\Big|\int_0^{kT} (\nabla F(t, u_n(t)), \tilde{u}_n(t)) dt\Big| \\
&\leq \int_0^{kT} f(t) h(|\bar{u}_n + \tilde{u}_n(t)|) |\tilde{u}_n(t)| dt + \int_0^{kT} g(t) |\tilde{u}_n(t)| dt \\
&\leq \int_0^{kT} f(t) [C^*(h(|\bar{u}_n|) + h(|\tilde{u}_n(t)|))] |\tilde{u}_n(t)| dt + \|\tilde{u}_n\|_\infty \int_0^{kT} g(t) dt \\
&\leq C^*[h(|\bar{u}_n|) + h(|\tilde{u}_n(t)|)] \|\tilde{u}_n\|_\infty \int_0^{kT} f(t) dt + \|\tilde{u}_n\|_\infty \int_0^{kT} g(t) dt \\
&\leq C^*\Big[\frac{1}{2pC^* C_0^p} \|\tilde{u}_n\|_\infty^p + 2pC^* C_0^p h^q(|\bar{u}_n|) \Big(\int_0^{kT} f(t) dt\Big)^q\Big] + \|\tilde{u}_n\|_\infty \int_0^{kT} g(t) dt \\
&\quad + C^* h(\|\tilde{u}_n\|_\infty) \|\tilde{u}_n\|_\infty \int_0^{kT} f(t) dt \qquad (3.1)\\
&\leq \frac{1}{2p} \int_0^{kT} |\dot{u}_n(t)|^p dt + C_2 h^q(|\bar{u}_n|) + C^*[K_1 \|\tilde{u}_n\|_\infty^\alpha + K_2] \|\tilde{u}_n\|_\infty \int_0^{kT} f(t) dt \\
&\quad + \|\tilde{u}_n\|_\infty \int_0^{kT} g(t) dt \\
&\leq \frac{1}{2p} \int_0^{kT} |\dot{u}_n(t)|^p dt + C_2 h^q(|\bar{u}_n|) + C_3 \Big(\int_0^{kT} |\dot{u}_n(t)|^p dt\Big)^{(\alpha+1)/p} \\
&\quad + C_4 \Big(\int_0^{kT} |\dot{u}_n(t)|^p dt\Big)^{1/p}.
\end{aligned}$$

Hence, we see that

$$\begin{aligned}
\|\tilde{u}_n\|_\infty &\geq |(\varphi_k'(u_n), \tilde{u}_n)| \\
&= \Big|\int_0^{kT} |\dot{u}_n(t)|^p dt - \int_0^{kT} (\nabla F(t, u_n(t)), \tilde{u}_n(t)) dt\Big| \qquad (3.2)\\
&\geq \Big(1 - \frac{1}{2p}\Big) \int_0^{kT} |\dot{u}_n(t)|^p dt - C_2 h^q(|\bar{u}_n|) - C_3 \Big(\int_0^{kT} |\dot{u}_n(t)|^p dt\Big)^{(\alpha+1)/p} \\
&\quad - C_4 \Big(\int_0^{kT} |\dot{u}_n(t)|^p dt\Big)^{1/p}
\end{aligned}$$

for large n. It follows from Wirtinger's inequality that

$$\|\tilde{u}_n\|_\infty \leq (C_0+1)^{1/p}\left(\int_0^{kT}|\dot{u}_n(t)|^p dt\right)^{1/p}$$

for all n, thus we obtain

$$C_5h^q(|\bar{u}_n|) \geq \int_0^{kT}|\dot{u}_n(t)|^p dt - C_6 \tag{3.3}$$

for all large n, which implies that

$$\begin{aligned}\|\tilde{u}_n\|_\infty &\leq \Big(C_0\int_0^{kT}|\dot{u}_n(t)|^p dt\Big)^{1/p}\\ &\leq [C_0(C_5h^q(|\bar{u}_n|)+C_6)]^{1/p}\\ &\leq [C_7(|\bar{u}_n|^{q\alpha}+1)]^{1/p}.\end{aligned}$$

Then one has

$$|u_n(t)| \geq |\bar{u}_n| - |\tilde{u}_n(t)| \geq |\bar{u}_n| - \|\tilde{u}_n(t)\|_\infty \geq |\bar{u}_n| - [C_7(|\bar{u}_n|^{q\alpha}+1)]^{1/p} \tag{3.4}$$

for all large n and every $t\in[0,kT]$.

We claim that $\{|\bar{u}_n|\}$ is bounded. Arguing indirectly, if $\{|\bar{u}_n|\}$ is unbounded, we may assume that, going to a subsequence if necessary,

$$|\bar{u}_n| \to +\infty \quad \text{as } n\to+\infty, \tag{3.5}$$

which, together with (3.4), implies

$$|u_n(t)| \geq \frac{1}{2}|\bar{u}_n|. \tag{3.6}$$

Then for all large n and every $t\in[0,kT]$, we have

$$h(|\bar{u}_n|) \leq h(2|u_n(t)|) \leq 2C^*h(|u_n(t)|). \tag{3.7}$$

Set $\delta = \frac{1}{2}\operatorname{meas}(E)$. In virtue of (H3) and Lemma 2.1, there exists a subset E_δ of E with $\operatorname{meas}(E\backslash E_\delta) < \delta$ such that

$$\frac{1}{h^q(|x|)}F(t,x) \to +\infty \quad \text{as } |x|\to+\infty \tag{3.8}$$

uniformly for all $t\in E_\delta$, which implies

$$\operatorname{meas}(E_\delta) = \operatorname{meas}(E) - \operatorname{meas}(E\backslash E_\delta) > \delta > 0, \tag{3.9}$$

and for every $\beta>0$, there exists $M\geq 1$ such that

$$\frac{1}{h^q(|x|)}F(t,x) \geq \beta \tag{3.10}$$

for all $|x|\geq M$ and all $t\in E_\delta$. By (3.5) and (3.6), one has $|u_n(t)|\geq M$ for all large n and every $t\in[0,kT]$. It follows from (3.3), (3.7), (3.9), (3.10) that

$$\begin{aligned}\varphi_k(u_n) &= \frac{1}{p}\int_0^{kT}|\dot{u}_n(t)|^p dt - \int_0^{kT}F(t,u_n(t))dt\\ &\leq \frac{1}{p}(C_5h^q(|\bar{u}_n|)+C_6) - \int_{[0,kT]\backslash E_\delta}\gamma(t)dt - \int_{E_\delta}\beta h^q(|u_n(t)|)dt\\ &\leq C_8h^q(|\bar{u}_n|)+C_9 - \beta\int_{E_\delta}\big(\frac{1}{2C^*}h(|\bar{u}_n|)\big)^q dt\end{aligned}$$

$$\le C_8 h^q(|\bar{u}_n|) + C_9 - \beta \frac{1}{(2C^*)^q} h^q(|\bar{u}_n|)\delta$$

for all large n. So, we obtain

$$\limsup_{n\to+\infty} \frac{1}{h^q(|\bar{u}_n|)} \varphi_k(u_n) \le C_8 - \beta \frac{1}{(2C^*)^q}\delta.$$

By the arbitrariness of $\beta > 0$, one has

$$\limsup_{n\to+\infty} \frac{1}{h^q(|\bar{u}_n|)} \varphi_k(u_n) = -\infty,$$

which contradicts the boundedness of $\varphi_k(u_n)$. Hence $\{|\bar{u}_n|\}$ is bounded. Furthermore, by (3.2) and (3.3), we know $\{u_n\}$ is bounded. Arguing then as in [11, Proposition 4.1], we conclude that (PS) condition is satisfied. $\square$

Lemma 3.2. *Assume that F satisfies assumption* (A1), (H2), (H4), (H5). *Then φ_k satisfies the* (PS) *condition.*

Proof. It follows from (3.1) and Lemma 2.2 that

$$\begin{aligned}
&\Big|\int_0^{kT} (\nabla F(t, u_n(t)), \tilde{u}_n(t))dt\Big| \\
&\le C^*\Big[\frac{1}{2pC^*C_0^p}\|\tilde{u}_n\|_\infty^p + 2pC^*C_0^p h^{*q}(|\bar{u}_n|)\Big(\int_0^{kT} f(t)dt\Big)^q\Big] + \|\tilde{u}_n\|_\infty \int_0^{kT} g(t)dt \\
&\quad + C^* h^*(\|\tilde{u}_n\|_\infty)\|\tilde{u}_n\|_\infty \int_0^{kT} f(t)dt \\
&\le \frac{1}{2p}\int_0^{kT} |\dot{u}_n(t)|^p dt + C_2 h^{*q}(|\bar{u}_n|) + C^*[\epsilon\|\tilde{u}_n\|_\infty^{p-1} + C_1]\|\tilde{u}_n\|_\infty \int_0^{kT} f(t)dt \\
&\quad + \|\tilde{u}_n\|_\infty \int_0^{kT} g(t)dt \qquad (3.11)\\
&\le \Big(\frac{1}{2p} + \epsilon C_{10}\Big)\int_0^{kT} |\dot{u}_n(t)|^p dt + C_2 h^{*q}(|\bar{u}_n|) + C_{11}\Big(\int_0^{kT} |\dot{u}_n(t)|^p dt\Big)^{1/p},
\end{aligned}$$

which implies

$$\begin{aligned}
&\|\tilde{u}_n\|_\infty \ge |(\varphi_k'(u_n), \tilde{u}_n)| \\
&\qquad \ge \Big(1 - \frac{1}{2p} - \epsilon C_{10}\Big)\int_0^{kT} |\dot{u}_n(t)|^p dt - C_2 h^{*q}(|\bar{u}_n|) - C_{11}\Big(\int_0^{kT} |\dot{u}_n(t)|^p dt\Big)^{1/p}
\end{aligned}$$

for large n. Thus, by (3.2), one has

$$C_{12} h^{*q}(|\bar{u}_n|) \ge \int_0^{kT} |\dot{u}_n(t)|^p dt - C_{13} \qquad (3.12)$$

for all large n and ϵ small enough, which implies

$$\|\tilde{u}_n\|_\infty \le [C_0(C_{12}h^{*q}(|\bar{u}_n|) + C_{13})]^{1/p} \le [C_{14}(\epsilon|\bar{u}_n|^p + 1)]^{1/p}$$

by Lemma 2.2. Consequently, we get

$$|u_n(t)| \ge |\bar{u}_n| - [C_{14}(\epsilon|\bar{u}_n|^p + 1)]^{1/p} \qquad (3.13)$$

for all large n, every $t \in [0, kT]$ and ϵ small enough.

Assume $\{|\bar{u}_n|\}$ is unbounded, by (3.13), for ϵ small enough, we obtain

$$|u_n(t)| \geq \frac{1}{2}|\bar{u}_n|. \tag{3.14}$$

Combine (3.14) with (H_4), one has

$$H^*(|\bar{u}_n|) \leq 2C^* H^*(|u_n(t)|).$$

With the same arguments of (3.8)-(3.10), by (H5), we know

$$\frac{1}{H^*(|x|)} F(t,x) \geq C_{15} \tag{3.15}$$

for all $|x| \geq M$ and all $t \in E_\delta$. We see that, jointly with (3.12), (3.15) and Lemma 2.2, for all large n,

$$\begin{aligned}
\varphi_k(u_n) &= \frac{1}{p}\int_0^{kT} |\dot{u}_n(t)|^p dt - \int_0^{kT} F(t,u_n(t))dt \\
&\leq \frac{1}{p}(C_{12}{h^*}^q(|\bar{u}_n|) + C_{13}) - \int_{[0,kT]\setminus E_\delta} \gamma(t)dt - \int_{E_\delta} C_{15} H^*(|u_n(t)|)dt \\
&\leq C_{16}{h^*}^q(|\bar{u}_n|) + C_{17} - C_{15}\int_{E_\delta} \frac{1}{2C^*} H^*(|\bar{u}_n|)dt \\
&\leq C_{16}{h^*}^q(|\bar{u}_n|) + C_{17} - C_{18}\delta H^*(|\bar{u}_n|),
\end{aligned}$$

which implies

$$\begin{aligned}
0 &= \limsup_{n\to+\infty} \frac{1}{H^*(|\bar{u}_n|)} \varphi_k(u_n) \\
&\leq \limsup_{n\to+\infty} \big[C_{16}\frac{{h^*}^q(|\bar{u}_n|)}{H^*(|\bar{u}_n|)} + \frac{C_{17}}{H^*(|\bar{u}_n|)} - C_{18}\delta\big] \leq -C_{18}\delta,
\end{aligned}$$

a contradiction. Hence $\{|\bar{u}_n|\}$ is bounded, moreover, we can get $\{u_n\}$ is bounded. So (PS) condition is satisfied, which completes the proof. $\square$

Now, we are ready to proof our main results.

Proof of Theorem 1.2. It follows from Lemma 3.1 that φ_k satisfies the (PS) condition. In order to use the saddle point theorem, we only need to verify the following conditions

(I1) $\varphi_k(u) \to +\infty$ as $\|u\| \to +\infty$ in $\tilde{W}^{1,p}_{kT}$,
(I2) $\varphi_k(x + e_k(t)) \to -\infty$ as $|x| \to +\infty$ in $\mathbb{R}^{\mathbb{N}}$,

where $\tilde{W}^{1,p}_{kT} := \{u \in W^{1,p}_{kT} : \bar{u} = 0\}$, $e_k(t) = k\cos(k^{-1}\omega t)x_0 \in \tilde{W}^{1,p}_{kT}$, $x_0 \in \mathbb{R}^{\mathbb{N}}$, $|x_0| = 1$ and $\omega = \frac{2\pi}{T}$. Next, we show that φ_k satisfies (I1) and (I2).

For all $x \in \mathbb{R}^{\mathbb{N}}$, it follows from (3.10) that

$$
\begin{aligned}
\varphi_k(x+e_k(t)) &= \frac{1}{p}\int_0^{kT}|\dot{e}_k(t)|^p dt - \int_0^{kT} F(t, x+k\cos(k^{-1}\omega t)x_0)dt \\
&\le \frac{1}{p}\int_0^{kT}|\omega(\sin k^{-1}\omega t)x_0|^p dt - \int_{[0,kT]\setminus E_\delta}\gamma(t)dt \\
&\quad - \beta\int_{E_\delta} h^q(|x+k\cos(k^{-1}\omega t)x_0|)dt \\
&\le C_{19}k - \int_{[0,kT]\setminus E_\delta}\gamma(t)dt - \beta h^q(M)\operatorname{meas}(E_\delta)
\end{aligned}
\tag{3.16}
$$

for all $|x| \ge M+k$. By the arbitrariness of β, one has

$$
\varphi_k(x+e_k(t)) \to -\infty \quad \text{as } |x| \to +\infty \text{ in } \mathbb{R}^{\mathbb{N}}.
$$

Thus (I2) is satisfied.

For all $u \in \tilde{W}_{kT}^{1,p}$, it follows from Sobolev's inequality that

$$
\begin{aligned}
&\Big|\int_0^{kT}[F(t,u(t)) - F(t,0)]dt\Big| = \Big|\int_0^{kT}(\nabla F(t, su(t)), u(t))dsdt\Big| \\
&\le \int_0^{kT}\int_0^1 f(t)h(|su(t)|)|u(t)|dsdt + \int_0^{kT}\int_0^1 g(t)|u(t)|dsdt \\
&\le \int_0^{kT} f(t)[K_1|u(t)|^\alpha + K_2]|u(t)|dt + \int_0^{kT} g(t)|u(t)|dt \\
&\le K_1\|u\|_\infty^{\alpha+1}\int_0^{kT} f(t)dt + K_2|u\|_\infty\int_0^{kT} f(t)dt + \|u\|_\infty\int_0^{kT} g(t)dt \\
&\le C_{20}\Big(\int_0^{kT}|\dot{u}(t)|^p dt\Big)^{(\alpha+1)/p} + C_{21}\Big(\int_0^{kT}|\dot{u}(t)|^p dt\Big)^{1/p}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\varphi_k(u) &= \frac{1}{p}\int_0^{kT}|\dot{u}(t)|^p dt - \int_0^{kT}[F(t,u(t)) - F(t,0)]dt - \int_0^{kT} F(t,0)dt \\
&\ge \frac{1}{p}\int_0^{kT}|\dot{u}(t)|^p dt - C_{20}\Big(\int_0^{kT}|\dot{u}(t)|^p dt\Big)^{(\alpha+1)/p} \\
&\quad - C_{21}\Big(\int_0^{kT}|\dot{u}(t)|^p dt\Big)^{1/p} - C_{22},
\end{aligned}
$$

then we can conclude that $\varphi_k(u) \to +\infty$ as $\|u\| \to +\infty$ in $\tilde{W}_{kT}^{1,p}$. Plainly, condition (I1) holds.

By (I1), (I2) and the saddle point theorem, there exists a critical point $u_k \in W_{kT}^{1,p}$ for φ_k such that

$$
-\infty < \inf_{\tilde{W}_{kT}^{1,p}} \varphi_k \le \varphi_k(u_k) \le \sup_{\mathbb{R}^{\mathbb{N}}+e_k} \varphi_k.
$$

For fixed $x \in \mathbb{R}^{\mathbb{N}}$, set

$$
A_k := \{t \in [0,kT] \,|\, |x + k(\cos k^{-1}\omega t)x_0| \le M\}.
$$

Using the same argument of [6], we have

$$
\operatorname{meas}(A_k) \le \frac{1}{2}k\delta. \tag{3.17}
$$

Let $E_k := \cup_{j=0}^{k-1}(jT + E_\delta)$, then it follows from (3.17) that

$$\operatorname{meas}(E_k \backslash A_k) \geq \frac{1}{2}k\delta$$

for large k. Taking into account (3.16), we have

$$\begin{aligned} k^{-1}\varphi_k(x + e_k(t)) &\leq C_{19} - k^{-1}\int_{[0,kT]\backslash(E_\delta\backslash A_k)} \gamma(t)dt - k^{-1}\beta h^q(M)\operatorname{meas}(E_\delta) \\ &\leq C_{19} + \int_0^T |\gamma(t)|dt - \frac{1}{2}\delta h^q(M)\beta \end{aligned}$$

for every $x \in \mathbb{R}^{\mathbb{N}}$ and all large k. Hence one has

$$\limsup_{k\to+\infty} \sup_{x\in\mathbb{R}^{\mathbb{N}}} k^{-1}\varphi_k(x + e_k) \leq C_{19} + \int_0^T |\gamma(t)|dt - \frac{1}{2}\delta h^q(M)\beta.$$

Observe the arbitrariness of β, we obtain

$$\limsup_{k\to+\infty} \sup_{x\in\mathbb{R}^{\mathbb{N}}} k^{-1}\varphi_k(x + e_k) = -\infty,$$

which implies

$$\limsup_{k\to+\infty} k^{-1}\varphi_k(u_k) = -\infty. \tag{3.18}$$

Finally, we prove that $\|u_k\|_\infty \to +\infty$ as $k \to +\infty$. If not, going to a subsequence if necessary, we may assume that

$$\|u_k\|_\infty \leq C_{23}$$

for all $k \in \mathbf{N}$. Hence we have

$$\begin{aligned} k^{-1}\varphi_k(u_k) &\geq -k^{-1}\int_0^{kT} F(t, u_k(t))dt \geq -k^{-1}\max_{0\leq s\leq C_{23}} a(s)\int_0^{kT} b(t)dt \\ &= -\max_{0\leq s\leq C_{23}} a(s)\int_0^T b(t)dt. \end{aligned}$$

It follows that

$$\liminf_{k\to+\infty} k^{-1}\varphi_k(u_k) > -\infty,$$

which contradicts (3.18). This completes the proof. $\square$

Proof of Theorem 1.5. By Lemma 3.2, φ_k satisfies the (PS) condition. By the argument of Theorem 1.2, we only need to check that φ_k satisfies (I1) and (I2). In fact from (3.15), for all $x \in \mathbb{R}^{\mathbb{N}}$ it follows that

$$\begin{aligned} \varphi_k(x + e_k(t)) &= \frac{1}{p}\int_0^{kT} |\dot{e}_k(t)|^p dt - \int_0^{kT} F(t, x + k\cos(k^{-1}\omega t)x_0)dt \\ &\leq C_{19}k - \int_{[0,kT]\backslash E_\delta} \gamma(t)dt - C_{15}\int_{E_\delta} H^*(|x + k\cos(k^{-1}\omega t)x_0|)dt \end{aligned}$$

for all $|x| \geq M + k$. Using the fact $H^*(t) \to +\infty$ as $t \to +\infty$ of Lemma 2.2 and (3.9), one has

$$\varphi_k(x + e_k(t)) \to -\infty \quad \text{as } |x| \to +\infty \text{ in } \mathbb{R}^{\mathbb{N}}.$$

Thus (I2) is safisfied.

For all $u \in \tilde{W}^{1,p}_{kT}$, we have

$$\begin{aligned}
&|\int_0^{kT} [F(t,u(t)) - F(t,0)]dt| \\
&\le \int_0^{kT}\int_0^1 f(t)h^*(|su(t)|)|u(t)|dsdt + \|u\|_\infty \int_0^{kT} g(t)dt \\
&\le \int_0^{kT} f(t)[\epsilon|u(t)|^{p-1} + C_1]|u(t)|dt + \|u\|_\infty \int_0^{kT} g(t)dt \\
&\le \epsilon C_{24}\int_0^{kT} |\dot{u}(t)|^p dt + C_{25}\Big(\int_0^{kT} |\dot{u}(t)|^p dt\Big)^{1/p},
\end{aligned}$$

which implies

$$\varphi_k(u) \ge \big(\frac{1}{p} - \epsilon C_{24}\big)\int_0^{kT} |\dot{u}(t)|^p dt - C_{25}\Big(\int_0^{kT} |\dot{u}(t)|^p dt\Big)^{1/p} - C_{22}.$$

Then for any ϵ small enough, we have $\varphi_k(u) \to +\infty$ as $\|u\| \to +\infty$ in $\tilde{W}^{1,p}_{kT}$. So condition (I1) holds, and the proof hereby is complete. $\square$

4. Example

In this section, we give a new example to illustrate our results. Consider function

$$F(t,x) = |\sin \omega t||x|^{1+\alpha},$$

where $0 < \alpha < p-1$ and $\omega = 2\pi/T$. We claim both Theorem 1.2 and Theorem 1.5 can handle this case.

Indeed, choose $h(t) = t^\alpha$, $\alpha \in (0, p-1)$, clearly, all conditions of Theorem 1.2 are satisfied. So that (1.1) has a kT-periodic solution $u_k \in W^{1,p}_{kT}$ for every positive integer k, and $\|u_k\|_\infty \to +\infty$ as $k \to +\infty$.

Let $h^*(t) = t^\alpha$, $H^*(t) = \int_0^t s^\alpha ds$ and C^* is a suitable positive constant. We know that

- (i*) $h^*(s) \le h^*(t)$ for all $s \le t$, $s,t \in \mathbb{R}^+$;
- (ii*) $h^*(s+t) = (s+t)^\alpha \le C^*(h^*(s) + h^*(t))$ for all $s,t \in \mathbb{R}^+$;
- (iii*) $th^*(t) - pH^*(t) = \big(1 - \frac{p}{1+\alpha}\big)t^{1+\alpha} \to -\infty$ as $t \to +\infty$;
- (iv*) $H^*(t)/t^p \to 0$ as $t \to +\infty$.

Moreover, we can check that (H2) and (H5) are satisfied. Therefore, by Theorem 1.5, problem (1.1) has kT-periodic solution $u_k \in W^{1,p}_{kT}$ for every positive integer k, and $\|u_k\|_\infty \to +\infty$ as $k \to +\infty$.

Acknowledgments. The author wants to thank Professor Xiangsheng Xu and the members of Department of Mathematics and Statistics at Mississippi State University for their warm hospitality and kindness. This Project is Supported by National Natural Science Foundation of China (Grant No.11026213, 10871096), Natural Science Foundation of the Jiangsu Higher Education Institutions (Grant No.10KJB110006) and Foundation of Nanjing University of Information Science & Technology (Grant No.20080280).

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