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EXISTENCE OF POSITIVE SOLUTIONS FOR A MULTI-POINT FOUR-ORDER BOUNDARY-VALUE PROBLEM

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Abstract. The article shows sufficient conditions for the existence of positive solutions to a multi-point boundary-value problem for a fourth-order differential equation. Our main tools are the Guo-Krasnoselskii fixed point theorem and the monotone iterative technique. We also show that the set of positive solutions is compact.

1. Introduction

Multi-point boundary-value problems for ordinary differential equations arise in a variety of areas in applied mathematics and physics. For this reason the have been investigated by several authors; see for example [2]-[4, 2, 3, 6, 7, 8, 9]. In this article, we study the existence of positive solutions for the problem

$$x^{(4)}(t) = \lambda f(t, x(t)), \quad 0 < t < 1, \tag{1.1}$$

$$x^{(2k+1)}(0) = 0, \quad x^{(2k)}(1) = \sum_{i=1}^{m-2} \alpha_{ki} x^{(2k)}(\eta_{ki}), \quad k = 0, 1, \tag{1.2}$$

where $\lambda > 0$, $0 < \eta_{k1} < \eta_{k2} < \dots < \eta_{k,m-2} < 1$, $(k = 0, 1)$ and α_{ki}, with $k = 0, 1$; $i = 1, 2, \dots, m-2$, are given positive constants satisfy the conditions

$$\sum_{i=1}^{m-2} \alpha_{1i}\eta_{1i} \le 1 < \sum_{i=1}^{m-2} \alpha_{1i}, \tag{1.3}$$

$$\sum_{i=1}^{m-2} \alpha_{0i}\eta_{0i}^2 < 1 < \sum_{i=1}^{m-2} \alpha_{0i}. \tag{1.4}$$

When $m = 3$; $\eta_{01} = \eta_0$, $\eta_{11} = \eta_1$; $\alpha_{01} = \alpha_0$, $\alpha_{11} = \alpha_1$; and the inhomogeneous term is $f(u(t))$, the problem (1.1)-(1.2) is studied in [1]. The authors in [1] obtained several existence results of positive solutions basing the computations of the fixed point index of open subsets of a Banach space relative to a cone and follow from a well-known theorem of Krasnosel'skii. One of the assumptions playing an important role in obtaining positive solution is that $1 < \alpha_i < \frac{1}{\eta_i}$, $i = 0, 1$.

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The rest of this paper is organized as follows. In section 2, we provide some results which are motivation for obtaining our main results. In section 3 we state and prove several existence results for at least one positive solution. Our main tools are the Guo-Krasnoselskii's fixed point theorem or the monotone iterative technique. Finally, section 4 devoted to the compactness of positive solutions set.

2. Preliminaries

In this article, $C([0,1])$ denotes the space of all continuous functions x from $[0,1]$ into $\mathbb{R}$ endowed with the supremum norm

$$\|x\| = \sup_{t\in[0,1]} |x(t)|, \quad x \in C([0,1]).$$

First we consider the auxiliary linear differential equation

$$-x''(t) = g(t), \quad 0 < t < 1, \tag{2.1}$$

with the boundary conditions

$$x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i), \tag{2.2}$$

where $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$ and α_i $(i = 1, 2, \dots, m-2)$ are given positive constants.

Lemma 2.1. *Let $g \in C[0,1]$ be non-negative (non-positive) and $\sum_{i=1}^{m-2} \alpha_i\eta_i \le 1 < \sum_{i=1}^{m-2} \alpha_i$. Then*

$$\begin{aligned} x(t) = &-\int_0^t (t-s)g(s)ds + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\int_0^1 (1-s)g(s)ds \\ &- \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} (\eta_i - s)g(s)ds\Big] \end{aligned} \tag{2.3}$$

is a unique non-positive (non-negative) solution of (2.1)–(2.2).

Proof. It is easy to see that (2.3) is a unique solution of (2.1)–(2.2). If $g(t) \ge 0$ on $[0,1]$ then

$$x'(t) = -\int_0^t g(s)ds \le 0$$

and

$$x(t) \le x(0) = \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\int_0^1 (1-s)g(s)ds - \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} (\eta_i - s)g(s)ds\Big]. \tag{2.4}$$

Let $F(\eta) = \frac{1}{\eta}\int_0^\eta (\eta - s)g(s)ds$. We have

$$F'(\eta) = \frac{\eta\int_0^\eta g(s)ds - \int_0^\eta (\eta-s)g(s)ds}{\eta^2} = \frac{\int_0^\eta sg(s)ds}{\eta^2} \ge 0.$$

This implies $F(\eta_i) \le F(1)$, for $i = 1, 2, \dots, m-2$; that is,

$$\int_0^{\eta_i} (\eta_i - s)g(s)ds \le \eta_i \int_0^1 (1-s)g(s)ds, \quad \text{for } i = 1, 2, \dots, m-2.$$

Hence

$$\sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i}(\eta_i-s)g(s)ds\le\sum_{i=1}^{m-2}\alpha_i\eta_i\int_0^1(1-s)g(s)ds\le\int_0^1(1-s)g(s)ds. \quad (2.5)$$

From (2.4) and (2.5), we conclude that $x(t)\le 0$, for all $t\in[0,1]$. In the case $g(t)\le 0$, by similar arguments, we obtain $x(t)\ge 0$, for all $t\in[0,1]$. This completes the proof. □

Lemma 2.2. *Let g be non-positive and non-increasing function in $C[0,1]$ and let $\sum_{i=1}^{m-2}\alpha_i\eta_i^2<1<\sum_{i=1}^{m-2}\alpha_i$. Then the unique solution (2.3) of (2.1)–(2.2) is nonnegative. Further we have*

$$\min_{0\le t\le 1}x(t)\ge\gamma\|x\|, \quad (2.6)$$

where

$$\gamma=\frac{1-\sum_{i=1}^{m-2}\alpha_i\eta_i^2}{\sum_{i=1}^{m-2}\alpha_i(1-\eta_i^2)}. \quad (2.7)$$

Proof. Because $g(t)\le 0$ for all $t\in[0,1]$, the unique solution (2.3) of (2.1)–(2.2) is non-decreasing and

$$x(t)\ge x(0)=\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\int_0^1(1-s)g(s)ds-\sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i}(\eta_i-s)g(s)ds\Big]. \quad (2.8)$$

Let $F_0(\eta)=\frac{1}{\eta^2}\int_0^\eta(\eta-s)g(s)ds$. Then we have

$$F_0'(\eta)=\frac{\eta\int_0^\eta g(s)ds-2\int_0^\eta(\eta-s)g(s)ds}{\eta^3}=\frac{\int_0^\eta(2s-\eta)g(s)ds}{\eta^3}$$

It is easy to check that the function $\eta\mapsto\int_0^\eta(2s-\eta)g(s)ds$ is non-increasing. Thus

$$\int_0^\eta(2s-\eta)g(s)ds\le 0,\quad \forall\eta\ge 0.$$

This implies that $F_0'(\eta)\le 0$, for all $\eta\ge 0$. Thus

$$\begin{aligned}\sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i}(\eta_i-s)g(s)ds&=\sum_{i=1}^{m-2}\alpha_i\eta_i^2F_0(\eta_i)\ge F_0(1)\sum_{i=1}^{m-2}\alpha_i\eta_i^2\\&\ge\int_0^1(1-s)g(s)ds.\end{aligned} \quad (2.9)$$

Combining (2.8) and (2.9), we deduce that $x(t)\ge 0$ for all $t\in[0,1]$. Finally, we need to check inequality (2.6), or equivalently,

$$x(0)\ge\gamma x(1). \quad (2.10)$$

Indeed, it follows from (2.3) that (2.10) is equivalent to

$$\sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i}(\eta_i-s)g(s)ds\ge\frac{1-\gamma\sum_{i=1}^{m-2}\alpha_i}{1-\gamma}\int_0^1(1-s)g(s)ds. \quad (2.11)$$

By the monotonicity of F_0, we have

$$\sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i}(\eta_i-s)g(s)ds=\sum_{i=1}^{m-2}\alpha_i\eta_i^2F_0(\eta_i)\ge\sum_{i=1}^{m-2}\alpha_i\eta_i^2\int_0^1(1-s)g(s)ds. \quad (2.12)$$

So, it is not difficult to obtain (2.11) from (2.12) and (2.7). The proof is completed. $\square$

Remark 2.3. For $t, s \in [0,1]$, we put

$$\begin{aligned} G(t,s,\alpha_i,\eta_i) &= \begin{cases} s-t, & 0 \le s \le t \le 1,\\ 0, & 0 \le t \le s \le 1, \end{cases} \\ &\quad + \overline{\alpha} \begin{cases} 1-\sum_{i=1}^{m-2}\alpha_i\eta_i + (\sum_{i=1}^{m-2}\alpha_i - 1)s, & 0 \le s \le \eta_1,\\ 1-\sum_{i=2}^{m-2}\alpha_i\eta_i + (\sum_{i=2}^{m-2}\alpha_i - 1)s, & \eta_1 \le s \le \eta_2,\\ \dots \\ 1-\sum_{i=k}^{m-2}\alpha_i\eta_i + (\sum_{i=k}^{m-2}\alpha_i - 1)s, & \eta_{k-1} \le s \le \eta_k,\\ \dots \\ 1-s, & \eta_{m-2} \le s \le 1, \end{cases} \end{aligned} \tag{2.13}$$

where $\overline{\alpha} = (1-\sum_{i=1}^{m-2}\alpha_i)^{-1}$. Then (2.3) can be rewrite as

$$u(t) = \int_0^1 G(t,s,\alpha_i,\eta_i)\, g(s)\, ds. \tag{2.14}$$

Now we consider the linearized equation

$$x^{(4)}(t) = g(t), \quad 0 < t < 1, \tag{2.15}$$

subject to the boundary conditions (1.2). We have the following lemma.

Lemma 2.4. *Let $g \in C[0,1]$ be non-negative and*

$$\sum_{i=1}^{m-2} \alpha_{1i}\eta_{1i} \le 1 < \sum_{i=1}^{m-2}\alpha_{1i}, \quad \sum_{i=1}^{m-2}\alpha_{0i}\eta_{0i}^2 < 1 < \sum_{i=1}^{m-2}\alpha_{0i}.$$

Then (2.15), (1.2) has a unique non-negative solution

$$x(t) = \int_0^1 \Phi(t,s)g(s)ds := Ag(t), \tag{2.16}$$

where $\Phi(t,s)$ is the Green function

$$\Phi(t,s) = \int_0^1 G(t,\tau,\alpha_{0i},\eta_{0i})G(\tau,s,\alpha_{1i},\eta_{1i})\, d\tau, \quad \text{for } t,s \in [0,1]. \tag{2.17}$$

Moreover, we have $\min_{t\in[0,1]} x(t) \ge \gamma_0\|x\|$, where

$$\gamma_0 = \frac{1-\sum_{i=1}^{m-2}\alpha_{0i}\eta_{0i}^2}{\sum_{i=1}^{m-2}\alpha_{0i}(1-\eta_{0i}^2)}.$$

Proof. It follows from Lemma 2.1 that

$$-x''(t) = \int_0^1 G(t,s,\alpha_{1i},\eta_{1i})g(s)\, ds \le 0$$

is non-positive non-increasing for all $t \in [0,1]$. Thus, by Lemma 2.2,

$$\begin{aligned} x(t) &= \int_0^1 G(t,s,\alpha_{0i},\eta_{0i}) \int_0^1 G(s,\tau,\alpha_{1i},\eta_{1i})g(\tau)\, d\tau\, ds \\ &= \int_0^1 \Big(\int_0^1 G(t,\tau,\alpha_{0i},\eta_{0i})G(\tau,s,\alpha_{1i},\eta_{1i})\, d\tau \Big) g(s)ds \end{aligned}$$

$$= \int_0^1 \Phi(t,s)g(s)ds \geq 0, \quad t \in [0,1],$$

and $\min_{t\in[0,1]} x(t) \geq \gamma_0\|x\|$. The proof is complete. $\square$

The following result is straightforward and we will omit its proof.

Lemma 2.5. *The operator $A : C([0,1]) \to C([0,1])$, defined by (2.16), be a completely continuous linear operator. If g is a nonnegative function in $C([0,1])$ then Ag is also nonnegative.*

Next we give some properties of the Green function $\Phi(t,s)$ which is used in the sequel.

Lemma 2.6. *Let*

$$\sum_{i=1}^{m-2} \alpha_{1i}\eta_{1i} \leq 1 < \sum_{i=1}^{m-2} \alpha_{1i}, \quad \sum_{i=1}^{m-2} \alpha_{0i}\eta_{0i}^2 < 1 < \sum_{i=1}^{m-2} \alpha_{0i}.$$

Then we have

1. $\Phi(t,s) \geq 0$, *for all* $s,t \in [0,1]$;
2. *there exists a continuous function* $\chi : [0,1] \to [0,+\infty)$ *such that*

$$\gamma_0\chi(s) \leq \Phi(t,s) \leq \chi(s), \quad \forall s,t \in [0,1].$$

Proof. From (2.13) and the assumptions $\sum_{i=1}^{m-2} \alpha_{1i}\eta_{1i} \leq 1 < \sum_{i=1}^{m-2} \alpha_{1i}$, it is easy to check that, for each $s \in [0,1]$, $\tau \mapsto G(\tau, s, \alpha_{1i}, \eta_{1i})$ is a non-positive, non-increasing and continuous function. So by using (2.17) and the Lemma 2.2, the function $\Phi(t,s) \geq 0$ for all $s,t \in [0,1]$ and

$$\min_{t\in[0,1]} \Phi(t,s) \geq \gamma_0\|\Phi(\cdot,s)\| = \gamma_0\Phi(1,s).$$

Let $\chi(s) = \Phi(1,s)$. Obviously we have $\gamma_0\chi(s) \leq \Phi(t,s) \leq \chi(s)$. The proof is complete. $\square$

To study (1.1)-(1.2), we use the assumption

(A1) $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous

Let K be the cone in $C([0,1])$, consisting of all nonnegative functions and

$$P = \{x \in K : \min_{t\in[0,1]} x(t) \geq \gamma_0\|x\|\}$$

It is clear that P is also a cone in $C([0,1])$. For each $x \in P$, denote $F(x)(t) = \lambda f(t, x(t))$, $t \in [0,1]$. By the assumption (A1), the operator $F : P \to K$ is continuous. Therefore the operator $T := A \circ F : P \to K$ is completely continuous. On the other hand it is not difficult to check that for $x \in P$ we have

$$\min_{0\leq t\leq 1} Tx(t) \geq \gamma_0\|Tx\|$$

using the Lemma 2.6, that is $TP \subset P$.

We note that the nonzero fixed points of the operator T are positive solutions of (1.1)-(1.2). To finish this section we state here the Guo-Krasnoselskii's fixed point theorem (see [5])

Theorem 2.7. *Let X be a Banach space and $P \subset X$ be a cone in X. Assume Ω_1, Ω_2 are two open bounded subsets of X with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$ and $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that*

(i) $\|Tu\| \le \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \ge \|u\|$, $u \in P \cap \partial\Omega_2$, *or*
(ii) $\|Tu\| \ge \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \le \|u\|$, $u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Existence of positive solutions

We introduce the notation

$$f_0 := \liminf_{z\to 0^+} \min_{t\in[0,1]} \frac{f(t,z)}{z}, \quad f^\infty := \limsup_{z\to +\infty} \max_{t\in[0,1]} \frac{f(t,z)}{z},$$
$$f^0 := \limsup_{z\to 0^+} \max_{t\in[0,1]} \frac{f(t,z)}{z}, \quad f_\infty := \liminf_{z\to +\infty} \min_{t\in[0,1]} \frac{f(t,z)}{z},$$
$$A = \Big(\int_0^1 \Phi(1,s)ds\Big)^{-1}, \quad B = \frac{A}{\gamma_0}.$$

Theorem 3.1. *Assume that* (A1) *holds. Then* (1.1)-(1.2) *has at least one positive solution for every $\lambda \in \big(\frac{B}{f_0}, \frac{A}{f^\infty}\big)$ if $f_0, f^\infty \in (0,\infty)$ satisfy $f_0\gamma_0 > f^\infty$; or $\lambda \in \big(\frac{B}{f_\infty}, \frac{A}{f^0}\big)$ if $f^0, f_\infty \in (0,\infty)$ satisfy $f_\infty\gamma_0 > f^0$.*

Proof. Set
$$\Omega_i = \{x \in C([0,1]) : \|x\| < R_i\}, \quad i = 1, 2.$$
Then Ω_1, Ω_2 are two open bounded of $C([0,1])$ and $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$.
Case 1: $f_0, f^\infty \in (0,\infty)$ and $f_0\gamma_0 > f^\infty$. Let $\lambda \in (\frac{B}{f_0}, \frac{A}{f^\infty})$. Then there exists $\varepsilon > 0$ such that
$$\frac{B}{f_0 - \varepsilon} < \lambda < \frac{A}{f^\infty + \varepsilon}.$$
Since $f_0 \in (0,\infty)$ there exists $R_1 > 0$ such that $f(t,z) \ge (f_0 - \varepsilon)z$ for all $t \in [0,1], z \in [0, R_1]$. So if $x \in P$ such that $\|x\| = R_1$, we have
$$f(t,x(t)) \ge (f_0 - \varepsilon)x(t) \ge \gamma_0(f_0 - \varepsilon)\|x\|, \quad \forall t \in [0,1].$$
This implies
$$Tx(t) = \lambda \int_0^1 \Phi(t,s)f(s,x(s))ds \ge \lambda\gamma_0(f_0 - \varepsilon)\|x\| \int_0^1 \Phi(t,s)ds, \quad \forall t \in [0,1].$$
Hence, for all $x \in P \cap \partial\Omega_1$,
$$\|Tx\| \ge \lambda\gamma_0(f_0 - \varepsilon) \max_{0\le t\le 1} \Big(\int_0^1 \Phi(t,s)ds\Big)\|x\| \ge \|x\|.$$
On the other hand, since $f^\infty \in (0,\infty)$, there exists $R > 0$ such that $f(t,z) \le (f^\infty + \varepsilon)z$ for all $t \in [0,1], z \in [R, +\infty]$. Set $R_2 = \max\{R_1 + 1, R\gamma_0^{-1}\}$. Let us $x \in P \cap \partial\Omega_2$. We have
$$x(t) \ge \gamma_0\|x\| = \gamma_0 R_2, \quad \forall t \in [0,1].$$
So
$$Tx(t) = \lambda \int_0^1 \Phi(t,s)f(s,x(s))ds \le \lambda(f^\infty + \varepsilon)\|x\| \int_0^1 \Phi(t,s)ds.$$
Consequently, $\|Tx\| \le \|x\|$ for all $x \in P \cap \partial\Omega_2$. Therefore, using the second part of Theorem 2.7, we conclude that T has a fixed point in $P \cap \overline{\Omega}_2 \setminus \Omega_1$.

Case 2: $f^0, f_\infty \in (0,\infty)$ and $f_\infty\gamma_0 > f^0$. Let $\lambda \in (\frac{B}{f_\infty}, \frac{A}{f^0})$. Then there exists $\varepsilon > 0$ such that

$$\frac{B}{f_\infty - \varepsilon} < \lambda < \frac{A}{f^0 + \varepsilon}.$$

Using the arguments as in Case 1, we can find $R_2 > R_1 > 0$ such that $\|Tx\| \le \|x\|$, for all $x \in P \cap \partial\Omega_1$ and $\|Tx\| \ge \|x\|$, for all $x \in P \cap \partial\Omega_2$. So T has a fixed point in $P \cap \overline{\Omega}_2 \setminus \Omega_1$ which is a positive solution of (1.1)-(1.2), using the Theorem 2.7. $\square$

Next, we add the following assumption

(A2) The function $f(t,x)$ is nondecreasing about x.

Using the monotone iterative technique, we get the following result.

Theorem 3.2. *Let* (A1) *and* (A2) *hold. Assume that there exist two positive numbers* $R_1 < R_2$ *such that*

$$0 < R_1 \sup_{t\in[0,1]} f(t,R_2) < \gamma_0 R_2 \inf_{t\in[0,1]} f(t,\gamma_0 R_1).$$

Then if

$$\lambda \in \Big[\frac{BR_1}{\inf_{t\in[0,1]} f(t,\gamma_0 R_1)}, \frac{AR_2}{\sup_{t\in[0,1]} f(t,R_2)}\Big]$$

then (1.1)-(1.2) *has positive solutions* x_1^*, x_2^* *(*x_1^* *and* x_2^* *may coincide) with*

$$R_1 \le \|x_1^*\| \le R_2 \quad \text{and} \quad \lim_{n\to\infty} T^n x_0 = x_1^*, \quad \text{where } x_0(t) = R_2, \quad \forall t \in [0,1];$$

and

$$R_1 \le \|x_2^*\| \le R_2 \quad \text{and} \quad \lim_{n\to\infty} T^n \overline{x}_0 = x_2^*, \quad \text{where } \overline{x}_0(t) = R_1, \quad \forall t \in [0,1].$$

Proof. Set

$$P_{[R_1,R_2]} = \{x \in P : R_1 \le \|x\| \le R_2\}.$$

Let $x \in P_{[R_1,R_2]}$. It's clear that $\gamma_0 R_1 \le \gamma_0\|x\| \le x(t) \le \|x\| \le R_2$, for all $t \in [0,1]$. So

$$Tx(t) = \lambda \int_0^1 \Phi(t,s) f(s,x(s))ds \le \lambda \int_0^1 \Phi(t,s) f(s,R_2)ds \le R_2,$$

and

$$Tx(t) \ge \lambda \int_0^1 \Phi(t,s) f(s,\gamma_0 R_1)ds \ge \frac{AR_1}{\gamma_0} \int_0^1 \Phi(t,s)ds \ge AR_1 \int_0^1 \Phi(1,s)ds = R_1.$$

This implies that $TP_{[R_1,R_2]} \subset P_{[R_1,R_2]}$.

Let $x_0(t) = R_2$ for all $t \in [0,1]$. It is evident that $x_0 \in P_{[R_1,R_2]}$. We consider the sequence in $P_{[R_1,R_2]}$, $\{x_n\}_{n\in\mathbb{N}}$, defined by

$$x_n = Tx_{n-1} = T^n x_0, \quad n = 1,2,\dots. \tag{3.1}$$

Because T is the completely continuous operator, there exists a subseqence $\{x_{n_k}\}$ of $\{x_n\}$ which uniformly converges to $x_1^* \in C([0,1])$. On the other hand we can see that $T : P_{[R_1,R_2]} \to P_{[R_1,R_2]}$ is a nondecreasing operator using the assumption (A2). Therefore, since

$$0 \le x_1(t) \le \|x_1\| \le R_2 = x_0(t), \quad \forall t \in [0,1],$$

we have $Tx_1 \le Tx_0$, that is $x_2 \le x_1$. Similarly by induction we deduce that $x_{n+1} \le x_n$ for all $n \in \mathbb{N}$. Therefore, we can conclude that the sequence $\{x_n\}$ uniformly converges to x^*. Letting $n \to +\infty$ in (3.1) yields $Tx_1^* = x_1^*$.

Let $\overline{x}_0(t) = R_1$ for all $t \in [0,1]$ and $\overline{x}_n = T\overline{x}_{n-1}$ for $n = 1, 2, \dots$. It is clear that $x_n \in P_{[R_1,R_2]}$ for all $n \in \mathbb{N}$. Moreover, by definition of the operator T, we have

$$\begin{aligned}\overline{x}_1(t) = T\overline{x}_0(t) &= \lambda \int_0^1 \Phi(t,s) f(s, \overline{x}_0(s)) ds \\ &\geq \lambda \int_0^1 \Phi(t,s) f(s, \gamma_0 R_1) ds \geq R_1 \equiv \overline{x}_0(t),\end{aligned}$$

for $t \in [0,1]$. Therefore, by using the arguments as above, we deduce that $\{\overline{x}_n\}$ converges uniformly to $x_2^* \in P_{[R_1,R_2]}$ and $Tx_2^* = x_2^*$. The proof is complete. $\square$

Example 3.3. Let a, b, c, d be positive numbers such that $5bc > 42ad$. We consider the boundary-value problem

$$\begin{gathered} x^{(4)}(t) = (t^2+1)\frac{ax^2(t) + bx(t)}{cx(t) + d}, \quad 0 < t < 1, \\ x'(0) = x^{(3)}(0) = 0, \\ x(1) = \frac{3}{2}x(\frac{3}{4}),\ x''(1) = \frac{4}{3}x''(\frac{1}{2}). \end{gathered}$$

We have $\gamma_0 = \frac{5}{21}$,

$$G(t,\tau,\alpha_{01},\eta_{01}) = \begin{cases} \tau - t & \text{if } 0 \leq \tau \leq t \leq 1 \\ 0 & \text{if } 0 \leq t \leq \tau \leq 1 \end{cases} + \begin{cases} \frac{1}{4} - \tau & \text{if } 0 \leq \tau \leq \frac{3}{4} \\ 2\tau - 2 & \text{if } \frac{3}{4} \leq \tau \leq 1 \end{cases}$$

and

$$G_1(\tau,s,\alpha_{11},\eta_{11}) = \begin{cases} s - \tau & \text{if } 0 \leq s \leq \tau \leq 1 \\ 0 & \text{if } 0 \leq \tau \leq s \leq 1 \end{cases} - \begin{cases} 1 + s & \text{if } 0 \leq s \leq \frac{1}{2} \\ 3(1-s) & \text{if } \frac{1}{2} \leq s \leq 1 \,. \end{cases}$$

By doing some calculating, $\Phi(t,s)$ is defined as follows: For $s \leq t$,

$$\begin{aligned}\Phi(t,s) &= -\frac{1}{6}(s-t)^3 \\ &\quad + \begin{cases} -\frac{5}{32}s + (\frac{1}{2}t^2 + \frac{5}{32})(s+1) - \frac{1}{8}s^2 + \frac{1}{6}s^3 + \frac{47}{384} & \text{if } 0 \leq s \wedge s \leq \frac{1}{2} \\ -\frac{5}{32}s - (3s-3)(\frac{1}{2}t^2 + \frac{5}{32}) - \frac{1}{8}s^2 + \frac{1}{6}s^3 + \frac{47}{384} & \text{if } \frac{1}{2} \leq s \wedge s \leq \frac{3}{4} \\ -(3s-3)(\frac{1}{2}t^2 + \frac{5}{32}) - \frac{1}{3}(s-1)^3 & \text{if } s \leq 1 \wedge \frac{3}{4} \leq s; \end{cases}\end{aligned}$$

and for $t \leq s$,

$$\Phi(t,s) = + \begin{cases} -\frac{5}{32}s + (\frac{1}{2}t^2 + \frac{5}{32})(s+1) - \frac{1}{8}s^2 + \frac{1}{6}s^3 + \frac{47}{384} & \text{if } 0 \leq s \wedge s \leq \frac{1}{2} \\ -\frac{5}{32}s - (3s-3)(\frac{1}{2}t^2 + \frac{5}{32}) - \frac{1}{8}s^2 + \frac{1}{6}s^3 + \frac{47}{384} & \text{if } \frac{1}{2} \leq s \wedge s \leq \frac{3}{4} \\ -(3s-3)(\frac{1}{2}t^2 + \frac{5}{32}) - \frac{1}{3}(s-1)^3 & \text{if } s \leq 1 \wedge \frac{3}{4} \leq s \end{cases}$$

So $A = \left(\int_0^1 \Phi(1,s) ds\right)^{-1} = 103/128$. Now we set

$$f(t,x) = (t^2+1)\frac{ax^2 + bx}{cx + d}.$$

Then $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and

$$\begin{gathered} f_0 = \lim_{x \to 0^+} \min_{0 \leq t \leq 1} \frac{f(t,x)}{x} = \lim_{x \to 0^+} \frac{ax^2 + bx}{cx^2 + dx} = \frac{b}{d}, \\ f^\infty = \lim_{x \to \infty} \max_{0 \leq t \leq 1} \frac{f(t,x)}{x} = 2 \lim_{x \to \infty} \frac{ax^2 + bx}{cx^2 + dx} = \frac{2a}{c}; \end{gathered}$$

that is, $\gamma_0 f_0 > f^\infty$. Thus, by Theorem 3.1, we conclude that for each $\lambda \in (\frac{2163d}{640b}, \frac{103c}{256a})$ our problem has at least one positive solution.

4. Compactness of the set of positive solutions

Theorem 4.1. *Let* (A1) *hold. Assume that we have*

$$f_0, f^\infty \in (0,\infty), \quad f_0\gamma_0 > f^\infty \quad \text{and} \quad \lambda \in \Big(\frac{B}{f_0}, \frac{A}{f^\infty}\Big). \tag{4.1}$$

Then the set of positive solutions of (1.1)-(1.2) *is nonempty and compact.*

Proof. Put $S = \{x \in P : x = Tx\}$. By Theorem 3.1 S is nonempty. We shall show that S is compact in $C([0,1])$.

First we claim that S is a closed subset of $C([0,1])$. Indeed, assume that $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in S and $\lim_{n\to\infty}\|x_n - x\| = 0$. Then for each $t \in [0,1]$, we have

$$\begin{aligned}
&\big|x(t) - \lambda \int_0^1 \Phi(t,s) f(s,x(s))ds\big| \\
&\le |x(t) - x_n(t)| + \big|x_n(t) - \lambda \int_0^1 \Phi(t,s) f(s,x_n(s))ds\big| \\
&\quad + \lambda \big|\int_0^1 \Phi(t,s) f(s,x(s))ds - \int_0^1 \Phi(t,s) f(s,x_n(s))ds\big|.
\end{aligned}$$

This implies

$$\begin{aligned}
&\big|x(t) - \lambda \int_0^1 \Phi(t,s) f(s,x(s))ds\big| \\
&\le |x(t) - x_n(t)| + \lambda \int_0^1 \Phi(t,s)|f(s,x(s)) - f(s,x_n(s))|ds,
\end{aligned}$$

because $x_n = Tx_n$ for all $n \in \mathbb{N}$. Let $n \to \infty$ in the last inequality we can deduce that

$$x(t) = \lambda \int_0^1 \Phi(t,s) f(s,x(s))ds, \quad \forall t \in [0,1],$$

using the continuity of the function f and the dominated convergence theorem. So $x \in S$ and S is closed in $C([0,1])$. It remains to check that S is relatively compact in $C([0,1])$. Let (4.1) holds. Choosing $\varepsilon^* > 0$ such that

$$\frac{B}{f_0 - \varepsilon^*} < \lambda < \frac{A}{f^\infty + \varepsilon^*}.$$

Clearly there exists a constant $R > 0$ such that $f(t,z) \le (f^\infty + \varepsilon^*)z$, for all $t \in [0,1]$ and $z \in [R,\infty)$. Hence

$$f(t,x(t)) \le (f^\infty + \varepsilon^*)x(t) + \beta,\ t \in [0,1],$$

where $\beta = \max\{f(t,z) : (t,z) \in [0,1]\times[0,R]\}$. So, for $x \in S$ and for every $t \in [0,1]$, we have

$$\begin{aligned}
x(t) &= \lambda \int_0^1 \Phi(t,s) f(s,x(s))ds \\
&\le \lambda \int_0^1 \Phi(t,s)[(f^\infty + \varepsilon^*)x(s) + \beta]ds \\
&\le \frac{\lambda}{A}(f^\infty + \varepsilon^*)\|x\| + \frac{\lambda\beta}{A}.
\end{aligned}$$

We can deduce from this inequality that $\|x\| \le \frac{\lambda\beta}{A-\lambda(f^\infty+\varepsilon^*)}$; that is, S is bounded in $C([0,1])$. By the compactness of the operator $T : P \to P$ we conclude that $S = T(S)$ is relatively compact. The proof is complete. $\square$

Remark 4.2. Assume that $f^0, f_\infty \in (0,\infty)$, $f_\infty\gamma_0 > f^0$, $f^\infty \le f^0$ and

$$\lambda \in \big(\frac{B}{f_\infty}, \frac{A}{f^0}\big).$$

Thanks to Theorem 2.7, the set of positive solutions S of the problem (1.1) (1.2) is nonempty. Moreover by the similar arguments we can show that S is compact in $C([0,1])$.

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