Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 135, pp. 1–27.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

BOUNDARY-VALUE PROBLEMS FOR NONAUTONOMOUS NONLINEAR SYSTEMS ON THE HALF-LINE

JASON R. MORRIS

Abstract. A method is presented for proving the existence of solutions for boundary-value problems on the half line. The problems under study are nonlinear, nonautonomous systems of ODEs with the possibility of some prescribed value at $t = 0$ and with the condition that solutions decay to zero as t grows large. The method relies upon a topological degree for proper Fredholm maps. Specific conditions are given to ensure that the boundary-value problem corresponds to a functional equation that involves an operator with the required smoothness, properness, and Fredholm properties (including a calculable Fredholm index). When the Fredholm index is zero and the solutions are bounded *a priori*, then a solution exists. The method is applied to obtain new existence results for systems of the form $\dot{v} + g(t, w) = f_1(t)$ and $\dot{w} + h(t, v) = f_2(t)$.

1. Introduction

Let $F = F(t, z)\colon [0,\infty) \times \mathbb{R}^N \to \mathbb{R}^N$ be a given function and P be the projection of $\mathbb{R}^N$ onto X_1 along a given splitting $\mathbb{R}^N = X_1 \oplus X_2$. This paper concerns the existence of solutions to problems of the type

$$\begin{gathered} \dot{u}(t) + F(t, u(t)) = f(t) \quad \text{for all } t \geq 0, \\ Pu(0) = \xi, \\ \lim_{t\to\infty} u(t) = 0, \end{gathered} \tag{1.1}$$

where $f \in C_0$ and $\xi \in X_1$ are given.

By choice of X_1 and X_2, different kinds of initial conditions are obtained. For example, the choice $X_1 = \mathbb{R}^N$ and $X_2 = \{0\}$ results in the usual initial value problem. If $X_1 = \{0\}$ and $X_2 = \mathbb{R}^N$, there is no initial condition. Using the usual representation of a second (or higher) order system by a larger first-order system, this framework also accommodates a second (or higher) order equation or system with a variety of different initial conditions including those of Dirichlet, Neumann, or mixed type.

2000 *Mathematics Subject Classification.* 34B40, 34B15, 34D09, 46E15, 47H11, 47N20.
Key words and phrases. Ordinary differential equation; half-line; infinite interval; boundary and initial value problem; Fredholm operator; degree theory; exponential dichotomy; properness; a priori bounds.

Submitted October 4, 2011. Published October 17, 2011.

This kind of problem was recently approached by Rabier and Stuart in the two papers [17] and [16]. As the authors point out in [17], boundary-value problems on infinite intervals have been studied by many. In particular, Andres, Gabor, and Gorniewicz [2] contains many references. The book by Agarwal and O'Regan [1] contains another good survey of such problems. As a rule of thumb, most results in the literature deal with very specific problems and/or involve weaker conditions as $t \to \infty$.

To discuss more recent results, some more detail is needed. In [17] and [16] the authors prove the existence of a solution in the Sobolev space $W^{1,p}\big((0,\infty),\mathbb{R}^N\big)$, under appropriate conditions on F. The authors Rabier and Stuart use a degree argument. Because the problem is posed on an unbounded interval, the underlying operator is not a compact perturbation of the identity and therefore the Leray-Schauder degree cannot be used. Instead, a degree for proper C^1 maps of Fredholm index zero is employed. This degree was developed for C^2 maps by Fitzpartick, Pejsachowicz, and Rabier [7], and later extended to C^1 maps by Pejsachowicz and Rabier [14]. Because of this degree argument, prominent roles are obviously played by the Fredholm and properness properties, and also by the issue of finding *a priori* bounds on solutions.

This work continued in the present author's dissertation [10], in which existence is obtained in the space $C_0^1\big([0,\infty),\mathbb{R}^N\big)$ of continuously differentiable functions that tend to zero (along with their derivatives)as $t \to \infty$. In practice, this allows for simpler *a priori* bounds analysis than is possible in the Sobolev space setting. Moreover, the author removed a key assumption from [17] and [16], namely that $F(t,u)$ have an autonomous limit $F^\infty(u)$ as $t \to \infty$.

More recently, Evéquoz [5, 4] considers problems of the form

$$
\begin{gathered}
\dot{u}(t) + F(t,u(t),\xi,\lambda) = 0 \quad \text{for all } t \geq 0, \\
Pu(0) = \phi(\xi,\lambda), \\
\lim_{t\to\infty} u(t) = 0,
\end{gathered}
$$

in which global continuation in the real parameter λ is explored, in both the continuous and Sobolev settings. This further expands the range of applicability of the technique. In particular, a more complicated dependence on ξ is allowed, as well as apparently more flexibility (via λ) in the path of solutions departing from the trivial solution. These results are applied by the same author in [6] to a third order ODE that arises in the study of free convection boundary layers in porous media.

In this paper, we will first provide all of the necessary background from the present author's dissertation [10]. These arguments are elaborated in some cases and simplified in others. Since this material has not been published elsewhere, we include all such background for completeness and ease of reference in this and future work.

Once this is complete, we prove the existence of solutions to a class of boundary value problems of the form

$$
\begin{gathered}
\dot{v} + g(t,w) = f_1, \\
\dot{w} + h(t,v) = f_2, \\
v(0) = \xi, \\
v(\infty) = w(\infty) = 0\,.
\end{gathered}
\tag{1.2}
$$

A version of this problem was treated in [16], in which the nonlinearities g and h were assumed to have autonomous limits as $t \to \infty$, and in which restrictions were placed on the magnitudes of the derivatives $D_w g$ and $D_v h$. Our results remove these restrictions.

We now briefly indicate the overall arrangement of this paper; each section includes its own detailed introduction and orientation. We approach problem (1.1) by writing it as $\mathcal{F}(u) = (f, \xi)$, where

$$\mathcal{F}(u) := (\dot{u} + F(\cdot, u), Pu(0)). \tag{1.3}$$

In Section 2, we provide the functional setting for $\mathcal{F}$ and prove that under suitable hypotheses concerning F, that $\mathcal{F}$ is a C^1 map of Banach spaces.

In Section 3, we turn to the question of properness. To determine whether $\mathcal{F}$ is proper on closed, bounded subsets of the domain, we provide a condition that involves checking for solutions of certain limits of (1.1) obtained by letting $t \to \infty$ in a suitable topology.

In Section 4, we discuss the desired Fredholm property, along with some connections with exponential dichotomies and with the properness property.

Having paved the way for the use of topological degree for proper C^1 maps of Fredholm index zero, in Section 5 we prove several existence theorems. This shows how to use the results of Sections 2, 3, and 4 (along with *a priori* bounds for solutions) to prove the existence of solutions to (1.1).

In Section 6, we turn to the specific problem (1.2), showing that under suitable conditions of g and h that this problem always has a solution.

Throughout this paper, we will use the following notation and definitions. Given an interval $I \subseteq \mathbb{R}$, we will denote by $C_{\mathrm{b}}(I)$ the Banach space of all continuous $\mathbb{R}^N$-valued functions on I, with the usual supremum norm

$$\|u\|_\infty = \sup_{t \in I} |u(t))|.$$

(Of course any convenient norm $|\xi|$ can be used in $\mathbb{R}^N$.) We will denote by $C^1_{\mathrm{b}}(I))$ the Banach space consisting of those functions in $C_{\mathrm{b}}(I)$ with bounded derivative. For this space, we use the norm

$$\|u\|_{1,\infty} = \|u\|_\infty + \|\dot{u}\|_\infty$$

We denote by $C_0(I)$ the closed subspace of $C_{\mathrm{b}}(I)$ that consists of that functions that tend to zero as $t \to \infty$. We denote by $C^1_0(I)$ the closed subspace of $C^1_{\mathrm{b}}(I)$ that consists of those functions such that both $u(t)$ and $\dot{u}(t)$ tend to zero as $t \to \infty$.

Almost always, we will use $I = [0, \infty)$. In those cases, we will simply write C_{b}, C^1_{b}, etc. Otherwise, we will explicitly specify the interval, by writing $C^1_0(\mathbb{R})$, $C_{\mathrm{b}}((-\infty, 0])$, etc.

Given a function such as $F = F(t, z)\colon [0, \infty) \times \mathbb{R}^N \to \mathbb{R}^N$, we will often have need of the so-called Nemytskiĭ operator N_F associated to F. The operator N_F acts on functions $u = u(t)\colon [0, \infty) \to \mathbb{R}^N$ through composition, as follows.

$$N_F(u) = v, \quad \text{where } v(t) = F(t, u(t)).$$

2. Smoothness of the Nemytskii Operator

In this section we provide the function setting and we give the conditions on $F = F(t, z)$ that ensure that the induced map $\mathcal{F}$ from (1.3) is a C^1 map of Banach spaces.

Let $F = F(t,z)\colon [0,\infty) \times \mathbb{R}^N \to \mathbb{R}^N$ be a function that satisfies the following conditions:

$$F \text{ is continuous, with } \lim_{t\to\infty} F(t,0) = 0, \tag{2.1}$$

$$D_z F \text{ exists and is continuous on } [0,\infty) \times \mathbb{R}^N, \tag{2.2}$$

and for each compact subset K of $\mathbb{R}^N$,

$$F \text{ and } D_z F \text{ are BUC on } [0,\infty) \times K, \tag{2.3}$$

where "BUC" means "bounded and uniformly continuous".

Lemma 2.1. *Let $G = G(t,z)\colon [0,\infty) \times \mathbb{R}^N \to \mathbb{R}^N$ be bounded and uniformly continuous on $[0,\infty) \times K$, for every compact subset K of $\mathbb{R}^N$. Then G has the following properties:*

(a) *For each $u \in C_{\rm b}$ and each $\epsilon > 0$, there is $\delta > 0$ such that*
$$|G(t,u(t)) - G(t,v(t))| < \epsilon$$
for all $v \in C_{\rm b}$ such that $\|u - v\|_\infty < \delta$.

(b) *For each $u \in C_0$,*
$$\lim_{t\to\infty} G(t,u(t)) - G(t,0) = 0.$$

Proof. (a) Let $R = 1 + \|u\|_\infty$. By assumption, the function G is uniformly continuous on $[0,\infty) \times \overline{B}_R(0)$. Thus, there is $\delta_1 > 0$ such that
$$|G(t,x) - G(t,y)| < \epsilon$$
as long as $x, y \in \overline{B}_R(0)$ with $|x - y| < \delta_1$. It follows at once that the choice $\delta = \min(1, \delta_1)$ is sufficient.

(b) By assumption, the function G is uniformly continuous on $[0,\infty) \times \overline{B}_1(0)$. Thus, there is $0 < \delta < 1$ such that
$$|G(t,x) - G(t,0)| < \epsilon$$
as long as $|x| < \delta < 1$. Since $\lim_{t\to\infty} u(t) = 0$, part (b) is proved. $\square$

Theorem 2.2. *Suppose that F satisfies* (2.1), (2.2), *and* (2.3). *Then the Nemytskiĭ operator N_F is a well defined C^1 map from C_0^1 to C_0, and $DN_F(u)v = N_G(u)v$ (where $G = D_z F$ and where the multiplication of $N_G(u)$ by v is pointwise in t).*

Proof. Let $u \in C_0^1$ be given. To see that $N_F(u) \in C_0$, consider a sequence $t_n \to t$ in $[0,\infty)$. It follows that $u(t_n) \to u(t)$ since u is continuous, and hence that $F(t_n, u(t_n)) \to F(t, u(t))$. This shows that $N_F(u)$ is continuous on $[0,\infty)$. To see that $N_F(u)(t) \to 0$ as $t \to \infty$, let $\epsilon > 0$ be given. Since $u \in C_0$ and F is uniformly continuous on $[0,\infty) \times \overline{B}_{\|u\|_\infty}(0)$ by (2.3), one finds for all sufficiently large t that $|F(t,u(t)) - F(t,0)| < \epsilon$. By (2.1), this is enough to show that $N_F(u) \in C_0$.

The next claim to verify is that N_F is differentiable, with $DN_F = N_G$. Let u and v be chosen members of C_0^1. By the using the mean value theorem once for each $t \geq 0$,
$$N_F(u+v) - N_F(u) - N_G(u)v = N_G(u + \tau v)v - N_G(u)v,$$
where the function $0 \leq \tau \leq 1$ of t depends on the choice of v (and of course on the choice of u). It thus follows from Lemma 2.1 (a) that
$$N_F(u+v) - N_F(u) - N_G(u)v = {\rm o}(\|v\|_\infty) = {\rm o}(\|v\|_{1,\infty})$$

in C_0 as v tends to zero in C_0^1.

Finally, to check that DN_F is continuous, let $u \in C_0^1$ and $\epsilon > 0$ be given. It follows from Lemma 2.1 (a) that

$$\|N_G(u)v - N_G(w)v\|_\infty < \epsilon$$

for all unit vectors $v \in C_0^1$, provided only that $\|u - w\|_{1,\infty}$ (or even $\|u - w\|_\infty$) is sufficiently small. This proves that $DN_F(u)$ varies continuously in $\mathcal{L}\big(C_0^1, C_0\big)$ with respect to $u \in C_0^1$. ☐

Remark 2.3. An examination of the above proof reveals that N_F is also of class C^1 as a map from C_0 into itself, and also as a map from $C_{\rm b}$ into itself.

Remark 2.4. With only the obvious changes, the result holds when $[0, \infty)$ is replaced by any closed interval, and $(-\infty, \infty)$ in particular.

We have the following corollary for the operator $\mathcal{F}\colon C_0^1 \to C_0 \times X_1$ defined by

$$\mathcal{F}(u) := (\dot{u} + N_F(u), Pu(0)) \quad \text{for } u \in C_0^1. \tag{2.4}$$

Corollary 2.5. *In the situation of Theorem 2.2, the map $\mathcal{F}$ is of class C^1. In addition, if we set $G := D_zF$, then*

$$D\mathcal{F}(u)v = (\dot{v} + N_G(u)v, Pv(0)).$$

Proof. Differentiation is a continuous linear map from C_0^1 into C_0, and the evaluation of v at $t = 0$ followed by the linear projection P is a continuous linear map from C_0^1 into X_1. Therefore, Corollary 2.5 is a direct result of Theorem 2.2. ☐

3. Properness

In this section, we establish a necessary and sufficient condition for $\mathcal{F}$ to be proper on the closed, bounded subsets of C_0^1. The essential idea is the following. When $\{u_n\}$ is a bounded sequence in C_0^1 such that $\mathcal{F}(u_n)$ converges in $C_0 \times X_1$, we are to show that $\{u_n\}$ has a convergent subsequence in C_0^1. To find a convergent subsequence of $\{u_n\}$, we show that the sequence forms a relatively compact set, by the use of a result from Rabier [15] that characterizes the relatively compact subsets of C_0. To use this result, one must show that the sequence $\{u_n(t)\}$ tends to zero uniformly (with respect to n) as $t \to \infty$.

This "equi-decay" is characterized in [15] by a condition involving sequences of the form $\{u_n(\cdot + \xi_n)\}$, where $\xi_n \to \infty$. It is this temporal translation to infinity that ultimately brings one to the condition (Theorem 3.16) that no equation of the form $\dot{u} + N_E(u) = 0$ have a nonconstant C^1 solution, where E is any uniform–on–compacta limits of temporal translations of F. It is a noteworthy artifact of the translation to infinity that this condition involves problems on the whole line, even though the original problem is posed on the half line.

Remark 3.1. In this section we will follow the following convention. Any vector valued function (in particular, any real valued function) that depends on a variable $t \geq 0$ is extended to negative values of t whenever convenient, by using the even extension. Of course, this convention is only used where needed and never for a function whose domain already includes negative values of t.

For example, let $u \in C_0^1$ and $\xi_n \to \infty$ in $\mathbb{R}$. Then the sequence $\{u(t + \xi_n)\}$ is well defined for all $t \in \mathbb{R}$ under this convention, and this sequence can hence have a well-defined pointwise or uniform–on–compacta limit on all of $\mathbb{R}$. Notice that this kind of sequence is locally eventually independent of the choice of extension (since $t+\xi_n$ is eventually positive), and so any limit function is independent of the choice of extension. This is characteristic of our later use of this "even extension convention", which will occur with minimal further comment. The purpose of this perhaps distracting convention is to avoid the alternate distraction of a prominently displayed extension operator.

3.1. **Topological preliminaries.** Let Z be a nonempty subset of $\mathbb{R}^N$, and define the following closed subspace of $C_{\rm b}(\mathbb{R}) = C_{\rm b}(\mathbb{R},\mathbb{R}^N)$:

$$C_Z(\mathbb{R}) = C_Z(\mathbb{R},\mathbb{R}^N) := \big\{u \in C_{\rm b}(\mathbb{R}) : \lim_{t\to\pm\infty} \operatorname{dist}(u(t), Z) = 0\big\}, \tag{3.1}$$

consisting of those functions $u = u(t)$ converging to Z as $|t|$ tends to infinity. We define the space $C_Z \subset C_{\rm b}$ of functions defined on $[0,\infty)$ by requiring that $\lim_{t\to\infty} \operatorname{dist}(u(t), Z) = 0$. It is clear that $C_0 = C_{\{0\}}$ and that $C_Z \subseteq C_W$ if and only if $\bar{Z} \subset \bar{W}$ (with equality if and only if $\bar{Z} = \bar{W}$).

We say that Z is *totally disconnected* if the connected components of Z are singletons. Examples include finite sets, sequences (with or without their limit point), and Cantor-like sets. If $Z \subset \mathbb{R}$, then Z is totally disconnected if and only if Z does not contain an interval.

If $\mathcal{H}$ is a subset of $C_{\rm b}$ or $C_{\rm b}(\mathbb{R})$ and if I is a subset of the corresponding domain $[0,\infty)$ or $\mathbb{R}$ respectively, then $\mathcal{H}(I) := \{u(t) : u \in \mathcal{H}, t \in I\}$, the union of the direct images of I under members of $\mathcal{H}$.

Here is the result that we will use from Rabier [15], followed by a simple corollary adapted for use on the half line.

Lemma 3.2 (Rabier [15, Corollary 7])**.** *A subset $\mathcal{H}$ of $C_0(\mathbb{R})$ is relatively compact if and only if the following three conditions hold:*

(a) *The set $\mathcal{H}(\mathbb{R})$ is bounded.*
(b) *The set $\mathcal{H}$ is uniformly equicontinuous.*
(c) *There is a compact and totally disconnected subset Z of $\mathbb{R}^N$ with the following property. If $\tilde{u} \in C_{\rm b}(\mathbb{R})$ and there are sequences $\{u_n\} \subset \mathcal{H}$ and $\{\xi_n\} \subset \mathbb{R}$ such that $|\xi_n| \to \infty$ and*

$$\lim_{n\to\infty} u_n(t+\xi_n) = \tilde{u}(t) \quad \text{for all } t \in \mathbb{R},$$

then $\tilde{u}(\mathbb{R}) \subset Z$.

Corollary 3.3. *A subset $\mathcal{H}$ of C_0 is relatively compact if and only if the following three conditions hold:*

(a) *The set $\mathcal{H}([0,\infty))$ is bounded.*
(b) *The set $\mathcal{H}$ is uniformly equicontinuous.*
(c) *There is a compact and totally disconnected subset Z of $\mathbb{R}^N$ with the following property. If $\tilde{u} \in C_{\rm b}(\mathbb{R})$ and there are sequences $\{u_n\} \subset \mathcal{H}$ and $\{\xi_n\} \subset \mathbb{R}$ such that $\xi_n \to \infty$ and*

$$\lim_{n\to\infty} u_n(t+\xi_n) = \tilde{u}(t) \quad \text{for all } t \in \mathbb{R},$$

then $\tilde{u}(\mathbb{R}) \subset Z$.

Proof. Use the even extension of the functions in C_0. Now apply Lemma 3.2. Also see the final paragraph of Section 2 in [15], where such generalizations are mentioned. □

Remark 3.4. In [15] Lemma 3.2 is proved in a more general setting. For one thing, the functions in C_0 are allowed to take values in a general metric space, and item (a) is that $\mathcal{H}(\mathbb{R})$ should be relatively compact. This suggests the question of whether item (c) is necessary when the metric space is $\mathbb{R}^N$. The example $\mathcal{H} = \{u_n : n \in \mathbb{N}\}$ where $u_n(t) = \min\big(1, \max(n-t, 0)\big)$ for $t \geq 0$ shows that item (c) is indeed necessary.

Remark 3.5. When using Lemma 3.2 or Corollary 3.3, one often finds that the set $\mathcal{H}$ consists of the terms of a sequence $\{v_n\}$. If so, one may assume that $\{u_n\}$ is a subsequence of $\{v_n\}$ when checking condition (c).

If $\{T_n\}$ is a sequence of continuous functions from a metric spaces M into a metric space N, we will write[1] $T = \text{co-lim}_{n\to\infty} T_n$ if the sequence $\{T_n\}$ converges uniformly to T on each compact subset of M. In each use of this notation, M and N will be clear from context.

Lemma 3.6. *Suppose that F satisfies* (2.1), (2.2), *and* (2.3). *Let $\{\xi_n\} \subset \mathbb{R}$ be a sequence such that $\xi_n \to \infty$ and put $E_n(t,z) := F(t+\xi_n, z)$. Then there exist a subsequence $\{E_{n_k}\}$ and a function $E\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ such that*

- (a) $E = \text{co-lim}_{k\to\infty} E_{n_k}$,
- (b) *E satisfies* (2.1), (2.2), *and* (2.3) *(with $[0,\infty)$ replaced by $\mathbb{R}$ and F replaced by E), and*
- (c) $D_zE = \text{co-lim}_{k\to\infty} D_zE_{n_k}$.

Proof. The classical Ascoli-Arzela Theorem applies for each compact subset K of $\mathbb{R}\times\mathbb{R}^N$. (The boundedness of $\{E_n(\alpha) : n \in \mathbb{N}, \alpha \in K\}$ and and the equicontinuity of $\{E_n\big|_K\}$ follow from (2.3).) Apply the Ascoli-Arzela theorem recursively for an increasing sequence of compact sets that exhausts $\mathbb{R}\times\mathbb{R}^N$. This diagonal process yields a function E and a subsequence $\{E_{n_k}\}$ such that $E = \text{co-lim}_{k\to\infty} E_{n_k}$. Repeat this process with $\{D_zE_{n_k}\}$ to obtain a further subsequence (still denoted $\{E_{n_k}\}$) and a function H such that $H = \text{co-lim}_{k\to\infty} D_zE_{n_k}$. As usual, since the convergence is uniform on compact sets, D_zE exists and $D_zE = H$. Thus, E satisfies (2.2). It is now easy to verify that (2.1) (with $E(t,0) \equiv 0$) and (2.3) are inherited by E from F. □

Definition 3.7. Given a function F that satisfies (2.1), (2.2), and (2.3), we define the *omega limit set* [2] of F by

$$\omega(F) = \{E : E = \text{co-lim}\, E_n \text{ for some sequence } \xi_n \to \infty\}.$$

Here, $E_n(t,z) := F(t+\xi_n, z)$, just as in Lemma 3.6. It is a corollary of Lemma 3.6 that $\omega(F)$ is nonempty. Before moving on, several examples may be helpful. The proofs of the following claims are left as exercises (but may also be found in [10]).

Example 3.8. The pointwise limit $F^\infty(x) = \lim_{t\to\infty} F(t,x)$ exists if and only if $\omega(F)$ is the singleton $\{F^\infty\}$ (which includes the autonomous situation $F(t,x) = F^\infty(x)$). This is the case considered in [17, 16].

Example 3.9. Suppose F is periodic or asymptotically periodic, in the sense that

$$\lim_{t\to\infty} |F(t,x) - G(t,x)| = 0$$

---

[1] As suggested by the usual name "compact-open" for the resulting topology.

[2] This terminology is suggested by its use in the linear setting in [19].

pointwise in x, for some G that is periodic in t. Then

$$\omega(F) = \{G(\cdot + \tau, \cdot) : 0 \le \tau < T\},$$

where T is a period of G.

Example 3.10. Suppose that for all $x \in \mathbb{R}^N$ and all $R > 0$,

$$\lim_{t\to\infty} \sup_{0\le\tau\le R} |F(t,x) - F(t+\tau,x)| = 0.$$

In this case we will say that F is *asymptotically autonomous*. This is the case if and only if $\omega(F)$ consists only of autonomous functions $G(t,x) = G(x)$. If F is C^1, a sufficient (but not necessary) condition for asymptotic autonomy is that $\lim_{t\to\infty} D_t F(t,x) = 0$ pointwise in x.

Example 3.11. Suppose F is of the quasilinear form $F(t,x) = A(t)q(x)$ where A is a $d \times d$ matrix function and $q(x) \in \mathbb{R}^N$. Let $\omega(A)$ denote the set of all uniform-on-compact-intervals limits of sequences $\{A(\cdot + \tau_k)\}$ where $\tau_k \to \infty$. Then $\omega(F)$ consists of all quasilinear functions $G(t,x) = B(t)q(x)$ where $B \in \omega(A)$. When F is quasilinear, the preceding examples correspond respectively to the cases that $A(t)$ has a limit as $t \to \infty$, or that A is asymptotically periodic, or that A is asymptotically constant. In this last case, note that

$$\omega(A) = \cap_{n\in\mathbb{N}} \overline{A\big([n,\infty)\big)}.$$

3.2. **Properness via solutions of omega limit equations.** The following definition is of key importance in Theorem 3.16. Recall that the definition of the term "totally disconnected" is provided on page 6.

Definition 3.12. Assume that F satisfies (2.1), (2.2), and (2.3). Let S denote the set of all (bounded) functions $u \in C^1_{\rm b}(\mathbb{R})$ such that $\dot u + E(t,u) = 0$ on $(-\infty,\infty)$, for some E in the omega-limit set $\omega(F)$. We will say that F has an admissible omega-limit set provided that S consists only of constant functions, and that these constants form a compact and totally disconnected subset of $\mathbb{R}^N$.

When F has an admissible omega-limit set, if "t goes to infinity" in the equation $\dot u + F(t,u) = f$, no resulting equation has a nonconstant solution that is bounded on $\mathbb{R}$. This will be the key to proving Theorem 3.16.

It is useful to record one more definition, if only to highlight the connection between the admissibility of the omega-limit set and the third item in Corollary 3.3.

Definition 3.13. Given a function F that satisfies (2.1), (2.2), and (2.3), we define the *omega zero set* of F by

$$Z(F) := \{z \in \mathbb{R}^N : E(\cdot, z) = 0 \text{ for some } E \in \omega(F)\}.$$

Remark 3.14. Notice that $u(t) = c$ is a constant solution of $\dot u + E(t,u) = 0$ (for some $E \in \omega(F)$) if and only if $c \in Z(F)$. This shows that if F has an admissible omega-limit set, then the set S that is mentioned in Definition3.12 coincides with $Z(F)$.

Remark 3.15. Notice that $Z(F)$ includes all $z \in \mathbb{R}^N$ such that $\lim_{t\to\infty} F(t,z) = 0$. However, $Z(F)$ may contain other points. As an illustration, consider $F(t,z) = (\sin\sqrt{t})z$. Then $z = 0$ is the only point such that $\lim_{t\to\infty} F(t,z) = 0$, but $Z(F) = \mathbb{R}^N$ because $0 \in \omega(F)$.

For Theorem 3.16, recall that $\mathcal{F}\colon C_0^1 \to C_0 \times X_1$ is defined by

$$\mathcal{F}(u) := (\dot{u} + N_F(u), Pu(0)),$$

and that it follows from Theorem 2.2 that $\mathcal{F} \in C^1\big(C_0^1, C_0 \times X_1\big)$ and that

$$D\mathcal{F}(u)v = (\dot{v} + D_zF(\cdot,u)v, Pv(0)) \quad \text{for } u, v \in C_0^1.$$

Theorem 3.16. *Assume that F satisfies* (2.1), (2.2), (2.3), *and that F has an admissible omega-limit set. Then $\mathcal{F}$ is proper on each subset of C_0^1 that is closed and bounded.*

Proof. Let $\{u_n\}$ be a bounded sequence in C_0^1 such that $\{(f_n, \xi_n)\} := \{\mathcal{F}(u_n)\}$ is convergent in $C_0 \times X_1$. We are to prove that $\{u_n\}$ has a C_0^1-convergent subsequence. To do so, we will first use Corollary 3.3 to find a C_0-convergent subsequence; take $\mathcal{H} = \{u_n : n \in \mathbb{N}\}$ (see Remark 3.5).

Item (a) of Corollary 3.3 follows immediately from the boundedness of $\{u_n\}$ in C_0^1. Item (b) also follows from this boundedness. To see this, since $\{u_n\}$ is bounded in C_0^1, the sequence $\{\dot{u}_n\}$ of derivatives is equibounded on $[0,\infty)$. Therefore, all of the functions in $\mathcal{H}$ are uniformly Lipshitz on $[0,\infty)$, and share a common Lipshitz constant. This implies that $\mathcal{H}$ is uniformly equicontinuous.

For item (c), take $Z = Z(F)$ (see Definition 3.13) which is compact and totally disconnected by Definition 3.12. Let $u \in C_{\rm b}(\mathbb{R})$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ be given. Let $\{\xi_k\}$ be a sequence of real numbers such that $\lim_{k\to\infty} \xi_k = \infty$. Put $v_k(t) := u_{n_k}(t + \xi_k)$. Assuming that $\{v_k\}$ converges pointwise to u, we are to show that $u(\mathbb{R}) \subset Z(F)$.

We make several observations:

(i) Since $\{f_n\} \subset C_0$, the translated sequence $\{f_{n_k}(\cdot + \xi_k)\}$ converges to zero as $k \to \infty$, uniformly on compact intervals.

(ii) By passing to a subsequence and relabeling, we may assume that the convergence of $\{v_k\}$ to u is uniform on compact intervals: $u = \text{co-lim}_{k\to\infty} v_k$. This follows from the boundedness and equicontinuity of $\{v_k\}$.

(iii) By again passing to a subsequence, there is some $E \in \omega(F)$ such that $E = \text{co-lim}_{k\to\infty} E_k$, where $E_k(t,z) = F(t+\xi_k, z)$. See Lemma 3.6 and Definition 3.7.

It now follows readily from (ii) and (iii) that the sequence $\{N_{E_k}(v_k)\}$ converges to $\{N_E(u)\}$ uniformly on compact intervals; we note that $N_{E_k}(v_k)(t) = E_k(t, v_k(t)) = F(t+\xi_k, u_{n_k}(t+\xi_k))$ and $N_E(u)(t) = E(t,u(t))$. As a result, the sequence $\{\dot{v}_k\}$ converges uniformly on compact sets to $-E(t,u)$. Indeed,

$$\begin{aligned}
\dot{v}_k(t) &= \dot{u}_{n_k}(t+\xi_k) \\
&= f_{n_k}(t+\xi_k) - F(t+\xi_k, u_{n_k}(t+\xi_k)) \\
&= f_{n_k}(t+\xi_k) - E_k(t, v_k(t)),
\end{aligned}$$

which converges to $-E(t,u(t))$ as $k \to \infty$, uniformly on compact sets. But uniform convergence of $\{\dot{v}_k\}$ on an interval implies that the limit function u is differentiable and that $\{\dot{v}_k\}$ converges to $\dot{u}$. Therefore,

$$\dot{u}(t) + E(t, u(t)) = 0.$$

Since u is bounded, it follows (from the admissibility of the omega-limit set of F) that u is a constant function, say $u(t) \equiv z$. Therefore, $E(t,z) = E(t,u(t)) = -\dot{u}(t) = 0$, so that $u(\mathbb{R}) = \{z\} \subset Z(F)$ by Definition 3.13.

Having completed the verification of all three items in Corollary 3.3, we conclude that $\{u_n\}$ has a subsequence (again denoted $\{u_n\}$) that converges in C_0 to a limit u^∞. To complete the proof, recall that N_F is continuous from C_0 to itself (Remark 2.3). Therefore, the sequence $\{\dot{u}_n\} = \{-N_F(u_n)\}$ converges to $-N_F(u^\infty)$ in C_0. In particular, u^∞ is differentiable and $\{\dot{u}_n\}$ converges to $\dot{u}^\infty$ in C_0. Since (the subsequence) $\{u_n\}$ is therefore convergent in C_0^1, the proof is complete. $\square$

4. The Fredholm property

4.1. **Linear systems.** Let $A = A(t)\colon [0,\infty) \to \mathbb{R}^{d\times d}$ be a bounded and continuous matrix function. In this section, we develop the Fredholm properties of the linear operator $D_A\colon C_0^1 \to C_0$ defined by

$$D_A u(t) := \dot{u}(t) + A(t)u(t),$$

as well as the augmented linear operator $\Lambda\colon C_0^1 \to C_0 \times X_1$ defined by

$$\Lambda u = (D_A u, Pu(0)).$$

Before we begin, we have a few remarks concerning autonomous systems (meaning that $A(t) \equiv A$ is constant). In this case, the Fredholm property and index is determined by the spectrum of A together with the dimensions of the associated invariant subspaces of $\mathbb{R}^N$. If A has any eigenvalue on the imaginary axis, then the range of D_A is not closed in C_0, whence neither D_A nor Λ is Fredholm. Otherwise both operators are Fredholm. Moreover, the index of D_A is the algebraic count of eigenvalues of A having positive real part. The index of Λ is diminished from the index of D_A by exactly the dimension of X_1.

These facts are proved in the Sobolev space setting in [16, Section 2]. That paper then reduces the nonautonomous case to the autonomous case by using a limit $A^\infty = \lim_{t\to\infty} A(t)$. We do not assume the existence of such a limit, and because of this our arguments will be significantly different. In the case A^∞ exists, our results reduce to those of [16, Theorem 4.1], albeit with respect to spaces of smooth functions rather than Sobolev spaces.

When there is no limit for $A(t)$ as $t \to \infty$, the situation is a bit more subtle. Merely to know the spectrum of $A(t)$ is no longer sufficient. In fact, the operator Λ turns out to be Fredholm exactly when the matrix $A = A(t)$ admits an *exponential dichotomy* on $[0,\infty)$. This was proved by Palmer [13, 12]. Palmer did not consider spaces of functions that tend to zero as $t \to \infty$, so we will provide proofs herein.

With respect to the linear system $D_A u := \dot{u} + Au = 0$, let $U = U(t)$ denote the fundamental matrix solution of the system $\dot{U} + AU = 0$ with $U(0) = I$. We recall that A admits an exponential dichotomy on $[0,\infty)$ if there exist a projection Π and positive constants K and α such that

$$|U(t)\Pi U(s)^{-1}| \le K e^{-\alpha(t-s)} \tag{4.1}$$

for all $t \ge s \ge 0$ and

$$|U(t)(I-\Pi)U(s)^{-1}| \le K e^{-\alpha(s-t)} \tag{4.2}$$

for all $s \ge t \ge 0$.

It is well known that the range of Π is uniquely determined in the sense that if A also admits an exponential dichotomy with projection Π', then $\operatorname{rge}\Pi' = \operatorname{rge}\Pi$. Conversely, if Π' is any projection with $\operatorname{rge}\Pi' = \operatorname{rge}\Pi$, then A admits an exponential dichotomy with projection Π'. (This converse is untrue if one replaces $[0,\infty)$ by the whole real line.) Also, $\operatorname{rge}\Pi$ coincides with the subspace of $\mathbb{R}^N$ consisting of those initial data $\xi \in \mathbb{R}^N$ such that the solution $u(t) = U(t)\xi$ of $D_A u = 0$, $u(0) = \xi$ remains bounded as $t \to \infty$. By (4.1) a solution with initial data in $\operatorname{rge}\Pi$ will tend to 0 exponentially as $t \to \infty$. In contrast, (4.2) shows that a solution with initial data outside of $\operatorname{rge}\Pi$ will tend to infinity exponentially as $t \to \infty$. For these properties (and others) see Coppel [3] or Massera and Schäffer [9].

Our first result in this section is that the Fredholm property of the operator D_A is equivalent to the property that A admit an exponential dichotomy on $[0,\infty)$, and that the Fredholm index is the same as the rank of any projection associated with the exponential dichotomy. In order to prove this result, it will help to know that when D_A has closed range, it follows D_A is onto C_0 and that a certain kind of *a priori* bound exists on solutions to $D_A u = f$. That is the content of the following lemma:

Lemma 4.1. *Assume that the operator $D_A : C_0^1 \to C_0$ has closed range. Let V_1 be the subspace of $\mathbb{R}^N$ consisting of the initial values $u(0)$ of bounded solutions u to the homogeneous equation $D_A u = 0$, and let V_2 be any direct complement of V_1. Then*

(a) *The operator D_A is onto C_0.*
(b) *There is a positive constant r such that for all $f \in C_0$, one has the estimate*
$$\|u\|_\infty \le r\|f\|_\infty, \tag{4.3}$$
where u is the unique $u \in C_0^1$ such that $D_A u = f$ and $u(0) \in V_2$.

Proof. Since the subspace C_0 of C_0 consisting of compactly supported functions is dense in C_0, it is enough to prove that the range of D_A contains C_0. Let $f \in C_0$, and suppose that f is supported in $[0,T]$. Let
$$\xi = -\int_0^T U(s)^{-1} f(s)\,\mathrm{d}s,$$
and let
$$u(t) = U(t)\Big(\xi + \int_0^t U(s)^{-1} f(s)\,\mathrm{d}s\Big).$$
Then u is supported in $[0,T]$ and $D_A u = f$. This completes the proof of the first assertion.

To prove the second assertion, let $f \in C_0$ be given. Since we now know that D_A is surjective, let $v \in C_0^1$ be such that $D_A v = f$. Let Π denote the projection onto V_1 along V_2. There is a unique $w \in C_0^1$ such that $D_A w = 0$ and $w(0) = \Pi v(0)$. It follows that if $u = v - w$, then $D_A u = f$ and $u(0) \in V_2$. To see that u is unique in C_0^1, the difference of two such functions will be a *bounded* solution to the homogeneous equation $D_A u = 0$ whose initial value lies in V_2. By definition of V_2, this initial value must be zero.

We will denote by $S: C_0 \to C_\mathrm{b}$ the linear map that carries f into u. We will use the closed graph theorem to prove that S is bounded, and the proof will then be complete. Take a sequence $\{(f_n, u_n)\}$ in the graph of S and suppose that this sequence converges in $C_0 \times C_\mathrm{b}$ to some (f, u). Fix $t > 0$. Using the uniform convergence of $\{(f_n, u_n)\}$ to (f, u) and the uniform continuity of A on $[0, t]$, we have

$$\begin{aligned} u(t) - u(0) &= \lim_{n\to\infty} u_n(t) - u_n(0) \\ &= \lim_{n\to\infty} \int_0^t -A(s)u_n(s) - f_n(s)\,\mathrm{d}s \\ &= \int_0^t -A(s)u(s) - f(s)\,\mathrm{d}s. \end{aligned}$$

Upon differentiating with respect to t, we find that $D_A u = f$. Since it also happens that $u(0) = \lim_{n\to\infty} u_n(0) \in V_2$, it follows that $Sf = u$ as desired. $\square$

Of course, the second assertion in Lemma 4.1 continues to hold if D_A is already known to be surjective. Here is the first main result in this section.

Theorem 4.2. *Assume that $A = A(t)\colon [0,\infty) \to \mathbb{R}^{d\times d}$ is bounded and continuous. Then the operator $D_A\colon C_0^1 \to C_0$ defined by $D_A u(t) = \dot{u}(t) + A(t)u(t)$ is a Fredholm operator if and only if A admits an exponential dichotomy on $[0,\infty)$. In this case, D_A is surjective and $\dim\ker D_A = \dim\operatorname{rge}\Pi$ so that*

$$\operatorname{ind} D_A = \dim\operatorname{rge}\Pi,$$

where Π is any projection associated with the exponential dichotomy. Additionally, for all $f \in C_0$,

$$\{u(0) : u \in C_0^1 \text{ and } \dot{u} + Au = f\} = \operatorname{rge}\Pi - \int_0^\infty (I-\Pi)U(s)^{-1}f(s)\,\mathrm{d}s. \tag{4.4}$$

Proof. First assume that D_A is Fredholm. In particular, the range of D_A is closed in C_0. According to Coppel [3, (Proposition 3 on page 22)], to prove that A admits an exponential dichotomy it is sufficient to show that the equation $\dot{u}(t)+A(t) = f(t)$ has a bounded solution (not necessarily in C_0^1) whenever $f \in C_\mathrm{b}$. So let $f \in C_\mathrm{b}$ be given, and let $\{f_n\}$ be a sequence of continuous functions on $[0,\infty)$ such that for each $n \in \mathbb{N}$,

(a) f_n agrees with f on $[0, n]$,
(b) f_n is supported in $[0, n+1]$, and
(c) $|f_n(t)| \le |f(t)|$ on $[0,\infty)$.

Since these functions are compactly supported, they are all in C_0. Let V_1 and V_2 be as in Lemma 4.1, and let u_n be the unique solution to $D_A u_n = f_n$ such that $u_n(0) \in V_2$. According to Lemma 4.1, the sequence $\{u_n\}$ is bounded because the sequence $\{f_n\}$ is bounded. There is hence a subsequence $\{u_{n_k}\}$ such that the initial values $\xi_k = u_{n_k}(0)$ converge to some $\xi \in V_2$. Define

$$u(t) = U(t)\Big(\xi + \int_0^t U(s)^{-1}f(s)\,\mathrm{d}s\Big)$$

so that $D_A u = f$. For the desired application of the mentioned result from Coppel [3], it remains to show that u is bounded on $[0,\infty)$. Fix $t > 0$. For all $n > t$, we have $f_n(t) = f(t)$. Therefore for all $n_k > t$,

$$u_{n_k}(t) - u(t) = U(t)(\xi_k - \xi).$$

In particular, $u(t) = \lim_{k\to\infty} u_{n_k}(t)$. Because the sequence $\{u_n\}$ is (uniformly) bounded, this shows that u is bounded on $[0,\infty)$.

Conversely, assume that A admits an exponential dichotomy on $[0,\infty)$. For each $f \in C_0$ and $\xi \in \mathbb{R}^N$, introduce the notation

$$u = u_{f,\xi}(t) := U(t)\Big(\xi + \int_0^t U(s)^{-1} f(s)\,\mathrm{d}s\Big)$$

for the solution to the initial value problem $D_A u = f$, $u(0) = \xi$, regardless of whether this solution is in C_0^1. Recall that the map S that carries ξ into $u_{0,\xi}$ is an isomorphism of $\mathbb{R}^N$ onto the vector space of all solutions to the homogeneous equation $D_A u = 0$. The kernel of $D_A : C_0^1 \to C_0$ is thus isomorphic to the set of all $\xi \in \mathbb{R}^N$ such that $S\xi = u_{0,\xi} \in C_0^1$. We claim that in fact

$$\ker D_A = S(\operatorname{rge}\Pi).$$

Indeed, the defining properties (4.1) and (4.2) of exponential dichotomy imply that $u_{0,\xi}$ has exponential decay as $t \to \infty$ when $\xi \in \operatorname{rge}\Pi$, and exponential growth otherwise. Therefore, $u_{0,\xi} \in C_0^1$ if and only if $\xi \in \operatorname{rge}\Pi$.

We next consider the range of D_A in C_0. For each choice of f and ξ, we decompose $u_{f,\xi}$ along the projection Π as follows:

$$\begin{aligned}
u_{f,\xi} &= U(t)(\Pi + I - \Pi)\Big(\xi + \int_0^t U(s)^{-1} f(s)\,\mathrm{d}s\Big)\\
&= U(t)\Pi\xi + \int_0^t U(t)\Pi U(s)^{-1} f(s)\,\mathrm{d}s\\
&\quad + U(t)\Big((I-\Pi)\xi + \int_0^t (I-\Pi)U(s)^{-1} f(s)\,\mathrm{d}s\Big)\\
&=: g_1(t) + g_2(t) + g_3(t).
\end{aligned}$$

First, $g_1 = u_{0,\Pi\xi} \in C_0^1$. Second, let $\epsilon > 0$ and let $T > 0$ be such that $|f(t)| < \epsilon$ when $t > T$. Because of (4.1), when $t > T$

$$\begin{aligned}
|g_2(t)| &\le \int_0^t K e^{-\alpha t - s}|f(s)|\,\mathrm{d}s\\
&\le K\|f\|_\infty \int_0^T e^{-\alpha(t-s)}\,\mathrm{d}s + K\epsilon \int_T^t e^{-\alpha(t-s)}\,\mathrm{d}s\\
&= K\alpha^{-1}\big(\|f\|_\infty(e^{\alpha(T-t)} - e^{-\alpha t}) + \epsilon(1 - e^{\alpha(T-t)})\big)
\end{aligned}$$

For sufficiently large t, this expression is no more than $2K\alpha^{-1}\epsilon$. Since $\epsilon > 0$ was arbitrary, this shows that $g_2 \in C_0$. As for g_3, note first that

$$\eta := \int_0^\infty (I-\Pi)U(s)^{-1} f(s)\,\mathrm{d}s$$

is a well-defined element of $\operatorname{rge}(I-\Pi)$; this is due to (4.2) and the boundedness of f. Indeed,

$$|(I-\Pi)U(s)^{-1}| \le K^{-1}e^{-\alpha s}$$

so that the integrand decays exponentially as $s \to \infty$. Notice now that

$$\begin{aligned}
g_3(t) &= U(t)\Big((I-\Pi)\xi + \eta - \int_t^\infty (I-\Pi)U(s)^{-1} f(s)\,\mathrm{d}s\Big)\\
&= U(t)(I-\Pi)(\xi+\eta) - \int_t^\infty U(t)(I-\Pi)U(s)^{-1} f(s)\,\mathrm{d}s
\end{aligned}$$

Because of (4.2), we have

$$\begin{aligned}\Big|\int_t^\infty U(t)(I-\Pi)U(s)^{-1}f(s)\,\mathrm{d}s\Big| &\le \int_t^\infty Ke^{-\alpha(s-t)}|f(s)|\,\mathrm{d}s\\ &\le K\alpha^{-1}\sup_{s\ge t}|f(s)|,\end{aligned}$$

which converges to zero as $t\to\infty$. Therefore, g_3 will be in C_0 if and only if $\lim_{t\to\infty}U(t)(I-\Pi)(\xi+\eta)=0$. For this, it is necessary and sufficient that one has $\xi+\eta\in\operatorname{rge}\Pi$. Since $\xi=u_{f,\xi}(0)$, this is the content of assertion (4.4). In any case, it is possible to choose ξ (say $\xi=-\eta$) so that g_3 is in C_0. In that case, $u_{f,\xi}=g_1+g_2+g_3$ is in C_0 as well. Because A is bounded and $\dot u=-Au+f$, it follows that $u\in C_0^1$. This shows that D_A is surjective, which completes the proof. □

Notice that in the first part of the proof, we used the fact that Fredholm maps have closed range (by definition), but we used no other property of Fredholm maps. Therefore, as soon as D_A is known to have closed range, it follows that D_A is Fredholm (and that A admits an exponential dichotomy on $[0,\infty)$). This observation amounts to the following corollary, which will be used later.

Corollary 4.3. *Assume that $A=A(t)\colon[0,\infty)\to\mathbb{R}^{d\times d}$ is bounded and continuous and that the operator $D_A\colon C_0^1\to C_0$ defined by $D_Au(t)=\dot u(t)+A(t)u(t)$ has closed range. Then D_A is Fredholm.*

We next consider the Fredholm properties of the differential operator with evaluation at zero.

Theorem 4.4. *Assume that $A=A(t)\colon[0,\infty)\to\mathbb{R}^{d\times d}$ is bounded and continuous. Let P be any linear projection in $\mathbb{R}^N$. Then the operator $\Lambda\colon C_0^1\to C_0\times X_1$ defined by $\Lambda u=(D_Au,Pu(0))$ is a Fredholm operator if and only if A admits an exponential dichotomy on $[0,\infty)$. In this case,*

$$\ker\Lambda=\{U(\cdot)\xi:\xi\in\operatorname{rge}\Pi\cap\ker P\},$$

$$\operatorname{rge}\Lambda=\{(f,\eta)\in C_0\times X_1:\eta+\int_0^\infty(I-\Pi)U(s)^{-1}f(s)\,\mathrm{d}s\in\operatorname{rge}\Pi+\ker P\},$$

and $\operatorname{ind}\Lambda=\dim\operatorname{rge}\Pi-\dim X_1$, *where Π is any projection associated to the exponential dichotomy admitted by A.*

Proof. We can append the zero map to D_A without changing the Fredholm property and index, as long as the target space is only trivially enlarged; this results in the map

$$(D_A,0)\colon C_0^1\to C_0\times\{0\}.$$

If we now enlarge the target space to $C_0\times X_1$, the codimension of the range increases by $\dim X_1$. The map Λ is then a finite rank perturbation of the result; recall that neither the Fredholm property nor the index are affected by perturbations of finite rank (nor even compact perturbations). The end result in changing D_A into Λ is that the Fredholm index decreases by $\dim X_1$, the only exception to this being that neither operator is Fredholm.

Next, $u\in\ker\Lambda$ if and only if $u\in C_0^1$ with $D_Au=0$ and $Pu(0)=0$. By Theorem 4.2 this is the case if and only if $u(t)=U(t)\xi$ (so that $D_A=0$) and $\xi\in\operatorname{rge}\Pi$ (so that $u\in C_0^1$), and $u(0)\in\ker P$.

Finally, by Theorem 4.2, (f,η) is in the range of Λ if and only if $\eta = P\xi$ and $\xi \in \operatorname{rge}\Pi - \int_0^\infty (I-\Pi)U(s)^{-1}f(s)\,\mathrm{d}s$. Since $\eta = P\xi$ means that η differs from ξ by a vector in $\ker P$, this proves the claimed characterization of $\operatorname{rge}\Lambda$. $\square$

Corollary 4.5. *In the situation of Theorem 4.4, the operator Λ is Fredholm of index zero from C_0^1 into $C_0 \times X_1$ if and only if Π and P have the same rank. In this case, the following are equivalent:*

(a) *The map Λ is an isomorphism.*
(b) $\operatorname{rge}\Pi \cap \ker P = \{0\}$.
(c) $\mathbb{R}^N = \operatorname{rge}\Pi \oplus \ker P$.

Proof. In the situation of Theorem 4.4 the Fredholm index of Λ is $\dim \operatorname{rge}\Pi - \dim X_1$. Of course, this is zero if and only if Π and P have the same rank. Furthermore, a map of Fredholm index zero is an isomorphism if and only if the map has trivial kernel. By Theorem 4.4, this kernel is $\operatorname{rge}\Pi \cap \ker P$. Finally, under the assumption that P and Π have the same rank, the conditions that $\operatorname{rge}\Pi \cap \ker P$ and $\mathbb{R}^N = \operatorname{rge}\Pi \oplus \ker P$ are equivalent. $\square$

By drawing upon what is known about exponential dichotomies, we can use Corollary 4.5 to quickly deduce a variety of specific conditions that are sufficient for Λ to be Fredholm of index zero. We present several as examples. Except where an alternate citation is given, all of these can be verified by consulting (for example) [3] or [9].

Example 4.6. If $A^\infty = \lim_{t\to\infty} A(t)$ exists (which includes the constant case), then Λ is Fredholm if and only if A^∞ has no eigenvalues on the imaginary axis. In this case, Λ has index zero if and only if the rank of P is equal to the algebraic count of eigenvalues of A^∞ that have positive real part.

Example 4.7. If A is asymptotically autonomous, then Λ is Fredholm if and only if the eigenvalues of A are eventually bounded away from the imaginary axis. In this case, Λ has index zero if and only if the rank of P is equal to the algebraic count of eigenvalues that stay to the right of the imaginary axis. We must remark that this kind of condition is not valid when A is not asymptotically autonomous.

Example 4.8. By the usual Floquet theory (see Hsieh and Sibuya [8, pages 87-89]), the study of periodic systems can be reduced to that of autonomous systems. As a result, if A has period T, then Λ is Fredholm if an only if $U(T)$ has no eigenvalues of unit modulus. In this case, Λ has index zero if and only if the rank of P is equal to the algebraic count of eigenvalues of $U(T)$ with modulus greater than unity. In fact, this example extends to asymptotically periodic systems.

Example 4.9. The operator Λ is Fredholm if and only if there are a bounded, continuously differentiable Hermitian matrix function $H = H(t)$ and a constant $\beta > 0$ such that

$$H(t)A(t) + A(t)^*H(t) - \dot{H}(t) \geq \beta I$$

for a.e. $t \geq 0$, in the sense of quadratic forms on $\mathbb{R}^N$. (See Coppel [3].) In this case (see [11, Corollary 4.4]), the algebraic count d of positive eigenvalues of $H(t)$ is eventually independent of t, and Λ has index zero if and only if the rank of P is equal to d.

4.2. **Properness and the Fredholm property.** There is an interesting and practical connection between Theorems 3.16 and 4.2. The connection is due to Yood's Criterion, which states that a bounded linear map of Banach spaces is proper on the closed and bounded subsets of the domain if and only if the kernel of the map is finite dimensional and the range of the map is closed. This leads to a test for the Fredholm property and index. To use this test, one can replace A by any element of $\omega(A)$, which may be easier to work with. We have the following theorem.

Theorem 4.10. *Assume that A is a bounded, uniformly continuous $N \times N$ matrix function on $[0,\infty)$. Assume that for all $B \in \omega(A)$, there are no bounded, nontrivial solutions to $\dot{u} + B(t)u = 0$ on $(-\infty,\infty)$. Then the dimension of the kernel of $D_B\colon C_0^1 \to C_0$ is independent of the choice of $B \in \omega(A)$. Moreover, Λ is Fredholm of index* $\dim\ker D_B - \dim X_1$.

Proof. By assumption, the set S in Definition 3.12 is $S = \{0\}$, which is certainly compact and totally disconnected. Therefore, A has an admissible omega-limit set, and it follows from Theorem 3.16 that Λ is proper on the subsets of C_0^1 that are closed and bounded. Thus, Yood's criterion guarantees that Λ has closed range in $C_0 \times X_1$. It follows from Theorem 4.2 (via Corollary 4.3) that D_A is a surjective Fredholm operator of index $k = \dim\ker D_A$ and that A admits an exponential dichotomy on $[0,\infty)$ with associated projection of rank $k = \dim\ker D_A$. By Theorem 4.4, the map Λ is Fredholm of dimension $\dim\ker D_A - \dim X_1$.

To complete the proof, we appeal to Remark 4 in Sacker [18], in which Sacker explains that under the current hypotheses, $\dim\ker D_B = \dim\ker D_A$ for all $B \in \omega(A)$. $\square$

Remark 4.11. In Theorem 4.10, it is assumed that there are no bounded, *nontrivial* solutions to $\dot{u}+B(t)u = 0$, while Definition 3.12 prohibits bounded *nonconstant* solutions. This is because Theorem 4.10 concerns a linear system, so that the set S in Definition 3.12 is automatically a vector space. Thus, for S to be compact, it is necessary to require that $S = \{0\}$.

4.3. **Nonlinear systems.** Recall that $\mathbb{R}^N = X_1 \oplus X_2$ is a given decomposition of $\mathbb{R}^N$ with associated projection P onto X_1 along X_2. If F satisfies (2.1), (2.2), and (2.3) recall once again that we define an operator $\mathcal{F}\colon C_0^1 \to C_0 \times X_1$ by

$$\mathcal{F}(u) := (\dot{u} + N_F(u), Pu(0)),$$

and that it follows from Theorem 2.2 that $\mathcal{F} \in C^1\big(C_0^1, C_0 \times X_1\big)$ with

$$D\mathcal{F}(u)v = (\dot{v} + D_zF(\cdot,u)v, Pv(0)) \quad \text{for } u, v \in C_0^1.$$

Lemma 4.12. *Assume that F satisfies* (2.1), (2.2), *and* (2.3). *Then for each $u \in C_0^1$, the bounded linear operator $D\mathcal{F}(u) - D\mathcal{F}(0)$ is compact.*

Proof. Let $\{v_n\}$ be a bounded sequence in C_0^1. To show that $\big(D\mathcal{F}(u) - D\mathcal{F}(0)\big)v_n = \big(\big(D_zF(\cdot,u) - D_zF(\cdot,0)\big)v_n, 0\big)$ has a convergent subsequence in $C_0 \times X_1$ is to show that

$$\{w_n\} := \{\big(D_zF(\cdot,u) - D_zF(\cdot,0)\big)v_n\}$$

has a convergent subsequence in C_0. Firstly, note that for each $N \in \mathbb{N}$, the (restriction of) the sequence $\{v_n\}$ is bounded and uniformly continuous on $[0,N]$. Thus, by the Ascoli-Arzela Theorem and a diagonal sequence argument, there is a subsequence of $\{v_n\}$ (again denoted $\{v_n\}$) and a function $v \in C_{\rm b}$ such that $v_n \to v$ uniformly on compact intervals.

We will now show that $\{w_n\}$ is a Cauchy sequence in C_0, and is hence convergent there. Let $\epsilon > 0$. The sequence $\{v_n\}$ is uniformly bounded on $[0,\infty) \times \mathbb{R}^N$. According to Lemma 2.1 (b), $\lim_{t\to\infty}\big(D_zF(t,u(t)) - D_zF(t,0)\big) = 0$. These two facts imply that there is $T > 0$ such that $|w_n(t)| < \epsilon/4$ for all $n \in \mathbb{N}$ and all $t > T$. Thus,

$$|w_n(t) - w_m(t)| < \epsilon/2, \quad \text{for all } n, m \in \mathbb{N} \text{ and } t > T. \tag{4.5}$$

On the other hand, $\{v_n\}$ is uniformly convergent (hence uniformly Cauchy) on $[0,T]$. Since $\big(D_zF(t,u(t)) - D_zF(t,0)\big)$ is bounded on $[0,T]$, the sequence $\{w_n\}$ is also uniformly Cauchy on $[0,T]$. With (4.5), this implies that for some $N \in \mathbb{N}$,

$$|w_n(t) - w_m(t)| < \epsilon, \quad \text{for all } n, m > N \text{ and } t \geq 0.$$

That is to say, the sequence $\{w_n\}$ is Cauchy in C_0, as advertised. □

Theorem 4.13. *Let F satisfy* (2.1), (2.2), *and* (2.3). *Then $\mathcal{F}\colon C_0^1 \to C_0 \times X_1$ is a Fredholm operator if and only if $D_zF(\cdot,0)$ admits an exponential dichotomy. In this case,* $\operatorname{ind}\mathcal{F} = 0$ *if and only if* $\dim X_1$ *equals the common rank of the projections associated to this exponential dichotomy.*

Proof. Recall that by definition, the nonlinear operator $\mathcal{F}$ is Fredholm if and only if $D\mathcal{F}(u)$ is Fredholm for some u. In this case, $D\mathcal{F}(u)$ is Fredholm for all u, and the index is independent of u, and the Fredholm index of the nonlinear operator $\mathcal{F}$ is defined to be this common value of $\operatorname{ind} D\mathcal{F}(u)$.

Lemma 4.12 shows that the Fredholm property and index (which do not vary under compact perturbations) of $D\mathcal{F}(u)$ agrees with that of $D\mathcal{F}(0)$. An application of Corollary 4.5 with $A(t) = D_zF(t,0)$ completes the proof. □

5. Existence theorems

In the first result the hypotheses are relatively abstract. This is where the topological degree argument is provided. Subsequent results will have more concrete (though less general) hypotheses.

Lemma 5.1. *Assume that $\mathcal{F}$ is of class C^1 from C_0^1 to $C_0 \times X_1$. Assume moreover that $\mathcal{F}$ is Fredholm of index zero and is proper on the subsets of C_0^1 that are closed and bounded. Assume as well that for a given pair $(f,\xi) \in C_0 \times X_1$, that* a priori *bounds exist in the sense that the set*

$$\{u \in C_0^1 : \mathcal{F}(u) = (sf, s\xi) \text{ for some } 0 \leq s \leq 1\} \tag{5.1}$$

is norm bounded in C_0^1. Finally, assume that $\mathcal{F}$ is odd.

Then there exists $u \in C_0^1$ such that $\mathcal{F}(u) = (f,\xi)$.

Proof. Let $R > 0$ be a norm bound for the set defined in (5.1), and let B be the open ball of radius $R+1$ and center 0 in C_0^1. By assumption, $\mathcal{F}\colon C_0^1 \to C_0 \times X_1$ is a C^1 map of Fredholm index zero that is proper on the closure of B. All of this ensures that $\mathcal{F}$ is B-*admissible*, in the sense of [14, Definition 4.1].

Next, the choice of B ensures that $(sf, s\xi) \in C_0 \setminus \mathcal{F}(\partial B)$ for all $0 \leq s \leq 1$. As it is introduced in [14, Corollary 5.5], the absolute degree $|d|(\mathcal{F}, B, (sf, s\xi))$ is a well-defined nonnegative integer for all $0 \leq s \leq 1$. Introduce the homotopy $h\colon [0,1] \times C_0^1 \to C_0 \times X_1$ by

$$h(s,u) := \mathcal{F}(u) - (sf, s\xi).$$

To conclude that the absolute degree is invariant along h, we must verify first that h is B-*admissible*, in the sense of [14, Definition 4.2]. It is clear that h is C^1. To see that h is Fredholm of index one, note that $Dh(s,u)$ is a rank one perturbation of the linear map $L := (0, D\mathcal{F}(u))$ from $\mathbb{R} \times C_0^1$ into $C_0 \times X_1$. Thus, the Fredholm properties of $Dh(s,u)$ coincide with those of L. Now, L has the same range and target space as $D\mathcal{F}(u)$, but the kernel of L is $\mathbb{R} \times \ker D\mathcal{F}(u)$. Since $D\mathcal{F}(u)$ is assumed to be Fredholm of index zero, it follows that L is Fredholm of index one, as desired.

For the B-admissibility of h, we must also verify that the restriction of h to $[0,1] \times \overline{B}$ is proper. That is to say, the preimage of a compact set in $C_0 \times X_1$ should have compact intersection with $[0,1] \times \overline{B}$. Suppose that $\{(s_n, u_n)\}$ is a sequence in $[0,1] \times \overline{B}$ such that $\{h(s_n, u_n)\}$ converges in $C_0 \times X_1$, say to a point (g, η). Since $[0,1]$ is compact, we may assume that s_n converges to some s_0. By definition of h, we see that the sequence $\{\mathcal{F}(u_n)\}$ converges to $(g + s_0 f, \eta + s_0 \xi)$. It follows at once from the assumed properness of $\mathcal{F}$ on the subsets of C_0^1 that are closed and bounded that $\{u_n\}$ possesses a convergent subsequence. Having verified that h is B-admissible, it now follows from [14, Theorem 5.1] that $|d|(h(s,\cdot), B, (0,0))$ is independent of $0 \le s \le 1$.

By Borsuk's Theorem, the assumed oddness of $\mathcal{F}$ implies that this degree is nonzero when $s = 0$. By homotopy invariance it follows that $|d|(h(1,\cdot), B, (0,0))$ is nonzero. The normalization property of the degree implies the existence of some $u \in B$ such that $\mathcal{F}(u) - (f, \xi) = (0,0)$. This completes the proof. $\square$

There are various tools available to ensure that the degree is nonzero at the $s = 0$ point of the homotopy h that is used in the above proof. In Lemma 5.1, the assumption that $\mathcal{F}$ be odd can be dispensed with if the degree is known to be nonzero. For this, one sufficient pair of conditions is that there be no nonzero solution $u \in C_0^1$ to the homogeneous equation $\mathcal{F}(u) = (0,0)$, and the zero solution be regular in the sense that $D\mathcal{F}(0)$ is an isomorphism. Briefly, the condition is that the trivial solution to $\mathcal{F}(u) = (0,0)$ be both unique and regular.

In that case, it follows from the definition of the degree at regular values that $|d|(\mathcal{F}, B, (0,0)) = 1$. An additional relevance of this situation is that isomorphisms are automatically Fredholm of index zero. Since Lemma 4.12 ensures the compactness of $D\mathcal{F}(u) - D\mathcal{F}(0)$ for all $u \in C_0^1$, it follows from the invariance of the Fredholm property under compact perturbations that $\mathcal{F}$ is Fredholm of index zero. All of this results in the following variation of Lemma 5.1:

Corollary 5.2. *Assume that $\mathcal{F}$ is of class C^1 from C_0^1 to $C_0 \times X_1$. Assume also that $\mathcal{F}$ is proper on the subsets of C_0^1 that are closed and bounded. Assume as well that for a given pair $(f, \xi) \in C_0 \times X_1$, that* a priori *bounds exist in the sense that the set* (5.1) *is norm bounded in C_0^1. Finally, assume that there is no nonzero solution $u \in C_0^1$ to the homogeneous equation $\mathcal{F}(u) = (0,0)$, and that $D\mathcal{F}(0)$ is an isomorphism.*

Then there exists $u \in C_0^1$ such that $\mathcal{F}(u) = (f, \xi)$.

Proof. The proof of Lemma 5.1 needs to be modified only according to the above remarks concerning nonzero degree and the Fredholm property. In particular, recall that compact perturbations of linear maps of Fredholm zero are again Fredholm of index zero, and that $\mathcal{F}$ is Fredholm of index zero if $D\mathcal{F}(u)$ is Fredholm of index zero at each $u \in C_0^1$. $\square$

Remark 5.3. There is no harm done (but perhaps no practical gain made) in replacing the linear path from $(0,0)$ to (f,ξ) in (5.1) by any C^1 path from $(0,0)$ to (f,ξ).

Before stating the next theorem, it may be helpful to bring in the following definition:

Definition 5.4. Let $f \in C_0$ and $\xi \in X_1$ be given. Let S be the set of all $u \in C_0^1$ such that

$$\begin{gathered} \dot{u}(t) + F(t,u(t)) = sf(t) \quad \text{for all } t \geq 0, \\ Pu(0) = s\xi \end{gathered} \tag{5.2}$$

for some $0 \leq s \leq 1$. If S is norm-bounded in C_0^1, we say that the pair (f,ξ) *satisfies the a priori bounds condition* for F.

Theorem 5.5. *Assume that F satisfies* (2.1)*,* (2.2)*, and* (2.3) *and that F has an admissible omega-limit set. Assume that $(f,\xi) \in C_0 \times X_1$ satisfies the* a priori *bounds condition for F, and that the homogeneous system associated to* (1.1) *has both uniqueness and regularity of the trivial solution. Then there is at least one $u \in C_0^1$ to solve* (1.1).

Proof. We apply Corollary 5.2. It follows from Corollary 2.5 that $\mathcal{F}$ (as defined in (2.4)) is a C^1 map from C_0^1 to $C_0 \times X_1$. It follows from Theorem 3.16 that $\mathcal{F}$ is proper on the subsets of C_0^1 that are closed and bounded. According to Definition 5.4, the *a priori* bounds condition of Corollary 5.2 is satisfied.

Finally, to say that the homogeneous system associated to (1.1) has both uniqueness and regularity of the trivial solution means precisely that the remaining requirements of Corollary 5.2 are met as well. We conclude that there is indeed at least one $u \in C_0^1$ to solve(1.1). □

Remark 5.6. As per Remark 5.3, there is no harm in replacing the linear path from $(0,0)$ to (f,ξ) in (5.2) by any C^1 path from $(0,0)$ to (f,ξ).

Perhaps less useful in practice, the following version (which avoids the isomorphism condition) is worth recording:

Theorem 5.7. *Assume that F satisfies* (2.1)*,* (2.2)*, and* (2.3) *and that F has an admissible omega-limit set. Assume that $(f,\xi) \in C_0 \times X_1$ satisfies the* a priori *bounds condition for F. Finally, assume that $\mathcal{F}$ is Fredholm of index zero and that $F(t,\cdot)$ is odd for each $t \geq 0$. Then there is at least one $u \in C_0^1$ to solve*(1.1).

Proof. This result follows from Lemma 5.1 instead of Corollary 5.2. □

Remark 5.8. In applications of Theorem 5.7, one must verify that $\mathcal{F}$ is Fredholm of index zero. By Lemma 4.12 it is enough to check that $D\mathcal{F}(0)$ is Fredholm of index zero, for which in turn it is sufficient to know that $D\mathcal{F}(0)$ is an isomorphism. In this context, the assumed oddness replaces the uniqueness of the trivial solution to the homogeneous problem.

Alternately, one can use the results of Section 4 to verify that $\mathcal{F}$ is Fredholm of index zero. According to Theorem 4.13, it is necessary and sufficient that $D_zF(\cdot,0)$ admit an exponential dichotomy with associated projection of the same rank as P. There are a number of examples given in Section 4 of how to do this. Of particular interest in the context of this paper is that it is sufficient to set $A = D_zF(\cdot,0)$ and to check that for all $B \in \omega(A)$, that there are no bounded, nontrivial solutions to

$\dot{u} + B(t)u = 0$ on $(-\infty, \infty)$. In this case, according to Theorem 4.10 it remains only to check that for any $B \in \omega(A)$ (and hence all $B \in \omega(A)$) that the dimensions of $\ker D_A$ and of X_1 are equal.

In principle, this approach allows for applications of Theorem 5.7 even when the trivial solution to the homogeneous system associated to (1.1) is neither unique nor regular.

6. Example

This section provides a new example that builds upon the results in [16, Example 7.2, culminating in Theorem 7.3]. (For a variety of other examples, please see [10].)

Let I be any closed interval containing zero, possibly as an endpoint. Let $g = g(t,s)\colon I \times \mathbb{R} \to \mathbb{R}$ and $h = h(t,s)\colon I \times \mathbb{R} \to \mathbb{R}$ be two real-valued functions with the following properties. (Of course, this means that each of the following is true also upon replacing g by h, possibly with different constants.)

$$g(t,0) = 0, \tag{6.1}$$

$$D_s g \text{ exists and is continuous on } I \times \mathbb{R}, \tag{6.2}$$

$$g, \text{ and } D_s g \text{ are B.U.C. on } I \times K \text{ for each compact interval } K, \tag{6.3}$$

$$D_s g \text{ is non-negative}, \tag{6.4}$$

$$\inf_{t \in I} D_s g(t,0) > 0, \tag{6.5}$$

and finally, there are positive constants α and s^* such that for all $t \in I$,

$$g(t,s)/s \geq \alpha \text{ whenever } |s| > s^*. \tag{6.6}$$

Of course, the significance of conditions (6.1)-(6.3) are that g and h satisfy (2.1)-(2.3) (with $N = 1$, with each of g and h in place of F, and with I in place of $[0, \infty)$). Also, if condition (6.6) holds for all $\alpha > 0$ (so that $s^* = s^*(\alpha)$), then g and h are super-linear.

The following are a few simple properties that will be used later.

Lemma 6.1. *Continue to assume that $g\colon I \times \mathbb{R} \to \mathbb{R}$ satisfies conditions (6.1)-(6.6). Then*

(a) *$g(t,s)$ has the same sign as s.*
(b) *$sg(t,s)$ and $g(t,s)/s$ are both positive when $s \neq 0$.*
(c) *If $s > 0$, then $\inf_{t \in I} g(t,s) > 0$.*
(d) *If $s < 0$, then $\sup_{t \in I} g(t,s) < 0$.*
(e) *The function $g^*(s) := \inf_{t \in I} g(t,s)$ is monotone increasing.*
(f) *$\lim_{|s| \to \infty} |g(t,s)| = \infty$ uniformly in $t \in I$.*

Proof. (a) Since $g(t,0) = 0$, this follows from the non-negativity of $D_s g$, along with the strict positivity of $D_s g(t,0)$.

(b) This follows from part (a).
(c) This follows from (6.5).
(d) This also follows from (6.5).
(e) Let $s_2 > s_1$, and choose $\epsilon > 0$. Let t_1 and t_2 be chosen so that both $|g^*(s_1) - g(t_1, s_1)|$ and $|g^*(s_2) - g(t_2, s_2)|$ are smaller than ϵ. Then

$$\begin{aligned} g^*(s_2) - g^*(s_1) &\geq g(t_2, s_2) - g(t_1, s_1) - 2\epsilon \\ &= \big(g(t_2, s_2) - g(t_2, s_1)\big) + \big(g(t_2, s_1) - g(t_1, s_1)\big) - 2\epsilon \end{aligned}$$

$$\geq 0 + g^*(s_1) - g(t_1, s_1) - 2\epsilon$$
$$\geq -3\epsilon,$$

where we have used the fact that for fixed t_2, the function $g(t_2, s)$ is monotone increasing in s. Since $\epsilon > 0$ was arbitrary, this shows that $g^*(s_2) - g^*(s_1) \geq 0$, as desired.

(f) This follows from (6.6).

□

Before stating and proving the main existence result, it helps to prove two lemmas that concern blowup and *a priori* bounds.

Lemma 6.2. *Let $g = g(t, s)$ and $h = h(t, s)$ be two real valued functions on $I \times \mathbb{R}$ that satisfy conditions* (6.1)-(6.5). *Let $u = (v, w)$ be any nontrivial C^1 solution to the homogeneous problem*

$$\begin{aligned} \dot{v} + g(t, w) &= 0, \\ \dot{w} + h(t, v) &= 0. \end{aligned} \tag{6.7}$$

We take u to be extended from $t = 0$ as far as possible as a solution, perhaps to all of I.

(a) *If $v(0)w(0) \leq 0$ and $[0, \infty) \subseteq I$, then u blows up as $t \to \infty$ (possibly in finite time).*

(b) *If $v(0)w(0) \geq 0$, and $(-\infty, 0] \subseteq I$, then u blows up as $t \to -\infty$ (possibly in finite time).*

Proof. At most one of $v(0)$ and $w(0)$ is zero, and it is no loss of generality to assume that $w(0) \neq 0$. Otherwise, we can exchange the names of g and h. This exchange results in a corresponding exchange in the names of v and w.

We first consider (a). We examine the case that $w(0) > 0$ and $v(0) \leq 0$; the remaining case has a similar proof. With reference to Lemma 6.1 part (a), we have

$$\dot{v}(0) = -g(0, w(0)) < 0,$$

and

$$\dot{w}(0) = -h(0, v(0)) \geq 0.$$

Thus v is decreasing and w is *not* decreasing at $t = 0$. Let J be the set of all $t \geq 0$ such that $w > 0$ on $[0, t)$. Note that in J, we have $\dot{v}(t) = -g(t, w) < 0$ so that v is decreasing on J. In particular, v is non-positive on J. Since $\dot{w}(t) = -h(t, v)$, it follows that w is non-decreasing on J. Unless the solution blows up in finite time (in which case there is nothing to prove), this shows that $[0, \infty) \subseteq J$. Therefore, $w \geq w(0)$ on $[0, \infty)$, from which it follows that $\dot{v}(t) = -g(t, w(t)) \leq -g(t, w(0)) \leq -\inf_{t \geq 0} g(t, w(0)) < 0$. (See Lemma 6.1, part (c).) Since $\dot{v}$ is negative and bounded away from zero, this proves that $v(t) \to -\infty$ as $t \to \infty$.

We now consider (b). It is once again without loss of generality that $w(0) \neq 0$. We consider the case that $w(0) > 0$ and $v(0) \geq 0$; once again, the omitted case is very similar. We let J be the set of all $t \leq 0$ such that w is positive on $(t, 0]$. By an analysis similar to that of part (a), we find that $(-\infty, 0] \subseteq J$ and $\dot{v}(t)$ is therefore positive and bounded away from zero on $(-\infty, 0]$. This proves that $v(t) \to \infty$ as $t \to -\infty$. □

The next lemma says that if $u = (v, w)$ is a C_0^1 solution, then neither v nor w can become too large relative to the other.

Lemma 6.3. *Let $g = g(t,s)$ and $h = h(t,s)$ be two real valued functions on $[0,\infty)\times\mathbb{R}$, and assume that g and h satisfy conditions* (6.1)-(6.6)*. Let $f_1, f_2 \in C_0$, and let $K \geq 0$ be given. There exists $R = R(K, f_1, f_2) \geq 0$ with the following property. Let $u = (v,w)$ be any C_0^1 solution to the nonhomogeneous problem*

$$\begin{aligned} \dot v + g(t,w) &= \sigma f_1, \\ \dot w + h(t,v) &= \sigma f_2, \end{aligned} \tag{6.8}$$

for some $0 \leq \sigma \leq 1$. For all $t_0 \geq 0$, if one of $|v(t_0)|$ or $|w(t_0)|$ is no larger than K, then the other is no larger than R.

Proof. Fix $0 \leq \sigma \leq 1$ and $t_0 \geq 0$. Let $u = (v,w)$ be any C_0^1 solution to (6.8). By the symmetry of (6.8), it suffices to assume that $|v(t_0)| \leq K$ and prove that $|w(t_0)| \leq R$. To be clear, we need to find $R = R(K, f_1, f_2)$ such that $|w(t_0)| \leq R$. It is important to ensure that the choice of R does not depend on the choice of solution u, parameter σ, nor time t_0.

We first consider the case that $v(t_0)$ and $w(t_0)$ are both nonnegative. We will show that if $w(t_0)$ is too large relative to K, f_1, and f_2, then $u \notin C_0^1$. It follows from Lemma 6.1, part (f) that there is s_1 such that $\inf_{t\geq 0} g(t,s) \geq 1$ for all $s > s_1$. Suppose that $w(t_0) > s_1$. (Otherwise, just take $R > s_1$.)

Define $h^*(s) := \inf_{t\geq 0} h(t,s)$. Let J be the set of all $t \geq t_0$ such that both $h^*(v(\cdot)) > -\|f_2\| - 1$ and $w > s_1$ on the interval $[t_0, t]$. Because $t_0 \in J$, it follows that J is an interval $[t_0, a)$ for some $t_0 < a \leq \infty$. Notice that as t increases, t remains in J only so long as v and w are large enough. We will find an upper bound for a by showing that v is ultimately decreasing. However, this is sure to occur only after f_1 has become negligible.

To this end, let $t_1 \geq t_0$ satisfy $|f_1(t)| \leq 1/2$ for all $t > t_1$. Then for all $t \in J$,

$$\dot v(t) = \sigma f_1(t) - g(t, w(t)) \leq \sigma f_1(t) - 1 \leq \begin{cases} \|f_1\|, & \text{if } t \leq t_1; \\ -1/2, & \text{if } t \geq t_1. \end{cases}$$

Therefore, for all $t \in J$,

$$v(t) \leq \begin{cases} K + (t - t_0)\|f_1\|, & \text{if } t \leq t_1; \\ K + (t_1 - t_0)\|f_1\| - (1/2)(t - t_1), & \text{if } t > t_1. \end{cases}$$

Since $h(t, v(t)) \geq h^*(v(t)) > -\|f_2\| - 1$ in J, it follows that J contains only values of $t > t_1$ (if any) such that

$$h\big(t, K + t_1\|f_1\| - (1/2)(t - t_1)\big) > -\|f_2\| - 1.$$

With reference to items (a) and (f) of Lemma 6.1 (with h in place of g), this implicitly bounds the right endpoint a of J, in a way that depends only on K, f_1, and f_2.

Now we can estimate $w(t)$ when $t \in J$. Notice that

$$\sup_{t\in J} h(t,v) \leq M := \sup_{t\in J} h\big(t, K + t_1\|f_1\|\big) < \infty.$$

Note that M depends only on K, f_1, and f_2. When $t \in J$,

$$
\begin{aligned}
w(t) &= w(t_0) + \int_{t_0}^{t} \dot{w}(\tau)\,\mathrm{d}\tau \\
&= w(t_0) + \int_{t_0}^{t} \sigma f_2(\tau) - h(\tau, v(\tau))\,\mathrm{d}\tau \\
&\geq w(t_0) - (a - t_0)(\|f_2\| + M).
\end{aligned} \tag{6.9}
$$

Now, what would happen if $w(t_0)$ were too large? From inequality (6.9) it follows that if

$$
w(t_0) \geq s_1 + (a - t_0)(\|f_2\| + M) + 1,
$$

then $w(t) \geq s_1 + 1$ for all $t \in J$. By definition of J, it follows that $h^*(v(a)) = -\|f_2\| - 1$. (Recall that $a < \infty$). Now let's look at the solution (v, w) as t increases beyond a. Let J' be the set of all $t \geq a$ such that both $w > s_1$ and $h^*(v(\cdot)) < -\|f_2\|$ on $[a, t)$. Note that theses inequalities hold at $t = a$, so that J' is nonempty. Let $a' = \sup J'$. Notice that for all $t \in J'$,

$$
\dot{v}(t) = \sigma f_1(t) - g(t, w) \leq -1/2.
$$

Therefore, it is necessary that $a' < \infty$, lest v be unbounded and the solution $u = (v, w)$ fails to be in C_0^1. However, the definition of J' allows for only two possibilities concerning a'. The first is that $w(a') = s_1$. This is impossible, because $w(a) > s_1$ and $\dot{w} = \sigma f_2 - h(t, v) > 0$ on J'. The only remaining possibility is that $h^*(v(a')) = -\|f_2\|$. However, v is decreasing in J', so item (e) implies that $h^*(v(\cdot))$ is non-increasing in J'. Since $h^*(v(a)) = -\|f_2\| - 1$, this is a contradiction. We conclude that for $R = s_1 + (a - t_0)(\|f_2\| + M) + 1$, if $w(t_0) > R$ then $u = (v, w) \notin C_0^1$. This completes the proof in case $v(t_0)$ and $w(t_0)$ are both non-negative.

Still assuming that $v(t_0) \geq 0$, we next consider those solutions such that $w(t_0) < 0$. We now let $s_1 < 0$ be such that $-\|f_1\| - g(t, s) > 2$ for all $t \geq t_0$ and all $s < s_1$. Suppose that $w(t_0) < s_1$. This time we take J to be the set of all $t \geq t_0$ such that $-\|f_1\| - g(\cdot, w(\cdot)) > 1$ on $[t_0, t]$. For all $t \in J$,

$$
\dot{v}(t) = \sigma f_1 - g(t, w(t)) > 1,
$$

so that $v(t) > v(t_0) + (t - t_0) \geq t - t_0$ and so

$$
h(t, v(t)) > h(t, t - t_0).
$$

According to item (f) of Lemma 6.1 there is $t_1 \geq t_0$ such that $h(t, t - t_0) \geq \|f_2\| + 1$ for all $t \geq t_1$. Thus, for all $t \in J$,

$$
\begin{aligned}
w(t) &= w(t_0) + \int_{t_0}^{t} \sigma f_2(\tau) - h(\tau, v(\tau))\,\mathrm{d}\tau \\
&\leq \begin{cases} w(t_0) + (t - t_0)\|f_2\|, & \text{if } t \leq t_1; \\ w(t_0) + (t_1 - t_0)\|f_2\| - (t - t_1), & \text{if } t \geq t_1 \end{cases} \\
&\leq w(t_0) + (t_1 - t_0)\|f_2\|.
\end{aligned}
$$

This shows that if $w(t_0) \leq s_1 - (t_1 - t_0)\|f_2\|$, then $w(t) \leq s_1$ for all $t \in J$. It follows that $-\|f_1\| - g(t, w(t)) > 2$ for all $t \in J$, so that $J = [t_0, \infty)$. Since $\dot{v} > 1$ on J, the solution $u = (v, w)$ is unbounded and is not in C_0^1. This proves that no C_0^1 solution $u = (v, w)$ satisfies $w(t_0) < s_1 - (t_1 - t_0)\|f_2\|$. The proof is complete in the case that $v(t_0) \geq 0$.

The argument when $v(t_0) \le 0$ is similar, in principle. However, it is probably more efficient to use the following reflection argument. Let $p(t,s) = -g(t,-s)$, and $q(t,s) = -h(t,-s)$. Note that $D_s p(t,s) = D_s g(t,-s)$ and $D_s q(t,s) = D_s h(t,-s)$, so that p and q are seen to satisfy conditions (6.1)-(6.6). Also, the following are equivalent:

- The pair $(v,w) = (x,y)$ solves
$$\begin{aligned} \dot v + g(t,w) &= \sigma f_1, \\ \dot w + h(t,v) &= \sigma f_2, \end{aligned} \tag{6.10}$$
- The pair $(v,w) = (-x,-y)$ solves
$$\begin{aligned} \dot v + p(t,w) &= -\sigma f_1, \\ \dot w + q(t,v) &= -\sigma f_2, \end{aligned} \tag{6.11}$$

Therefore, if $v(t_0) \le 0$, we can apply the case that has already been proved to the reflected problem to obtain the desired bound. $\square$

With the help of the preceding technical lemmas, it remains only to see how Theorem 5.5 can be applied.

Theorem 6.4. *Let g and h be real-valued functions on $[0,\infty) \times \mathbb{R}$ that satisfy conditions* (6.1)-(6.6). *Let $\xi \in \mathbb{R}$, and let $f_1, f_2 \in C_0$ be such that $\|f_1\| + \|f_2\| < \alpha$, where α is the bound that appears in condition* (6.6). *Then the system*
$$\begin{gathered} \dot v + g(t,w) = f_1, \\ \dot w + h(t,v) = f_2, \\ v(0) = \xi \end{gathered} \tag{6.12}$$
has at least one solution $(v,w) \in C_0^1$.

Proof. To apply Theorem 5.5, we set $u = \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$, $F(t,z) = F(t,u) = \begin{bmatrix} g(t,w) \\ h(t,v) \end{bmatrix}$, $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, and $P\left[\begin{smallmatrix} s_1 \\ s_2 \end{smallmatrix}\right] = s_1$. Using the variable $z = \left[\begin{smallmatrix} s_1 \\ s_2 \end{smallmatrix}\right]$, it follows that
$$D_z F(t,z) = \begin{bmatrix} 0 & D_s g(t,s_2) \\ D_s h(t,s_1) & 0 \end{bmatrix}.$$
It follows immediately from the conditions placed upon g and h that F satisfies (2.1), (2.2), and (2.3).

Next, we must show that F has an admissible omega-limit set. Let E be any member of $\omega(F)$. It follows that $E = \begin{bmatrix} \tilde g \\ \tilde h \end{bmatrix}$ for some $\tilde g \in \omega(g)$ and some $\tilde h \in \omega(h)$, and that these are functions that satisfy conditions (6.1)-(6.6) with $I = \mathbb{R}$. (Some of these conditions are verified via Lemma 3.6; the rest follow easily using the uniform convergence on compact sets.) Assume that $u \in C_{\rm b}^1(\mathbb{R})$ is a (bounded) solution to $\dot u + E(t,u) = 0$. For the admissibility of the omega-limit set of F, we are to show that u is constant, and that the set of all such constant functions (over all choices of $E \in \omega(F)$) forms a compact, totally disconnected subset of $\mathbb{R}^2$. Note that $\dot u + E(t,u) = 0$ means that
$$\begin{aligned} \dot v + \tilde g(t,w) &= 0, \\ \dot w + \tilde h(t,v) &= 0. \end{aligned}$$

Since $\tilde{g}$ and $\tilde{h}$ satisfy conditions (6.1)-(6.6), the two parts of Lemma 6.2 together imply that the bounded (v, w) must be the trivial solution. This proves that u must be zero, which is indeed constant. Moreover, the set of all possible constant solutions is again just $\{0\}$, which is compact and totally disconnected. This verifies the admissibility of the omega-limit set of F.

In our application of Theorem 5.5, we next verify that the homogeneous system

$$\begin{gathered}\dot{v} + g(t, w) = 0,\\ \dot{w} + h(t, v) = 0\\ v(0) = 0\end{gathered}$$

has both uniqueness and regularity of the trivial solution in C_0^1. Uniqueness follows from the first part of Lemma 6.2. For regularity, we must show that the linearized system

$$\begin{gathered}\dot{v} + D_s g(t, 0)w = 0,\\ \dot{w} + D_s h(t, 0)v = 0\\ v(0) = 0\end{gathered}$$

has only the trivial solution in C_0^1. Notice that the maps $s \mapsto D_s g(t, 0)s$ and $s \mapsto D_s h(t, 0)s$ satisfy conditions (6.1)-(6.6). In particular, the fact that g and h satisfy condition (6.5) is what ensures that $D_s g$ and $D_s h$ satisfy both (6.5) and (6.6). Therefore, the first part of Lemma 6.2 implies that the linearized system has no nontrivial solutions in C_0^1.

Finally, we must verify that (f, ξ) satisfies the *a priori* bounds condition; see Definition 5.4. Thus, let S be the set of all pairs $(v, w) \in C_0^1$ such that

$$\begin{gathered}\dot{v} + g(t, w) = \sigma f_1,\\ \dot{w} + h(t, v) = \sigma f_2,\\ v(0) = \sigma\xi\end{gathered}$$

for some $0 \le \sigma \le 1$. We are to show that S is norm bounded in C_0^1.

Let $(v, w) \in S$, and let $r(t) = v(t)^2 + w(t)^2$. Since $r \in C_0^1$, this function achieves its maximum at some $t_0 \ge 0$ at which $u = (v, w)$ also achieves its maximum Euclidean norm, which we will use to establish the *a priori* bounds. If $t_0 = 0$, then Lemma 6.3 provides an *a priori* bound $R(K, f_1, f_2)$ for $w(0)$, where $K = \xi$. In this case,

$$\sup_{t\ge 0}|u(t)| = |u(0)| = \sqrt{v(0)^2 + w(0)^2} \le \sqrt{K^2 + R^2}. \tag{6.13}$$

Before using inequality (6.13), we derive an analogous inequality in case $t_0 > 0$. If at least one of $v(t_0)$ and $w(t_0)$ is no greater than $K = 1$ in absolute value, we use Lemma 6.3 to bound the other by $R = R(1, f_1, f_2)$. It follows that

$$\sup_{t\ge 0}|u(t)| = |u(t_0)| = \sqrt{v(t_0)^2 + w(t_0)^2} \le \sqrt{1 + R^2}. \tag{6.14}$$

It remains to consider the case that both $|v(t_0)|$ and $|w(t_0)|$ are greater than 1, while $t_0 > 0$. Since r attains its maximum at the interior point $t_0 > 0$, it follows that $\dot{r}(t_0) = 0$. Thus,

$$v(t_0)\dot{v}(t_0) + w(t_0)\dot{w}(t_0) = 0,$$

from which it follows that

$$v(t_0)\big(\sigma f_1(t_0) - g(t_0, w(t_0))\big) + w(t_0)\big(\sigma f_2(t_0) - h(t_0, v(t_0))\big) = 0.$$

Division by $v(t_0)w(t_0) \neq 0$ results in

$$\frac{g(t_0, w(t_0))}{w(t_0)} + \frac{h(t_0, v(t_0))}{v(t_0)} = \frac{\sigma f_1(t_0)}{w(t_0)} + \frac{\sigma f_2(t_0)}{v(t_0)} \tag{6.15}$$

Because $|v(t_0)|$ and $|w(t_0)|$ are greater than 1, the expression on the right side of equation (6.15) is bounded by $\|f_1\| + \|f_2\|$ in absolute value. On the other hand, both terms of the left side of (6.15) are positive; this follows from item (a) of Lemma 6.1. Thus, each of the two terms is independently bounded by $\|f_1\| + \|f_2\|$, which is assumed to be less than the bound α from (6.6); this implies that both $|v(t_0)|$ and $|w(t_0)|$ are no greater than the value s^* from (6.6). Together with inequalities (6.13) and (6.14) from the other two cases, we have verified that there is a constant $R = R(\xi, f_1, f_2)$ such that

$$\sup_{t\geq 0}|u(t)| \leq R$$

for all $u \in S$. This proves that S is norm bounded in C_0. To achieve a bound in the norm of C_0^1, it remains only to use the equations

$$\begin{aligned} \dot{v} &= \sigma f_1 - g(t, w), \\ \dot{w} &= \sigma f_2 - h(t, v). \end{aligned}$$

Having bounded v and w, the right sides of these equations serve to bound $\dot{v}$ and $\dot{w}$. This completes the verification that F has an admissible omega-limit set, which in turn completes the application of Theorem 5.5. We conclude that (6.12) has at least one solution in C_0^1, as advertised. □

As a simple corollary, if g and h are strictly super-linear, then we need not assume that f_1 and f_2 are small. Specifically, we have the following corollary.

Corollary 6.5. *Let g and h be real-valued functions on $[0,\infty) \times \mathbb{R}$ that satisfy conditions* (6.1)-(6.6)*. However, assume moreover that g and h are super linear, in the sense that* (6.6) *holds for all $\alpha > 0$ (meaning that $s^* = s^*(\alpha)$).*

Let $\xi \in \mathbb{R}$, and let $f_1, f_2 \in C_0$ be arbitrary. Then the system

$$\begin{gathered} \dot{v} + g(t, w) = f_1, \\ \dot{w} + h(t, v) = f_2, \\ v(0) = \xi \end{gathered} \tag{6.16}$$

has at least one solution $(v, w) \in C_0^1$.

Proof. Apply Theorem 6.4, with $\alpha = \|f_1\| + \|f_2\| + 1$. □

Remark 6.6. Some of the results that appear in this paper are elaborations of arguments that first appeared in the author's Ph. D. dissertation, completed under the guidance of Professor Patrick J. Rabier at The University of Pittsburgh.

References

[1] R. P. Agarwal and D. O'Regan, *Infinite interval problems for differential, difference and integral equations*, Kluwer Academic Publishers, Dordrecht, 2001.

[2] J. Andres, G. Gabor, and L. Górniewicz, *Boundary value problems on infinite intervals*, Trans. Amer. Math. Soc.

[3] W. A. Coppel, *Dichotomies in stability theory*, Lecture Notes in Mathematics, Vol. 629, Springer-Verlag, Berlin, 1978.

[4] G. Evéquoz, *Erratum: Global continuation for first order systems over the half-line involving parameters [MR2547710]*, Math. Nachr.

[5] G. Evéquoz, *Global continuation for first order systems over the half-line involving parameters*, Math. Nachr.

[6] G. Evéquoz, *Parameter-dependent systems on the half-line and free convection boundary-layers in porous media*, NoDEA Nonlinear Differential Equations Appl.

[7] P. M. Fitzpatrick, J. Pejsachowicz, and P.J. Rabier, *The degree of proper C^2 Fredholm mappings. I*, J. Reine Angew. Math. **427** (1992), 1–33.

[8] P. Hsieh and Y. Sibuya, *Basic theory of ordinary differential equations*, Universitext, Springer-Verlag, New York, 1999.

[9] J. L. Massera and J. J. Schäffer, *Linear differential equations and function spaces*, Pure and Applied Mathematics, Vol. 21, Academic Press, New York, 1966.

[10] J. R. Morris, *Nonlinear ordinary and partial differential equations on unbounded domains*, Ph. D. thesis, University of Pittsburgh, 2005, available at http://etd.library.pitt.edu /ETD/available/etd-03022005-112519/.

[11] J. R. Morris and P. J. Rabier, *Riccati inequality and functional properties of differential operators on the half line*, J. Differential Equations.

[12] K. J. Palmer, *Exponential dichotomies and Fredholm operators*, Proc. Amer. Math. Soc.

[13] K. J. Palmer, *Exponential dichotomies and transversal homoclinic points*, J. Differential Equations.

[14] J. Pejsachowicz and P. J. Rabier, *Degree theory for C^1 Fredholm mappings of index* 0, J. Anal. Math. **76** (1998), 289–319.

[15] P. J. Rabier, *Ascoli's theorem for functions vanishing at infinity and selected applications*, J. Math. Anal. Appl. **290** (2004), no. 1, 171–189.

[16] P. J. Rabier and C. A. Stuart, *Boundary value problems for first order systems on the half-line*, Topol. Methods Nonlinear Anal. **25** (2005), no. 1, 101–133.

[17] P. J. Rabier and C. A. Stuart, *A Sobolev space approach to boundary value problems on the half-line*, Commun. Contemp. Math. **7** (2005), no. 1, 1–36.

[18] R. J. Sacker, *The splitting index for linear differential systems*, J. Differential Equations.

[19] R. J. Sacker and G. R. Sell, *A spectral theory for linear differential systems*, J. Differential Equations **27** (1978), no. 3, 320–358.

Jason R. Morris

Department of Mathematics, The College at Brockport, State University of New York, 350 New Campus Drive, Brockport, NY 14420, USA

E-mail address: jrmorris@brockport.edu