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LONGITUDINAL LIBRATIONS OF A SATELLITE

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Abstract. Furi, Martelli and Landsberg gave a theoretical explanation of the chaotic longitudinal librations of Hyperion, a satellite of Saturn. The analysis was made under the simplifying assumption that the spin axis remains perpendicular to the orbit plane. Here, under the same assumption, we investigate the behavior of the longitudinal librations of any satellite. Also we show that they are possibly chaotic depending on two parameters: a constant k related to the principal moments of inertia of the satellite, and the eccentricity e of its orbit. We prove that the plane k-e contains an open region Ω with the property that the longitudinal librations of any satellite are possibly chaotic if the point (k, e) belongs to this region. Since Hyperion's point is inside Ω, the results of this paper are more general than those obtained previously.

1. Introduction

The differential equation that governs the motion of a satellite, when its spin axis remains perpendicular to the orbit plane, takes the form

$$\ddot{x} = \Big(\frac{a}{r(t)}\Big)^3\Big(\frac{16c^2 e\sin\theta(t)}{a^4(1-e^2)} - 3\frac{B-A}{C}\sin x\Big), \tag{1.1}$$

which, as shown in [7], is equivalent to the one in [8, 16]. The variable x is equal to twice the angle φ between the longest axis of the satellite and the planet-satellite center line, and the symbol $\ddot{x}$ denotes the second derivative of x with respect to the time t. The constants a and e are, respectively, the *semimajor axis* and the *eccentricity* of the elliptical orbit described by the center of mass of the satellite. The function $r(t)$ denotes the distance between the satellite and the planet. The constant c is the *areolar velocity* of the satellite, that is the instantaneous area swept by the segment joining the centers of mass of the planet and of the satellite. The function $\theta(t)$ indicates the *polar angle*. Measured counterclockwise and expressed in radians, it provides the angle between the major diameter of the elliptical orbit oriented towards the *periapsis* (i.e., the point when the satellite is closest to the planet) and the planet-satellite center line (see figure 1). The constants A, B, and C, with $0 < A \le B \le C$, are the principal *moments of inertia* of the satellite, with C being the moment about the spin axis.

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Equation (1.1) was studied in [7] in the case when all the constants involved are referred to Hyperion, one of the many satellites of the planet Saturn. Theoretical results and numerical estimates were used to explain why the longitudinal librations of this satellite are chaotic.

In this paper we investigate the behavior of the longitudinal librations of any satellite with motion governed by (1.1), and we show that these librations are possibly chaotic depending on two parameters: the ratio $k = (B - A)/C$ and the eccentricity e of the orbit of the satellite. Both constants are assumed to be positive and less than 1. By means of theoretical considerations we show that the plane k-e contains an open region Ω (see figure 2) with the property that the longitudinal librations of any satellite (with spin axis perpendicular to the orbit plane) are possibly chaotic provided that the corresponding pair of parameters (k, e) drops in this region. After having determined, numerically, the boundary of Ω, we realized that the pair (k, e) associated with Hyperion belongs to Ω, as expected. Thus, the results in this paper extend those in [7].

Some historical remarks are in order. Wisdom, Peale, and Mignard [16] investigated the irregular oscillations of Hyperion's longest axis with respect to the planet-satellite center line. The conclusions of the three scientists were derived from:

(1) Hyperion's images transmitted by Voyager 2 [14];
(2) a mostly numerical analysis of a differential equation modeling a planetary motion and proposed by P. Goldreich and S. Peale [8].

Martelli and VignoliThe modified the equation proposed by Danby [4] to study the longitudinal librations of the moon.

In an interesting paper Wisdom, Peale, and Mignard [16] provided numerical evidence that Hyperion's longitudinal librations are chaotic. Denoting by φ the angle between Hyperion's longest axis and the planet-satellite center line, they plotted the pairs $(\varphi, \dot\varphi)$ when Hyperion crosses the *periapsis*, obtaining a large cloud of points which dominates the 1/2 and 2 spin-orbit states.

We point out that, besides Wisdom, Peale, and Mignard [16], other authors (see, for example, [1, 2, 3, 5, 6, 9, 10, 11, 12, 13, 15, 17, 18]) have analyzed problems similar to the one investigated here.

2. The model

According to [8, 16] the motion of a tri-axial satellite describing an elliptical orbit around a planet, with spin axis perpendicular to the orbit plane, can be modelled by the second-order nonlinear differential equation

$$\ddot\varphi + \ddot\theta = -\frac{3(B - A)}{2C}\Big(\frac{a}{r(t)}\Big)^3 \sin(2\varphi), \tag{2.1}$$

where the quantities involved are as in Section 1. Thus, $\theta + \varphi$ is the angle between the satellite's longest axis and the longest diameter of the elliptical orbit (see figure 1).

Since the planet lies at one focus of the orbit, its distance r from the satellite is expressed by the following function of the polar angle θ:

$$r(\theta) = \frac{p}{1 + e\cos\theta}, \tag{2.2}$$

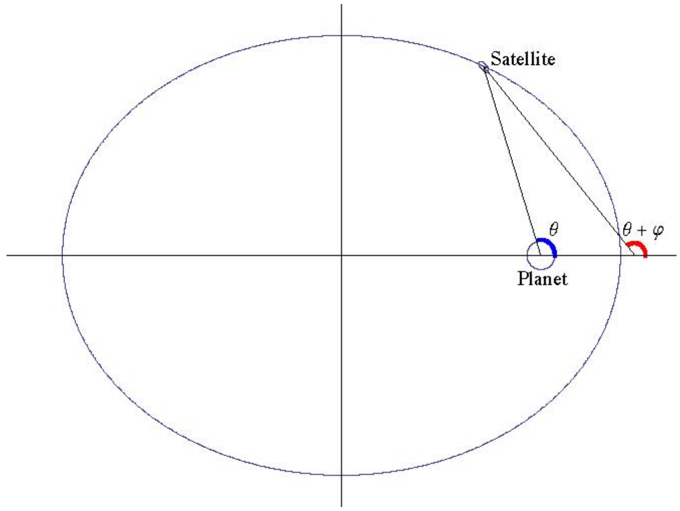

Figure 1. The planet-satellite system

where $p = a(1 - e^2)$ is the *parameter* of the elliptical orbit. The angle θ and the time t are related by Kepler's second law of planetary motion and by the initial condition $\theta(0) = 0$, obtained by selecting $t = 0$ when the planet is at the periapsis. Hence, $\theta(t)$ is the solution of the initial value problem

$$\begin{gathered} r^2(\theta)\dot{\theta} = 2c, \\ \theta(0) = 0. \end{gathered} \tag{2.3}$$

Differentiating (2.2) and (2.3), and recalling the equality $p = a(1 - e^2)$, we obtain

$$\ddot{\theta}(t) = -\frac{8c^2}{r^3(t)}\frac{e\sin\theta(t)}{a(1 - e^2)}.$$

Hence, (2.1) takes the form

$$\ddot{\varphi} = \big(\frac{a}{r(t)}\big)^3\big(\frac{8c^2 e\sin\theta(t)}{a^4(1 - e^2)} - \frac{3(B - A)}{2C}\sin 2\varphi\big).$$

Setting $x = 2\varphi$, we obtain the differential equation

$$\ddot{x} = \big(\frac{a}{r(t)}\big)^3\big(\frac{16c^2 e\sin\theta(t)}{a^4(1 - e^2)} - \frac{3(B - A)}{C}\sin x\big).$$

We set $a = 1$ and normalize the time so that the period of revolution of the satellite is 2π. Consequently, t and $\theta(t)$ coincide when $t = n\pi$, $n \in \mathbb{Z}$, and the difference $\eta(t) := t - \theta(t)$ is a 2π-periodic odd function. Moreover, the area swept during a full turn by the segment joining the satellite with the planet is $\pi ab = \pi\sqrt{1 - e^2}$. It follows that

$$c = \frac{1}{2}\sqrt{1 - e^2}.$$

Moreover, since $p = a(1 - e^2)$ and $r^2(\theta)\dot{\theta} = 2c$, with the above setting we obtain the differential equation

$$\ddot{x} = \big(\frac{1 + e\cos\theta(t)}{1 - e^2}\big)^3\big(4e\sin\theta(t) - 3k\sin x\big), \tag{2.4}$$

and the initial value problem (2.3), which defines the function $\theta(t)$, becomes

$$\begin{gathered}\dot\theta = \frac{(1+e\cos\theta)^2}{(1-e^2)^{3/2}},\\ \theta(0)=0.\end{gathered}\tag{2.5}$$

Notice that (2.4) can be regarded as governing the motion of a *point mass* constrained on a vertical circle and acted on by two periodic forces of the same period: an oscillating gravitational force, and a forcing term with zero mean acting, alternately, clockwise and counterclockwise. To emphasize this interpretation, we observe that (2.4) can be rewritten in the form

$$\ddot x + b(t)\sin x = a(t)\sin(t+\eta(t)),$$

where the functions $a(t)$ and $b(t)$ are strictly positive, even, and 2π-periodic. Moreover $\eta(t)$, which can be considered as the *phase* of a forcing term with oscillating *amplitude* $a(t)$, is odd and 2π-periodic.

3. Preliminary Results

In this section we begin our investigation of equation (2.4). We already mentioned that the dependent variable x can be regarded as the location (measured in radians) of a point mass on S^1. The lowest position of this point corresponds to $x=2k\pi$, $k\in\mathbb{Z}$, and will be called *South Pole*. The *North Pole* is identified with $x=(2k+1)\pi$, $k\in\mathbb{Z}$.

To make the statements and proofs of Sections 4 and 5 more concise and transparent we now present some terminology, lemmata, propositions, and appropriate remarks.

Remark 3.1. Regard (2.4) as the motion equation of a point x of unitary mass, constrained on S^1, and acted on by the force

$$f(t,x)=\Big(\frac{1+e\cos\theta(t)}{1-e^2}\Big)^3\big(4e\sin\theta(t)-3k\sin x\big).$$

Assume that $4e<3k$. Then $f(t,x)$ pushes towards the South Pole (for any t) whenever x lies in each one of the two opposite arcs

$$\begin{gathered}A_W=(-\pi+\arcsin(4e/3k),-\arcsin(4e/3k)),\\ A_E=(\arcsin(4e/3k),\pi-\arcsin(4e/3k)),\end{gathered}$$

centered at $-\pi/2$ and $\pi/2$ and defined by $-4e-3k\sin x>0$ and $4e-3k\sin x<0$, respectively.

For the rest of this article, we assume that the two positive parameters k and e satisfy the inequality $4e<3k$ ensuring the existence of the opposite arcs defined in Remark 3.1. We call A_W and A_E the *western arc* and *eastern arc*, with $A_W=(\alpha_W,\beta_W)$ and $A_E=(\beta_E,\alpha_E)$. Moreover, the arc containing the North Pole with extremes α_W and α_E will be called the *northern arc* and denoted by A_N, while the arc containing the South Pole with extremes β_W and β_E will be called the *southern arc* and denoted by A_S. Accordingly, the points *West, East, North Pole* and *South Pole* are the centers of the four arcs A_W,A_E,A_N and A_S.

Given $x_0,x_1\in\mathbb{R}$, $x_0\neq x_1$, we denote by $\overline{x_0x_1}$ the closed interval

$$[\min\{x_0,x_1\},\,\max\{x_0,x_1\}].$$

We need the following physically meaningful result (see e.g. [7]). For completeness' sake we include its elementary proof.

Lemma 3.2. *Let $f:\mathbb{R}^2\to\mathbb{R}$, $f_-:\mathbb{R}\to\mathbb{R}$ and $f_+:\mathbb{R}\to\mathbb{R}$ be continuous and such that*
$$f_-(x)\le f(t,x)\le f_+(x)$$
for all $(t,x)\in\mathbb{R}^2$. Let $x(t)$, $t\ge t_0$, be any solution of the initial value problem
$$\begin{aligned}\ddot x&=f(t,x),\\ x(t_0)&=x_0,\\ \dot x(t_0)&=v_0.\end{aligned}$$
Define the real functions g_- and g_+ by
$$g_-(x)=\frac{v_0^2}{2}+\int_{x_0}^x f_-(s)\,ds,\quad g_+(x)=\frac{v_0^2}{2}+\int_{x_0}^x f_+(s)\,ds.$$
Given $x_1\ne x_0$, denote by u_- and u_+ the numbers
$$u_-=\min\big\{g_-(x):x\in\overline{x_0x_1}\big\},\quad u_+=\min\big\{g_+(x):x\in\overline{x_0x_1}\big\}.$$
The following four assertions hold.

- *If $x_1>x_0$, $v_0>0$, and $u_->0$, then $\dot x(t)\ge\sqrt{2u_-}$ for all $t\ge t_0$ such that $x_0\le x(t)\le x_1$. In particular, $x(t)$ reaches x_1 without stopping.*
- *If $x_1<x_0$, $v_0<0$, and $u_+>0$, then $\dot x(t)\le-\sqrt{2u_+}$ for all $t\ge t_0$ such that $x_0\le x(t)\le x_1$. In particular, $x(t)$ reaches x_1 without stopping.*
- *If $x_1>x_0$, $v_0>0$, and $u_+<0$, then $x(t)$ does not reach x_1 before it stops.*
- *If $x_1<x_0$, $v_0<0$, and $u_-<0$, then $x(t)$ does not reach x_1 before it stops.*

Proof. Observe that
$$\frac{\dot x^2(t)}{2}=\frac{v_0^2}{2}+\int_{t_0}^t f(\tau,x(\tau))\dot x(\tau)\,d\tau,\quad\forall\,t\ge t_0,\tag{3.1}$$
as can be easily verified by differentiating both members of (3.1) and noticing that they coincide when $t=t_0$.

We have to examine four cases, and in each one of them the initial velocity $v_0=\dot x(t_0)$ is different from 0. Therefore, it makes sense to consider the maximal interval containing t_0 and contained in the relatively open subset $\{t\ge t_0:\dot x(t)\ne 0\}$ of $[t_0,+\infty)$. This nonempty interval will be denoted by $[t_0,t_*)$, with $t_0<t_*\le+\infty$.

Consider the first case, namely $x_1>x_0$, $v_0>0$, and $u_->0$. From equation (3.1) and from the inequality $f_-(x)\le f(t,x)$, we obtain
$$\frac{\dot x^2(t)}{2}\ge\frac{v_0^2}{2}+\int_{t_0}^t f_-(x(\tau))\dot x(\tau)\,d\tau=\frac{v_0^2}{2}+\int_{x_0}^{x(t)} f_-(s)\,ds=g_-(x(t))$$
for every $t\in[t_0,t_*)$. Since $\dot x(t_0)>0$, we obtain $\dot x(t)\ge\sqrt{2u_-}$ for all $t\in[t_0,t_*)$ such that $x(t)\le x_1$. Hence, the first of the four assertions follows.

The second case can be analyzed in a similar manner by taking into account that $\dot x(\tau)$ is negative for all $\tau\in[t_0,t]\subset[t_0,t_*)$.

Consider the third case: $x_1>x_0$, $v_0>0$, and $u_+<0$. From equation (3.1) and the inequality $f(t,x)\le f_+(x)$, we obtain
$$\frac{\dot x^2(t)}{2}\le\frac{v_0^2}{2}+\int_{x_0}^{x(t)} f_+(s)\,ds=g_+(x(t)),\quad\forall\,t\in[t_0,t_*).$$

Since $u_+ < 0$, there exists a point $\bar{x}$ in the open interval (x_0, x_1) such that $g_+(\bar{x}) < 0$. Thus, because of the above inequality, $x(t)$ cannot cross the point $\bar{x}$ before it stops.

The last case can be studied in a similar manner using the fact that now $\dot{x}(\tau) < 0$, $\forall\, \tau \in [t_0, t]$. □

Remark 3.3. The equation (2.4) is of the type $\ddot{x} = f(t, x)$, with

$$f_-(x) \le f(t,x) \le f_+(x),$$

for all $(t, x) \in \mathbb{R}^2$, where

$$\begin{aligned} f_-(x) &= \frac{-4e - 3k\sin x}{(1 + e\,\mathrm{sign}(-4e - 3k\sin x))^3}, \\ f_+(x) &= \frac{4e - 3k\sin x}{(1 - e\,\mathrm{sign}(4e - 3k\sin x))^3}. \end{aligned} \tag{3.2}$$

For the rest of this article, we shall assume that $f_-(x)$ and $f_+(x)$ are as in (3.2). Observe that in this case,

$$f_-(-x) = -f_+(x), \quad \forall x \in \mathbb{R}.$$

Moreover, under the assumption $4e < 3k$, which implies the existence of the arcs $A_W = (\alpha_{_W}, \beta_{_W})$ and $A_E = (\beta_{_E}, \alpha_{_E})$, we have $\alpha_{_W} = -\alpha_{_E}$ and $\beta_{_W} = -\beta_{_E}$. Consequently,

$$\int_{\alpha_{_W}}^{\beta_{_W}} f_-(x)dx = \int_{\alpha_{_E}}^{\beta_{_E}} f_+(x)dx\,, \quad \int_{\beta_{_W}}^{0} f_-(x)dx = \int_{\beta_{_E}}^{0} f_+(x)dx\,.$$

The following function of k and e, defined on the open triangle

$$T = \big\{(k,e) \in \mathbb{R}^2 : 0 < 4e < 3k < 3\big\},$$

plays an important role in the sequel:

$$h(k,e) := \int_{\alpha_{_W}}^{0} f_-(x)dx = \int_{\alpha_{_E}}^{0} f_+(x)dx. \tag{3.3}$$

Notice that $h(k,e) = h_+(k,e) + h_-(k,e)$, with

$$\begin{aligned} h_+(k,e) &:= \int_{\alpha_{_W}}^{\beta_W} f_-(x)dx = \int_{\alpha_{_E}}^{\beta_E} f_+(x)dx, \\ h_-(k,e) &:= \int_{\beta_{_W}}^{0} f_-(x)dx = \int_{\beta_{_E}}^{0} f_+(x)dx\,. \end{aligned}$$

Moreover, $h_+(k,e) > 0$ and $h_-(k,e) < 0$ for all (k,e) in the triangle T, since $f_-(x)$ is positive for $x \in (\alpha_{_W}, \beta_{_W})$ and negative for $x \in (\beta_{_W}, 0)$. Actually, elementary computations show that

$$h_+(k,e) = \frac{-4e(\pi - 2\arcsin(4e/3k)) + 6k\sqrt{1 - (4e/3k)^2}}{(1+e)^3}$$

and

$$h_-(k,e) = \frac{-4e\arcsin(4e/3k) + 3k(1 - \sqrt{1 - (4e/3k)^2})}{(1-e)^3}.$$

Proposition 3.4. *Let $h(k,e) > 0$, and assume that when $t = t_0$ the point mass crosses one of the positions α_W or α_E and it travels towards the South Pole. Then it reaches it at some $t_1 > t_0$ and $\dot{x}(t) \neq 0$ for every $t \in [t_0, t_1]$.*

Proof. We consider first the case when the point mass crosses α_W with positive velocity. With the notation of Lemma 3.2, let $x_0 = \alpha_W$ be the highest point of the west arc and let $x_1 = 0$ be the South Pole. Consider the function $g_- : [x_0, x_1] \to \mathbb{R}$ defined by

$$g_-(x) = \frac{v_0^2}{2} + \int_{x_0}^{x} f_-(s)\, ds,$$

where $v_0 := \dot{x}(t_0) > 0$ and $f_-(s)$ is as in Remark 3.3. As pointed out before, $f_-(s)$ is positive in the west arc (x_0, β_W) and negative between β_W and x_1 (the South Pole). Therefore, the function $g_-(x)$ is increasing up to β_W and then decreasing. Moreover, at the extremes of the interval $[x_0, x_1]$ we have

$$g_-(x_0) = \frac{v_0^2}{2} > 0 \quad \text{and} \quad g_-(x_1) = \frac{v_0^2}{2} + \int_{x_0}^{x_1} f_-(s)\, ds = \frac{v_0^2}{2} + h(k,e) > 0.$$

Thus, the number

$$u_- = \min\left\{g_-(x) : x \in \overline{x_0 x_1}\right\}$$

is positive. Consequently, the first assertion of Lemma 3.2 shows that $x(t)$ reaches the South Pole with positive velocity without stopping.

The situation in which the point mass crosses α_E clockwise can be analyzed as in the previous case, using the equality

$$h(k,e) = \int_{\alpha_E}^{0} f_+(x) dx$$

and the second assertion of Lemma 3.2. $\square$

4. Over or not

The two results of this section deal with *going over the top* (see Theorem 4.4) and *not going over the top* (see Theorem 4.5). We establish results similar to the ones proved in [6] for the satellite Hyperion. The procedure, however, is quite different since we are dealing with any satellite having its motion around a planet described by (2.4). Recall that we are assuming $4e < 3k$, which is verified for Hyperion.

As a first change, in equation (2.4) we substitute the independent variable t with θ, which can be regarded as well as an independent variable since the correspondence $t \mapsto \theta(t)$ is invertible. The reason of this substitution is that (2.4) contains the function $\theta(t)$ which depends on the satellite under consideration, and our numerical approach requires that we decide when $f(t) = \sin\theta(t)$ is equal to $+1$ or to -1. By using θ as the independent variable we avoid the complication of redetermining the values of t such that $\theta(t) = \pm\pi/2$ each time we change the parameters k and e.

For the rest of this article, we denote by x' and x'' the first and second derivatives of x with respect to θ. By abuse of terminology, those will be called *velocity* and *acceleration* of the point mass as if θ were the time. Recall that $\dot{x}$ and $\ddot{x}$ denote the first and second derivatives of x with respect to t.

Observe that

$$\frac{d}{d\theta} = \frac{1}{g(\theta)} \frac{d}{dt},$$

where, according to (2.5),

$$g(\theta) = \frac{d\theta}{dt} = \frac{(1 + e\cos\theta)^2}{(1 - e^2)^{3/2}}.$$

Consequently,

$$x'' = \frac{\ddot{x}}{g^2(\theta)} - \frac{g'(\theta)}{g(\theta)} x'.$$

Since (see (2.4))

$$\ddot{x} = \Big(\frac{1 + e\cos\theta(t)}{1 - e^2}\Big)^3 \big(4e\sin\theta(t) - 3k\sin x\big),$$

we obtain

$$x'' = \frac{1}{1 + e\cos\theta}(2e(x'+2)\sin\theta - 3k\sin x). \tag{4.1}$$

Remark 4.1. Notice that whenever $x(\theta)$ is a solution of (4.1), every function of the form

$$z_k(\theta) := x(\theta + 2k\pi) \tag{4.2}$$

is also a solution for every $k \in \mathbb{Z}$. Moreover, additional solutions are provided by the function $w(\theta) := -x(-\theta)$ (*equivariance*) and by its translates according to (4.2).

The following definition will be used repeatedly.

Definition 4.2. Given two events regarding the point mass $x(\theta)$, the expression *the second event happens immediately after the first one* means that the last event occurs for the first time after the previous one and, between the two events, the derivative $x'(\theta)$ never vanishes.

Notice that, with the new independent variable θ, Proposition 3.4 can be reformulated as follows.

Proposition 4.3. *Let $h(k, e) > 0$, and assume that when $\theta = \theta_0$ the point mass crosses one of the positions α_W or α_E and it travels downward. Then it reaches the South Pole immediately after θ_0.*

We now introduce symbols to be used in this section. In the definition of any of them, one of the following two events is always considered: the point mass $x(\theta)$ crosses S (the South Pole) when the variable θ is either $\theta_S = \pi/2$ or $\theta_S = -\pi/2$. The crossing can be counterclockwise or clockwise and must happen or not happen immediately before or after one of the following events:

(1) leaving the northern arc A_N,
(2) entering A_N,
(3) crossing A_N *strictly* (meaning that $x'(\theta)$ never vanishes during the crossing).

The plus sign is used when $\theta_S = \pi/2$, while the minus sign is used when $\theta_S = -\pi/2$. The letter P stands for *past* and it denotes that the event under consideration takes place before the crossing of S. The letter F stands for *future* and it denotes that the event happens after crossing S. The letter v always stands for the absolute value of $x'(\theta_S)$. An arrow located over the letter v indicates the supremum of v and an arrow located below v indicates the infimum of v. The orientation of the arrow is important, since it indicates the direction of the point mass when it crosses the South Pole. For example, $\leftarrow$ means that (for $\theta = \theta_s = \pm\pi/2$) we have $x(\theta_s) = 0$ and $x'(\theta_s) < 0$; that is, the motion is clockwise.

Clearly, there are $2^4 = 16$ possibilities. However, only 8 of them are of interest to us. Four of them regard the counterclockwise motion; the other ones the clockwise motion. We give their description below.

Recall first that in all cases, even if not specified, the crossing of S happens when the variable θ is either $\theta_S = \pi/2$ or $\theta_S = -\pi/2$ according to the plus or minus sign, and the crossing is counterclockwise or clockwise according to the direction of the arrow.

The symbols

$$\underrightarrow{v}_p^+ \quad \text{and} \quad \underleftarrow{v}_p^-$$

denote the *infimum* of $v = |x'(\theta_S)|$ such that the point mass crosses S immediately after leaving the northern arc A_N.

Similarly,

$$\underleftarrow{v}_F^+ \quad \text{and} \quad \underrightarrow{v}_F^-$$

indicate the *infimum* of $|x'(\theta_S)|$ sufficient for the point mass to arrive at the northern arc immediately after crossing S. Consequently, with a smaller v the point mass cannot reach A_N immediately after leaving S.

Likewise

$$\overleftarrow{v}_p^+ \quad \text{and} \quad \overrightarrow{v}_p^-$$

denote the *supremum* of $|x'(\theta_S)|$ such that the point mass crosses S immediately after coming from A_N, which, however, should not have been crossed *strictly*.

Finally,

$$\overrightarrow{v}_F^+ \quad \text{and} \quad \overleftarrow{v}_F^-$$

stand for the *supremum* of $|x'(\theta_S)|$ such that the point mass does not cross *strictly* the northern arc A_N immediately after leaving S.

We now set

$$\begin{aligned} \delta_{\rightarrow}^+ &= \underrightarrow{v}_p^+ - \overrightarrow{v}_F^+, \quad \delta_{\rightarrow}^- = \underrightarrow{v}_F^- - \overrightarrow{v}_p^-, \\ \delta_{\leftarrow}^+ &= \underleftarrow{v}_F^+ - \overleftarrow{v}_p^+, \quad \delta_{\leftarrow}^- = \underleftarrow{v}_p^- - \overleftarrow{v}_F^-. \end{aligned} \tag{4.3}$$

The equivariance mentioned in Remark 4.1 implies that

$$\begin{aligned} \underrightarrow{v}_P^+ &= \underrightarrow{v}_F^-, \quad \overrightarrow{v}_F^+ = \overrightarrow{v}_P^-, \\ \underleftarrow{v}_F^+ &= \underleftarrow{v}_P^-, \quad \overleftarrow{v}_P^+ = \overleftarrow{v}_F^-. \end{aligned} \tag{4.4}$$

From (4.3) and (4.4) we obtain

$$\begin{aligned} \delta_{\rightarrow}^+ &= \delta_{\rightarrow}^-, \\ \delta_{\leftarrow}^+ &= \delta_{\leftarrow}^-. \end{aligned} \tag{4.5}$$

To emphasize the dependence of (4.5) on the ratio k and the eccentricity e, we shall write

$$\delta_{\rightarrow}(k, e), \quad \delta_{\leftarrow}(k, e).$$

Moreover, we set

$$\delta(k, e) = \min\{\delta_{\rightarrow}(k, e),\ \delta_{\leftarrow}(k, e)\}. \tag{4.6}$$

Given a pair (k, e), to estimate the four numbers of (4.4) we use a program, written by the first author, that simulates the motion of the point mass on the circle S^1 satisfying the differential equation (4.1). On this circle the marks α_W, β_W, β_E, α_E are indicated.

For example, to compute $\underleftarrow{v}_F^+$ we use the shooting method starting from the South Pole S when $\theta = \pi/2$ and the parameter $v = x'(\pi/2)$ is negative; that is, the motion is clockwise. The corresponding solution is observed in the *future* (i.e., for $\theta > \pi/2$). Equivalently, because of the equality $\underset{\leftarrow}{v}{}^{+}_{F} = \underset{\leftarrow}{v}{}^{-}_{P}$, one can evaluate the same number by starting from S when $\theta = -\pi/2$ and $v = x'(-\pi/2) < 0$. The motion is again clockwise, but in this case the point mass must be observed in the *past* (so that the point appears to be moving counterclockwise).

The curves $h(k,e) = 0$ and $\delta(k,e) = 0$ intersect at two points (see figure 2) that have been evaluated to be

$$P_l = (0.179 \pm 10^{-3}, 0.088 \pm 10^{-3}), \quad P_r = (0.753 \pm 10^{-3}, 0.279 \pm 10^{-3}).$$

For P_l the four velocities are

$$\underset{\rightarrow}{v}{}^{+}_{P} = \underset{\rightarrow}{v}{}^{-}_{F} = 1.689 \pm 10^{-3}, \quad \overrightarrow{v}{}^{+}_{F} = \overrightarrow{v}{}^{-}_{P} = 1.161 \pm 10^{-3},$$
$$\underset{\leftarrow}{v}{}^{+}_{F} = \underset{\leftarrow}{v}{}^{-}_{P} = 1.444 \pm 10^{-3}, \quad \overleftarrow{v}{}^{+}_{P} = \overleftarrow{v}{}^{-}_{F} = 1.444 \pm 10^{-3}.$$

For P_r we obtain

$$\underset{\rightarrow}{v}{}^{+}_{P} = \underset{\rightarrow}{v}{}^{-}_{F} = 4.337 \pm 10^{-3}, \quad \overrightarrow{v}{}^{+}_{F} = \overrightarrow{v}{}^{-}_{P} = 1.526 \pm 10^{-3},$$
$$\underset{\leftarrow}{v}{}^{+}_{F} = \underset{\leftarrow}{v}{}^{-}_{P} = 2.970 \pm 10^{-3}, \quad \overleftarrow{v}{}^{+}_{P} = \overleftarrow{v}{}^{-}_{F} = 1.970 \pm 10^{-3}.$$

For Hyperion $k = 0.26$, $e = 0.11$, and we obtain

$$\underset{\rightarrow}{v}{}^{+}_{P} = \underset{\rightarrow}{v}{}^{-}_{F} = 2.177 \pm 10^{-3}, \quad \overrightarrow{v}{}^{+}_{F} = \overrightarrow{v}{}^{-}_{P} = 1.308 \pm 10^{-3},$$
$$\underset{\leftarrow}{v}{}^{+}_{F} = \underset{\leftarrow}{v}{}^{-}_{P} = 1.787 \pm 10^{-3}, \quad \overleftarrow{v}{}^{+}_{P} = \overleftarrow{v}{}^{-}_{F} = 1.729 \pm 10^{-3}.$$

It follows that $\delta(0.26, 0.11) = 0.058 \pm 2 \cdot 10^{-3}$.

The following theorem deals with *going over the top*.

Theorem 4.4. *Assume $\delta(k,e) > 0$ and consider a solution $x(\theta)$ of the differential equation* (4.1) *which, for some $\theta^0 \in \mathbb{R}$, satisfies the condition $x(\theta^0) \in A_N$. Suppose that, for some $k \in \mathbb{Z}$ and immediately after leaving A_N, the point mass $x(\theta)$ reaches counterclockwise the South Pole when $\theta = \theta_S = \pi/2 + 2k\pi$ or clockwise when $\theta = \theta_S = -\pi/2 + 2k\pi$. Then the point mass crosses A_N immediately after θ_S.*

Proof. First of all notice that the equation (4.1) is 2π periodic. Thus, if $x(\theta)$ is a solution, so is $y(\theta) := x(\theta + 2k\pi)$ for any $k \in \mathbb{Z}$. Therefore, we can set $\theta_S = \pi/2$ in the first case, and $\theta_S = -\pi/2$ in the second one.

Let us assume that point mass crosses the arc A_W reaching the South Pole for $\theta_S = \pi/2$. Then, from the condition $\delta_{\rightarrow}(k,e) > 0$ we derive that $\underset{\rightarrow}{v}{}^{+}_{p} > \overrightarrow{v}{}^{+}_{F}$. In other words, the speed of the point mass at the South Pole is sufficient for *going over the top*.

Let us now assume that the point mass crosses the arc A_E reaching the South Pole for $\theta_S = -\pi/2$. Then from the condition $\delta_{\leftarrow}(k,e) > 0$ we derive $\underset{\leftarrow}{v}{}^{-}_{p} > \overleftarrow{v}{}^{-}_{F}$. Hence, the speed of arrival of the point mass at S is larger than the maximum speed such that A_N is not crossed. Consequently, A_N is crossed. $\square$

Our next result deals with the case of *not going over the top*.

Theorem 4.5. *Suppose $\delta(k,e) > 0$ and let $x(\theta)$ be a solution of* (4.1) *that, for some $\theta_0 \in \mathbb{R}$, verifies the following conditions:*

- *$x(\theta_0) \in A_N$;*
- *$x'(\theta_0) = 0$;*
- *for some $k \in \mathbb{Z}$ and immediately after θ_0, the point mass $x(\theta)$ reaches the South Pole when $\theta = \theta_S = -\pi/2 + 2k\pi$ in the counterclockwise case or when $\theta = \theta_S = \pi/2 + 2k\pi$ in the clockwise case.*

Then the point mass will not cross the northern arc immediately after θ_S.

Proof. The proof repeats almost verbatim the one of Theorem 4.4. In the counterclockwise case, the condition $\delta_{\rightarrow}(k,e) > 0$ implies that $\underset{\rightarrow}{v}{}_F^- > \overrightarrow{v}{}_p^-$. Therefore, the point mass cannot reach A_N immediately after leaving S. A similar conclusion is obtained in the clockwise case, since the condition $\delta_{\leftarrow}(k,e) > 0$ implies $\underset{\leftarrow}{v}{}_F^+ > \overleftarrow{v}{}_p^+$. $\square$

5. Chaotic behavior

This section is similar to the corresponding portion of [7]. There are however, some important differences, since the independent variable t has been replaced by the independent variable θ. Moreover, throughout this section, we assume that the pair (k,e) belongs to the open triangle T given by $0 < 4e < 3k < 3$, so that the four arcs A_W, A_E, A_N, A_S are well defined, together with the function $h(k,e)$. In particular, we are interested in the following open subset of T (see figure 2):

$$\Omega = \big\{(k,e) \in T : h(k,e) > 0,\, \delta(k,e) > 0\big\}.$$

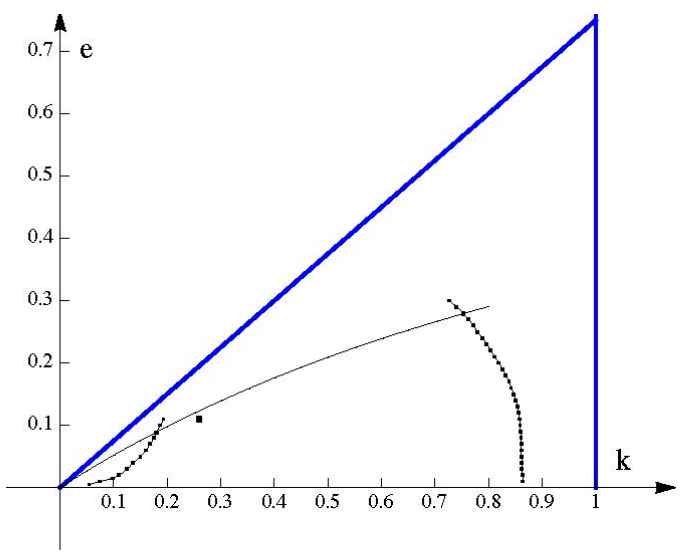

Figure 2. The region Ω is inside the triangle T. It is bounded to the right and to the left by the curve $\delta(k,e) = 0$ (represented as a sequence of dots, obtained numerically), and above by the curve $h(k,e) = 0$ (see equation (3.3)). The dot in Ω, namely the point with coordinates $(0.26, 0.11)$, corresponds to Hyperion

Let $\theta_N \in \mathbb{R}$ and $x_N \in A_N$ be fixed. Given any $v \in \mathbb{R}$, let $x_v(\theta)$ denote the solution of the initial value problem

$$\begin{gathered} x'' = \frac{1}{1 + e\cos\theta}\big(2e(x' + 2)\sin\theta - 3k\sin x\big) \\ x(\theta_N) = x_N, \quad x'(\theta_N) = v. \end{gathered} \tag{5.1}$$

Definition 5.1. Whenever, for $\theta > \theta_N$, $x_v(\theta)$ crosses A_W with counterclockwise velocity or A_E with clockwise velocity, we say that the crossing is a *significant event*. In the first case we label the crossing with the number 1; and in the second case, with -1.

Definition 5.2. Denote by Σ the set of all sequences in $\{-1, 1\}$. Given $\sigma \in \Sigma$ and $n \in \mathbb{N}$, we say that $x_v(\theta)$ *n-fulfills* σ if after θ_N it has at least n significant events and the list of labels associated to them coincides with the first n elements of σ, with their order preserved. We say that $x_v(\theta)$ *fulfills* $\sigma \in \Sigma$ if n-fulfills σ for every $n \in \mathbb{N}$.

Let $\sigma \in \Sigma$ and $n \in \mathbb{N}$ be given.

- We denote by $I_n^\sigma \subset \mathbb{R}$ the set of $v \in \mathbb{R}$ such that the solution $x_v(\theta)$ of (5.1) n-fulfills σ.
- Given $v \in I_n^\sigma$, denote by $\theta_n^\sigma(v)$ the value of θ such that $x_v(\theta)$ crosses the South Pole immediately after n-significant events.

Notice that continuity with respect to initial conditions implies that any I_n^σ is open. Moreover, as a consequence of Proposition 4.3, the function $\theta_n^\sigma : I_n^\sigma \to \mathbb{R}$ is well defined and continuous.

We are now ready to state and prove the main result of this paper.

Theorem 5.3. *Suppose $(k, e) \in \Omega$. Then, given any $\sigma \in \Sigma$, there exists $v \in \mathbb{R}$ such that the solution $x_v(\theta)$ of* (5.1) *fulfills σ.*

Proof. We shall determine a sequence $\{J_n : n \in \mathbb{N}\}$ of nonempty bounded open intervals such that the following two conditions are satisfied:

(a_n) $J_n \subset I_n^\sigma$;
(b_n) the closure of J_{n+1} is contained in J_n.

The two properties just mentioned imply that

$$J_\infty = \cap_{n\geq 1} J_n \neq \emptyset, \tag{5.2}$$

and a solution of (5.1) with initial velocity $v \in J_\infty$ fulfills σ.

We now describe how to define J_1, J_2 and J_3 for any $\sigma \in \Sigma$. An induction procedure can be used to define J_n for all $n \in N$ so that (5.2) holds.

Without loss of generality we can assume that the first element of σ is 1. Notice that a point mass with a large and counterclockwise initial speed will first cross A_W. Thus, the open set I_1^σ is nonempty. Analogously, if the initial speed is large enough and clockwise, the point mass will first cross A_E. Hence, I_1^σ is bounded below. Therefore, it contains a bounded interval $J_1 = (\omega_1, v_1)$ such that $\omega_1 \notin I_1^\sigma$. Continuity with respect to initial conditions implies that the unique solution of (5.1) with initial velocity ω_1 cannot have -1 as the first significant event. Thus, this solution will always remain in A_N oscillating indefinitely. Hence, continuity with respect to initial conditions implies

$$\lim_{v\to\omega_{1+}} \theta_1^\sigma(v) = +\infty. \tag{5.3}$$

Suppose now that the second element of σ is different from the previous one; i.e., it is -1. The continuity of θ_1^σ, equation (5.3) and Theorem 4.4 imply the existence of an initial velocity $v_2 \in J_1$ such that the corresponding solution of (5.1) crosses A_W a second time without stopping. Now observe that continuity with respect to initial conditions implies that any solution of (5.1) with initial velocity $v \in J_1$ close to ω_1 will stop inside A_N for some value $\theta_N \geq 0$ before the first significant event (recall that the solution of (5.1) with $v = \omega_1$ oscillates indefinitely inside A_N). Thus, equation (5.3) and Theorem 4.5 imply the existence of an initial $w_2 \in (\omega_1, v_2)$ such that the corresponding solution of (5.1) does not reach the northern arc A_N immediately after the first significant event. Therefore, continuity with respect to initial conditions implies the existence of a solution of (5.1) with initial velocity $u_2 \in (w_2, v_2)$ which enters A_N (immediately after the first significant event) and then goes back crossing A_E with clockwise velocity. Consequently, its second significant event is -1. This shows that the open set $I_2^\sigma \cap (u_2, v_2)$ is nonempty. Moreover, since this set is clearly strictly contained in (u_2, v_2), it contains an interval $J_2 = (u_2, \omega_2)$ with $\omega_2 \notin I_2^\sigma$. Continuity with respect to initial conditions implies that the solution of (5.1) with initial velocity ω_2 cannot have 1 or -1 as the second significant event. Thus, this solution, after the first significant event, will enter A_N and remain there oscillating indefinitely.

At this point, the situation is as follows:

- (a_2) the interval $J_2 = (u_2, \omega_2)$ is contained in I_2^σ;
- (b_2) the closure of J_2 is contained in J_1;
- (c_2) ω_2 is the initial velocity of a solution of (5.1) that after the first significant event enters A_N and remains there oscillating indefinitely.

Continuity with respect to initial conditions implies that

$$\lim_{v \to \omega_{2}-} \theta_2^\sigma(v) = +\infty. \tag{5.4}$$

Let us assume that the third element of σ is the same as the previous one; i. e., it is again -1. Observe that, because of (5.4) and condition (c_2), we can find in J_2 an initial velocity of a solution of (5.1) that satisfies the assumptions of Theorem 4.4 for some θ^0 after the first significant event and before the second (which, we point out, is labelled -1). Similarly, in J_2 we can find an initial velocity of a solution that satisfies the assumptions of Theorem 4.5 for some θ_0 between the first and the second significant events. Therefore, with a procedure analogous to the one described for the construction of J_2, we can select an open interval J_3 with the following properties:

- (a_3) all velocities of the interval J_3 3-fulfill the sequence σ;
- (b_3) the closure of J_3 is contained in J_2;
- (c_3) one of the extremes of J_3 is the initial velocity of a solution of (5.1) that, after the second significant event, enters A_N and remains there oscillating indefinitely.

An induction argument can now be used to complete the proof. □

Figure 3 below contains a quadrilateral in which, according to our numerical estimates, the assumptions of Theorem 5.3 are satisfied.

The vertices are $(0.15, 0.01)$, $(0.85, 0.01)$, $(0.75, 0.27)$, $(0.19, 0.09)$. Thus, a point (k, e) belongs to the quadrilateral if and only if

$$\begin{gathered}
(0.85 - 0.15)(e - 0.01) - (0.01 - 0.01)(k - 0.15) \geq 0, \\
(0.75 - 0.85)(e - 0.01) - (0.27 - 0.01)(k - 0.85) \geq 0, \\
(0.19 - 0.75)(e - 0.27) - (0.09 - 0.27)(k - 0.75) \geq 0, \\
(0.15 - 0.19)(e - 0.09) - (0.01 - 0.09)(k - 0.19) \geq 0.
\end{gathered}$$

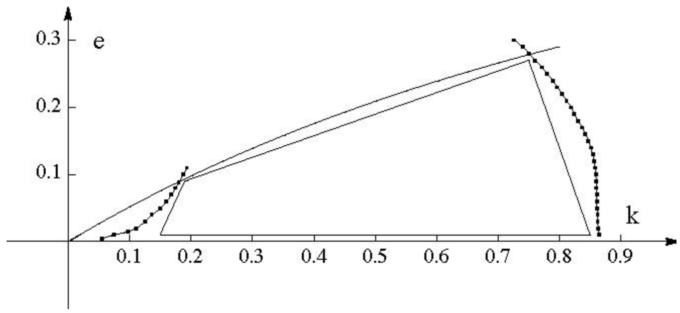

FIGURE 3. Quadrilateral in which the assumptions of Theorem 5.3 are satisfied

Notice that the above conditions are satisfied by the point (k, e) corresponding to the satellite Hyperion.

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