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CENTERS ON CENTER MANIFOLDS IN A QUADRATIC SYSTEM OBTAINED FROM A SCALAR THIRD-ORDER DIFFERENTIAL EQUATION

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Abstract. We give affirmative answers to two questions concerning the existence of centers on local center manifolds at equilibria of a quadratic system in the three dimensional space. These questions were posed by Dias and Mello [1] when studying a scalar third-order differential equation.

1. Introduction

Dias and Mello [1] studied the stability and bifurcations in the dynamics of the third-order differential equation

$$x''' + f(x)\,x'' + g(x)x' + h(x) = 0, \tag{1.1}$$

where $f, g, h : \mathbb{R} \to \mathbb{R}$ are

$$f(x) = a_1x + a_0, \quad g(x) = b_1x + b_0, \quad h(x) = c_2x^2 + c_1x + c_0, \tag{1.2}$$

with $a_1, a_0, b_1, b_0, c_2, c_1, c_0 \in \mathbb{R}$, $c_2 \neq 0$. From the natural definition of the variables $y = x'$ and $z = x''$, differential equation (1.1) can be written as the system of nonlinear differential equations

$$\begin{gathered} x' = P(x, y, z) = y, \\ y' = Q(x, y, z) = z, \\ z' = R(x, y, z) = -\big((a_1x + a_0)z + (b_1x + b_0)y + c_2x^2 + c_1x + c_0\big), \end{gathered} \tag{1.3}$$

where $(x, y, z) \in \mathbb{R}^3$ are the state variables and $(a_0, a_1, b_0, b_1, c_0, c_1, c_2) \in \mathbb{R}^7$, $c_2 \neq 0$, are real parameters. The choice of real affine functions f and g and a quadratic function h implies that the vector field that defines (1.3),

$$\mathcal{X}(x, y, z) = \left(P(x, y, z), Q(x, y, z), R(x, y, z)\right), \tag{1.4}$$

is a quadratic vector field. So, system (1.3) is a quadratic system of differential equations in $\mathbb{R}^3$.

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Despite its simplicity, (1.3) has a rich local dynamical behavior presenting several degenerate bifurcations. See [1] for more details. Define the following two curves in the space of parameters of system (1.3) (see [1, figures 1 and 2])

$$\mathcal{L}_2 = \{a_0 = 1/b_0, a_1 = 0, b_0 > 0, b_1 = 2b_0, c_0 = 0, c_1 = c_2 = 1\},$$
$$\mathcal{L}_3 = \{a_0 = 0, a_1 > 0, b_0 = 1/a_1, b_1 = 0, c_0 = 0, c_1 = c_2 = 1\}.$$

It was shown in [1] that for parameters in $\mathcal{L}_2$ the Jacobian matrix of $\mathcal{X}$ at the equilibrium point $E_0 = (0, 0, 0)$ presents one negative real eigenvalue and a pair of purely imaginary eigenvalues,

$$\lambda_1 = -\frac{1}{b_0}, \quad \lambda_{2,3} = \pm i\sqrt{b_0},$$

and the first four Lyapunov coefficients vanish. Analogously, for parameters in $\mathcal{L}_3$ the Jacobian matrix of $\mathcal{X}$ at the equilibrium point $E_1 = (-1, 0, 0)$ presents one positive real eigenvalue and a pair of purely imaginary eigenvalues,

$$\theta_1 = a_1, \quad \theta_{2,3} = \pm i/\sqrt{a_1},$$

and the first four Lyapunov coefficients vanish too.

In the study of local and global bifurcations of system (1.3) in [1], the following two questions were posed.

Question 1.1. Consider system (1.3) with parameters in $\mathcal{L}_2$. Is the equilibrium point E_0 a center for the flow of system (1.3) restricted to the center manifold?

Question 1.2. Consider system (1.3) with parameters in $\mathcal{L}_3$. Is the equilibrium point E_1 a center for the flow of system (1.3) restricted to the center manifold?

The study of stability of equilibrium points is an interesting subject of research; for recent developments see [4, 5]. However, the stability of degenerate equilibrium points is very difficult. The present article may contribute to the understanding of degenerate equilibrium points of system (1.3), by giving affirmative answers the two questions above.

2. Answers to Questions 1.1 and 1.2

For parameters in $\mathcal{L}_2$ ($\mathcal{L}_3$, respectively) system (1.3) has a nonhyperbolic equilibrium point at E_0 (E_1, respec.). By the Center Manifold Theorem, see [2], there is a two dimensional invariant manifold W_0^c (W_1^c, respec.) in a neighborhood of E_0 (E_1, respec.) that is tangent to the center eigenspace E_0^c at E_0 (E_1^c at E_1, respec.) and contains all the local recurrent behavior of the system. The center manifold W_0^c (W_1^c, respec.) is attracting (repelling, respec.) since $\lambda_1 < 0$ ($\theta_1 > 0$, respec.).

Our answers to Questions 1.1 and 1.2 are based on the existence of invariant algebraic surfaces for system (1.3): a polynomial $F(x, y, z)$ defines an invariant algebraic surface $\mathcal{A} = F^{-1}(0)$ for system (1.3) if and only if there exists a polynomial $K(x, y, z)$, called the cofactor of F, such that $\mathcal{X}F = KF$. See [3] and the references therein.

Theorem 2.1. *For parameters in $\mathcal{L}_2$ system* (1.3) *has an invariant algebraic surface $\mathcal{A}_{b_0} = F_{b_0}^{-1}(0)$, $b_0 > 0$, where*

$$F_{b_0}(x, y, z) = b_0 x + z + b_0 x^2. \tag{2.1}$$

Furthermore, $W_0^c \subset \mathcal{A}_{b_0}$ and the flow of system (1.3) *restrict to $\mathcal{A}_{b_0}$ has a center at E_0.*

Proof. For parameters in $\mathcal{L}_2$ we have

$$\mathcal{X}_{b_0} = \Big(y, z, -\big(x + b_0 y + \frac{1}{b_0} z + x^2 + 2b_0 xy\big)\Big). \tag{2.2}$$

It is simple to see that $\mathcal{X}_{b_0} F_{b_0} = K F_{b_0}$ for F_{b_0} in (2.1) and the cofactor $K(x,y,z) = -1/b_0$. Therefore, $\mathcal{A}_{b_0} = F_{b_0}^{-1}(0)$ is an invariant algebraic surface of the system defined by (2.2) for each $b_0 > 0$. It is immediate that $E_0 \in \mathcal{A}_{b_0}$. The center eigenspace E_0^c at E_0 is spanned by the vectors

$$V_{b_0}^1 = \big(-1/b_0, 0, 1\big), \quad V_{b_0}^2 = \big(0, -1/\sqrt{b_0}, 0\big).$$

The gradient of F_{b_0} at E_0 is given by $\nabla F_{b_0}(E_0) = (b_0, 0, 1)$. Hence $\nabla F_{b_0}(E_0)$ is orthogonal to $V_{b_0}^1$ and $V_{b_0}^2$. This implies that $W_0^c \subset \mathcal{A}_{b_0}$.

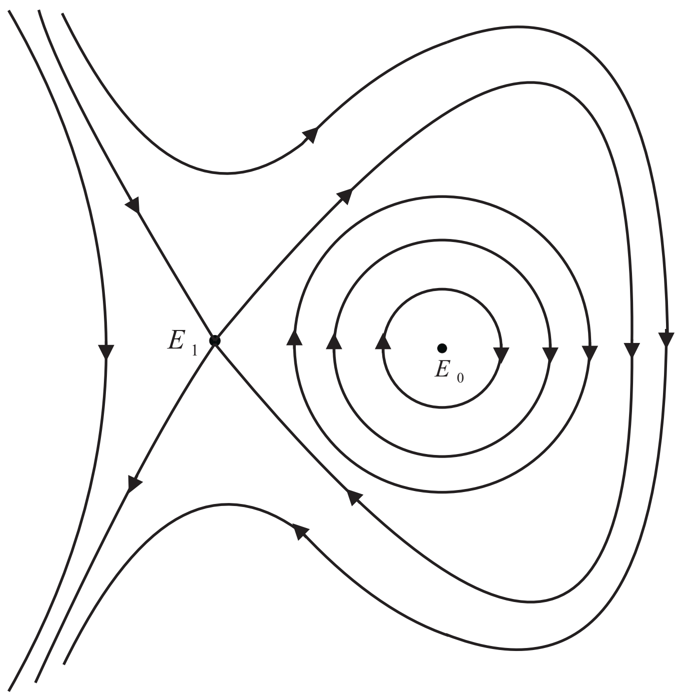

FIGURE 1. Phase portrait of system (2.3). The equilibrium E_0 is a center while the equilibrium E_1 is a saddle. Note a homoclinic loop at E_1 bounding the center region

Solving $F_{b_0} = 0$ for the variable z in terms of x and substituting into the first and second equations of the system defined by (2.2) we have the differential equations

$$x' = y, \quad y' = -b_0 x - b_0 x^2, \tag{2.3}$$

which is a Hamiltonian system with Hamiltonian function

$$H(x,y) = \frac{b_0}{2} x^2 + \frac{1}{2} y^2 + \frac{b_0}{3} x^3.$$

The phase portrait of this system is illustrated in Figure 1 which can be viewed as the projection in the plane xy of the phase portrait of the system defined by (2.2) on the invariant algebraic surface $\mathcal{A}_{b_0}$ for each $b_0 > 0$. The phase portrait of the system defined by (2.2) on $\mathcal{A}_{b_0}$ is depicted in Figure 2. The proof is complete. $\square$

The affirmative answer to Question 1.1 follows from Theorem 2.1.

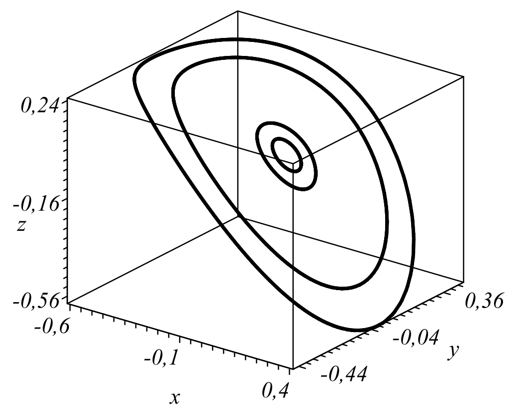

FIGURE 2. Phase portrait of the system defined by (2.2) on $\mathcal{A}_{b_0}$ in a neighborhood of the equilibrium E_0

To give an affirmative answer to Question 1.2 we make the change of variables $(\bar{x}, \bar{y}, \bar{z}) = (x, y, z) - (-1, 0, 0)$; that is, we translate the equilibrium $E_1 = (-1, 0, 0)$ to $\bar{E}_1 = (0, 0, 0)$.

Theorem 2.2. *For parameters in $\mathcal{L}_3$ system* (1.3) *with the above change of variables has an invariant algebraic surface $\mathcal{A}_{a_1} = F_{a_1}^{-1}(0)$, $a_1 > 0$, where*

$$F_{a_1}(x, y, z) = x + a_1 z. \tag{2.4}$$

Furthermore, $W_1^c \subset \mathcal{A}_{a_1}$ and the flow of system (1.3)*, with the above change of variables, restrict to $\mathcal{A}_{a_1}$ has a center at $\bar{E}_1$.*

Proof. For parameters in $\mathcal{L}_3$, with the change of variables $(\bar{x}, \bar{y}, \bar{z}) = (x, y, z) - (-1, 0, 0)$ and dropping the bars we have

$$\mathcal{X}_{a_1} = \Big(y, z, -\big(-x + \frac{1}{a_1} y - a_1 z + x^2 + a_1 x z\big)\Big). \tag{2.5}$$

It is simple to see that $\mathcal{X}_{a_1} F_{a_1} = K F_{a_1}$ for F_{a_1} in (2.4) and the cofactor $K(x, y, z) = a_1 - a_1 x$. Therefore, $\mathcal{A}_{a_1} = F_{a_1}^{-1}(0)$ is an invariant algebraic surface of the system defined by (2.5) for each $a_1 > 0$. It is immediate that $\bar{E}_1 \in \mathcal{A}_{a_1}$. The center eigenspace E_1^c at $\bar{E}_1$ is spanned by the vectors

$$V_{a_1}^1 = (-a_1, 0, 1), \quad V_{a_1}^2 = (0, -\sqrt{a_1}, 0).$$

The gradient of F_{a_1} at $\bar{E}_1$ is given by $\nabla F_{a_1}(\bar{E}_1) = (1, 0, a_1)$. Hence $\nabla F_{a_1}(\bar{E}_1)$ is orthogonal to $V_{a_1}^1$ and $V_{a_1}^2$. This implies that $W_1^c \subset \mathcal{A}_{a_1}$.

Solving $F_{a_1} = 0$ for the variable z in terms of x and substituting into the first and second equations of the system defined by (2.5) we have the differential equations

$$x' = y, \quad y' = -\frac{1}{a_1}x, \tag{2.6}$$

which is a Hamiltonian linear system with Hamiltonian function

$$H(x,y) = \frac{1}{2a_1}x^2 + \frac{1}{2}y^2.$$

The phase portrait of the system defined by (2.5) on $\mathcal{A}_{a_1}$ is depicted in Figure 3. The proof is complete. $\square$

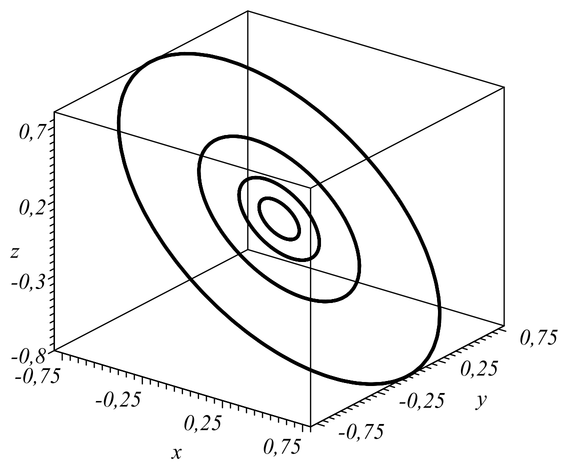

FIGURE 3. Phase portrait of the system defined by (2.5) on $\mathcal{A}_{a_1}$ in a neighborhood of the equilibrium $\bar{E}_1$

The affirmative answer to Question 1.2 follows from Theorem 2.2.

Concluding remarks. This paper provides a stability analysis that accounts for the characterization, in the space of parameters, of the structural as well as Lyapunov stability of the equilibria of system (1.3). Concerning the vanishing of the Lyapunov coefficients in a quadratic system two questions about the stability of the equilibria E_0 and E_1 are answered. See Questions 1.1 and 1.2 and Theorems 2.1 and 2.2.

Our proofs of Theorems 2.1 and 2.2 show that the local center manifolds of equilibria E_0 and E_1 are algebraic ruled surfaces. In particular, the local center manifolds of equilibrium E_1 are planes coincident with the center eigenspaces E_1^c for each parameter $a_1 > 0$. These are unexpected results.

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