PERIODIC SOLUTIONS FOR A SECOND-ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATION WITH VARIABLE DELAY

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ABSTRACT. In this work, the hybrid fixed point theorem of Krasnoselskii is used to prove the existence of periodic solutions of the second-order nonlinear neutral differential equation

$$\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)x(t) = \frac{d}{dt}g(t, x(t - \tau(t))) + f(t, x(t), x(t - \tau(t))).$$

We transform the problem into an integral equation and uniqueness of the periodic solution, by means of the contraction mapping principle.

1. INTRODUCTION

Due to their importance in numerous application in physics, population dynamics, industrial robotics, and other areas, many authors have studying the existence, uniqueness, stability and positivity of solutions for delay differential equations; see the references in this article and references therein.

The primary motivation for this work is the work by Dib et al. [9] and Wang et al. [19]. In these papers, the authors used Krasnoselskii's fixed point theorem to establish the existence of periodic solutions for the nonlinear neutral differential equations

$$\frac{d}{dt}x(t) = -a(t)x(t) + \frac{d}{dt}g(t, x(t - \tau(t))) + f(t, x(t), x(t - \tau(t))),$$

and

$$\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)x(t) = r(t)\frac{d}{dt}x(t - \tau(t)) + f(t, x(t), x(t - \tau(t))).$$

Some authors have used the contraction mapping principle to show the uniqueness of periodic solutions of these equations.

In this work, we show the existence and uniqueness of solutions for the second-order nonlinear neutral differential equation

$$\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)x(t) = \frac{d}{dt}g(t, x(t - \tau(t))) + f(t, x(t), x(t - \tau(t))), \quad (1.1)$$

where p and q are positive continuous real-valued functions. The function $g :$ $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is differentiable and $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ continuous in their respective arguments. To show the existence of periodic solutions, we transform (1.1) into an integral equation and then use Krasnoselskii's fixed point theorem. The obtained integral equation is the sum of two mappings, one is a contraction and the other is compact. Also, the transformation of equation (1.1) enables us to show the uniqueness of the periodic solution by the contraction mapping principle.

Note that in our consideration the neutral term $\frac{d}{dt}g(t, x(t - \tau(t)))$ of (1.1) produces nonlinearity in the derivative term $\frac{d}{dt}x(t - \tau(t))$. While, the neutral term $\frac{d}{dt}x(t - \tau(t))$ in [19] enters linearly. As a consequence, our analysis is different from that in [19].

The organization of this article is as follows. In Section 2, we introduce some notation and state some preliminary results needed in later sections. Then we give the Green's function of (1.1), which plays an important role in this paper. Also, we present the inversion of (1.1) and Krasnoselskii's fixed point theorem. For details on Krasnoselskii theorem we refer the reader to [18]. In Section 3, we present our main results on existence and uniqueness.

2. Preliminaries

For $T > 0$, let P_T be the set of continuous scalar functions x that are periodic in t, with period T. Then $(P_T, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t \in \mathbb{R}} |x(t)| = \sup_{t \in [0,T]} |x(t)|.$$

Since we are searching for periodic solutions for (1.1), it is natural to assume that

$$p(t + T) = p(t), \quad q(t + T) = q(t), \quad \tau(t + T) = \tau(t), \tag{2.1}$$

with τ being scalar function, continuous, and $\tau(t) \geq \tau^* > 0$. Also, we assume

$$\int_0^T p(s)ds > 0, \quad \int_0^T q(s)ds > 0. \tag{2.2}$$

Functions $g(t, x)$ and $f(t, x, y)$ are periodic in t with period T. They are also supposed to be globally Lipschitz continuous in x and in x and y, respectively. That is,

$$g(t + T, x) = g(t, x), \quad f(t + T, x, y) = f(t, x, y), \tag{2.3}$$

and there are positive constants k_1, k_2, k_3 such that

$$|g(t, x) - g(t, y)| \leq k_1 \|x - y\|, \tag{2.4}$$

and

$$|f(t, x, y) - f(t, z, w)| \leq k_2 \|x - z\| + k_3 \|y - w\|. \tag{2.5}$$

Lemma 2.1 ([14]). *Suppose that* (2.1) *and* (2.2) *hold and*

$$\frac{R_1[\exp(\int_0^T p(u)du) - 1]}{Q_1 T} \geq 1, \tag{2.6}$$

where

$$R_1 = \max_{t \in [0,T]} \Big| \int_t^{t+T} \frac{\exp(\int_t^s p(u)du)}{\exp(\int_0^T p(u)du) - 1} q(s)ds \Big|,$$

$$Q_1 = \Big(1 + \exp\Big(\int_0^T p(u)du\Big)\Big)^2 R_1^2.$$

Then there are continuous and T-periodic functions a and b such that b(t) > 0,
$\int_0^T a(u)du > 0$, *and*

$$a(t) + b(t) = p(t), \quad \frac{d}{dt}b(t) + a(t)b(t) = q(t), \quad for \ t \in \mathbb{R}.$$

Lemma 2.2 ([19])**.** *Suppose the conditions of Lemma 2.1 hold and $\phi \in P_T$. Then the equation*

$$\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)x(t) = \phi(t),$$

has a T-periodic solution. Moreover, the periodic solution can be expressed as

$$x(t) = \int_t^{t+T} G(t,s)\phi(s)ds,$$

where

$$G(t,s) = \frac{\int_t^s \exp[\int_t^u b(v)dv + \int_u^s a(v)dv]du + \int_s^{t+T} \exp[\int_t^u b(v)dv + \int_u^{s+T} a(v)dv]du}{[\exp\big(\int_0^T a(u)du\big) - 1][\exp\big(\int_0^T b(u)du\big) - 1]}.$$

Corollary 2.3. [19] *Green's function G satisfies the following properties*

$$G(t, t+T) = G(t,t), \quad G(t+T, s+T) = G(t,s),$$

$$\frac{\partial}{\partial s}G(t,s) = a(s)G(t,s) - \frac{\exp\big(\int_t^s b(v)dv\big)}{\exp\big(\int_0^T b(v)dv\big) - 1},$$

$$\frac{\partial}{\partial t}G(t,s) = -b(t)G(t,s) + \frac{\exp\big(\int_t^s a(v)dv\big)}{\exp\big(\int_0^T a(v)dv\big) - 1}.$$

The following lemma is essential for our results.

Lemma 2.4. *Suppose (2.1)-(2.3) and (2.6) hold. If $x \in P_T$, then x is a solution of (1.1) if and only if*

$$x(t) = \int_t^{t+T} E(t,s)g(s, x(s - \tau(s)))ds$$

$$+ \int_t^{t+T} G(t,s)[-a(s)g(s, x(s - \tau(s))) + f(s, x(s), x(s - \tau(s)))]ds, \tag{2.7}$$

where

$$E(t,s) = \frac{\exp(\int_t^s b(v)dv)}{\exp(\int_0^T b(v)dv) - 1}. \tag{2.8}$$

Proof. Let $x \in P_T$ be a solution of (1.1). From Lemma 2.2, we have

$$x(t) = \int_t^{t+T} G(t,s)[\frac{\partial}{\partial s}g(s, x(s - \tau(s))) + f(s, x(s), x(s - \tau(s)))]ds. \tag{2.9}$$

Integrating by parts, we have

$$
\int_t^{t+T} G(t,s)\frac{\partial}{\partial s}g(s,x(s-\tau(s)))ds
$$
$$
= -\int_t^{t+T} [\frac{\partial}{\partial s}G(t,s)]g(s,x(s-\tau(s)))ds \tag{2.10}
$$
$$
= \int_t^{t+T} g(s,x(s-\tau(s)))[E(t,s)-a(s)G(t,s)]ds,
$$

where E is given by (2.8). Then substituting (2.10) in (2.9) completes the proof. $\square$

Lemma 2.5. [19] *Let* $A = \int_0^T p(u)du$, $B = T^2 \exp\left(\frac{1}{T}\int_0^T \ln(q(u))du\right)$. *If*

$$
A^2 \ge 4B, \tag{2.11}
$$

then

$$
\min\left\{\int_0^T a(u)du, \int_0^T b(u)du\right\} \ge \frac{1}{2}(A - \sqrt{A^2-4B}) := l,
$$
$$
\max\left\{\int_0^T a(u)du, \int_0^T b(u)du\right\} \le \frac{1}{2}(A + \sqrt{A^2-4B}) := m.
$$

Corollary 2.6. [19] *Functions* G *and* E *satisfy*

$$
\frac{T}{(e^m-1)^2} \le G(t,s) \le \frac{T\exp\left(\int_0^T p(u)du\right)}{(e^l-1)^2}, \quad |E(t,s)| \le \frac{e^m}{e^l-1}.
$$

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of periodic solutions to (1.1). For its proof we refer the reader to [18].

Theorem 2.7 (Krasnoselskii). *Let* $\mathbb{M}$ *be a closed convex nonempty subset of a Banach space* $(\mathbb{B}, \|\cdot\|)$. *Suppose that* $\mathcal{A}$ *and* $\mathcal{B}$ *map* $\mathbb{M}$ *into* $\mathbb{B}$ *such that*

  (i) $x, y \in \mathbb{M}$, *implies* $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$,
  (ii) $\mathcal{A}$ *is compact and continuous*,
  (iii) $\mathcal{B}$ *is a contraction mapping*.

Then there exists $z \in \mathbb{M}$ *with* $z = \mathcal{A}z + \mathcal{B}z$.

3. MAIN RESULTS

We present our existence results in this section. To this end, we first define the operator $H : P_T \to P_T$ by

$$
(H\varphi)(t) = \int_t^{t+T} G(t,s)[-a(s)g(s,\varphi(s-\tau(s))) + f(s,\varphi(s),\varphi(s-\tau(s)))]ds
$$
$$
+ \int_t^{t+T} E(t,s)g(s,\varphi(s-\tau(s)))ds. \tag{3.1}
$$

From Lemma 2.4, we see that fixed points of H are solutions of (1.1) and vice versa. To use Theorem 2.7 we need to express the operator H as the sum of two

operators, one of which is compact and the other of which is a contraction. Let $(H\varphi)(t) = (\mathcal{A}\varphi)(t) + (\mathcal{B}\varphi)(t)$ where

$$(\mathcal{A}\varphi)(t) = \int_t^{t+T} G(t,s)[-a(s)g(s,\varphi(s-\tau(s))) + f(s,\varphi(s),\varphi(s-\tau(s)))]ds, \quad (3.2)$$

$$(\mathcal{B}\varphi)(t) = \int_t^{t+T} E(t,s)g(s,\varphi(s-\tau(s)))ds. \quad (3.3)$$

To simplify notation, we introduce the constants

$$\alpha = \frac{T\exp\left(\int_0^T p(u)du\right)}{(e^l-1)^2}, \quad \beta = \frac{e^m}{e^l-1}, \quad \gamma = \max_{t\in[0,T]}|a(t)|, \quad \lambda = \max_{t\in[0,T]}|b(t)|. \quad (3.4)$$

Lemma 3.1. *Suppose that conditions (2.1)-(2.6) and (2.11) hold. Then $\mathcal{A}: P_T \to P_T$ is compact.*

Proof. Let $\mathcal{A}$ be defined by (3.2). Obviously, $\mathcal{A}\varphi$ is continuous and it is easy to show that $(\mathcal{A}\varphi)(t+T) = (\mathcal{A}\varphi)(t)$. To see that $\mathcal{A}$ is continuous, we let $\varphi, \psi \in P_T$. Given $\varepsilon > 0$, take $\eta = \varepsilon/N$ with $N = \alpha T(\gamma k_1 + k_2 + k_3)$ where k_1, k_2 and k_3 are given by (2.4) and (2.5). Now, for $\|\varphi - \psi\| < \eta$, we obtain

$$\|\mathcal{A}\varphi - \mathcal{A}\psi\| \leq \alpha \int_t^{t+T} [\gamma k_1\|\varphi-\psi\| + (k_2+k_3)\|\varphi-\psi\|]ds \leq N\|\varphi-\psi\| < \varepsilon.$$

This proves that $\mathcal{A}$ is continuous. To show that the image of $\mathcal{A}$ is contained in a compact set, we consider $\mathbb{D} = \{\varphi \in P_T : \|\varphi\| \leq L\}$, where L is a fixed positive constant. Let $\varphi_n \in \mathbb{D}$, where n is a positive integer. Observe that in view of (2.4) and (2.5), we have

$$|g(t,x)| = |g(t,x) - g(t,0) + g(t,0)|$$
$$\leq |g(t,x) - g(t,0)| + |g(t,0)|$$
$$\leq k_1\|x\| + \rho_1.$$

Similarly,

$$|f(t,x,y)| = |f(t,x,y) - f(t,0,0) + f(t,0,0)|$$
$$\leq |f(t,x,y) - f(t,0,0)| + |f(t,0,0)|$$
$$\leq k_2\|x\| + k_3\|y\| + \rho_2,$$

where $\rho_1 = \max_{t\in[0,T]}|g(t,0)|$ and $\rho_2 = \max_{t\in[0,T]}|f(t,0,0)|$. Hence, if $\mathcal{A}$ is given by (3.2) we obtain

$$\|\mathcal{A}\varphi_n\| \leq D,$$

for some positive constant D. Next we calculate $\frac{d}{dt}(\mathcal{A}\varphi_n)(t)$ and show that it is uniformly bounded. By making use of (2.1), (2.2) and (2.3) we obtain by taking the derivative in (3.2) that

$$\frac{d}{dt}(\mathcal{A}\varphi_n)(t) = \int_t^{t+T} [-b(t)G(t,s) + \frac{\exp\left(\int_t^s a(v)dv\right)}{\exp\left(\int_0^T a(v)dv\right) - 1}]$$
$$\times [-a(s)g(s,\varphi_n(s-\tau(s))) + f(s,\varphi_n(s),\varphi_n(s-\tau(s)))]ds.$$

Consequently, by invoking (2.4), (2.5) and (3.4), we obtain

$$|\frac{d}{dt}(\mathcal{A}\varphi_n)(t)| \leq T(\lambda\alpha + \beta)[\gamma(k_1L + \rho_1) + (k_2+k_3)L + \rho_2] \leq M,$$

for some positive constant M. Hence the sequence $(\mathcal{A}\varphi_n)$ is uniformly bounded and equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $(\mathcal{A}\varphi_{n_k})$ of $(\mathcal{A}\varphi_n)$ converges uniformly to a continuous T-periodic function. Thus $\mathcal{A}$ is continuous and $\mathcal{A}(\mathbb{D})$ is a compact set. $\square$

Lemma 3.2. *If $\mathcal{B}$ is given by* (3.3) *with*

$$k_1 \beta T < 1, \tag{3.5}$$

then $\mathcal{B} : P_T \to P_T$ is a contraction.

Proof. Let $\mathcal{B}$ be defined by (3.3). It is easy to show that $(\mathcal{B}\varphi)(t + T) = (\mathcal{B}\varphi)(t)$. To see that $\mathcal{B}$ is a contraction. Let $\varphi, \psi \in P_T$ we have

$$\|\mathcal{B}\varphi - \mathcal{B}\psi\| = \sup_{t \in [0,T]} |(\mathcal{B}\varphi)(t) - (\mathcal{B}\psi)(t)| \le k_1 \beta T \|\varphi - \psi\|.$$

Hence $\mathcal{B} : P_T \to P_T$ is a contraction. $\square$

Theorem 3.3. *Let α, β and γ be given by* (3.4). *Suppose that conditions* (2.1)-(2.6), (2.11) *and* (3.5) *hold. Suppose there exists a positive constant J satisfying the inequality*

$$[(\alpha\gamma + \beta)\rho_1 + \alpha\rho_2]T + [\alpha(\gamma k_1 + k_2 + k_3) + k_1\beta]TJ \le J.$$

Then (1.1) *has a solution $x \in P_T$ such that $\|x\| \le J$.*

Proof. Define $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \le J\}$. By Lemma 3.1, the operator $\mathcal{A} : \mathbb{M} \to P_T$ is compact and continuous. Also, from Lemma 3.2, the operator $\mathcal{B} : \mathbb{M} \to P_T$ is a contraction. Conditions (i) and (ii) of Theorem 2.7 are satisfied. We need to show that condition (iii) is fulfilled. To this end, let $\varphi, \psi \in \mathbb{M}$. Then

$$|(\mathcal{A}\varphi)(t) + (\mathcal{B}\psi)(t)|$$
$$\le \alpha \int_t^{t+T} [\gamma(k_1\|\varphi\| + \rho_1) + (k_2 + k_3)\|\varphi\| + \rho_2]ds + \beta \int_t^{t+T} (k_1\|\psi\| + \rho_1)ds$$
$$\le [(\alpha\gamma + \beta)\rho_1 + \alpha\rho_2]T + [\alpha(\gamma k_1 + k_2 + k_3) + k_1\beta]TJ \le J.$$

Thus $\|\mathcal{A}\varphi + \mathcal{B}\psi\| \le J$ and so $\mathcal{A}\varphi + \mathcal{B}\psi \in \mathbb{M}$. All the conditions of Theorem 2.7 are satisfied and consequently the operator H defined in (3.1) has a fixed point in $\mathbb{M}$. By Lemma 2.4 this fixed point is a solution of (1.1) and the proof is complete. $\square$

Theorem 3.4. *Let α, β and γ be given by* (3.4). *Suppose that conditions* (2.1)-(2.6), (2.11) *and* (3.5) *hold. If*

$$[\alpha(\gamma k_1 + k_2 + k_3) + k_1\beta]T < 1,$$

then (1.1) *has a unique T-periodic solution.*

Proof. Let the mapping H be given by (3.1). For $\varphi, \psi \in P_T$, we have

$$|(H\varphi)(t) - (H\psi)(t)|$$
$$\le \alpha \int_t^{t+T} [\gamma(k_1\|\varphi - \psi\|) + (k_2 + k_3)\|\varphi - \psi\|]ds + \beta \int_t^{t+T} k_1\|\varphi - \psi\|ds.$$

Hence, $\|H\varphi - H\psi\| \le [\alpha(\gamma k_1 + k_2 + k_3) + k_1\beta]T\|\varphi - \psi\|$. By the contraction mapping principle, H has a fixed point in P_T and by Lemma 2.4, this fixed point is a solution of (1.1). The proof is complete. $\square$

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