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KWONG-WONG-TYPE INTEGRAL EQUATION ON TIME SCALES

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Abstract. Consider the second-order nonlinear dynamic equation

$$[r(t)x^{\Delta}(\rho(t))]^{\Delta} + p(t)f(x(t)) = 0,$$

where $p(t)$ is the backward jump operator. We obtain a Kwong-Wong-type integral equation, that is: If $x(t)$ is a nonoscillatory solution of the above equation on $[T_0, \infty)$, then the integral equation

$$\frac{r^{\sigma}(t)x^{\Delta}(t)}{f(x^{\sigma}(t))} = P^{\sigma}(t) + \int_{\sigma(t)}^{\infty} \frac{r^{\sigma}(s)[\int_0^1 f'(x_h(s))dh][x^{\Delta}(s)]^2}{f(x(s))f(x^{\sigma}(s))}\Delta s$$

is satisfied for $t \geq T_0$, where $P^{\sigma}(t) = \int_{\sigma(t)}^{\infty} p(s)\Delta s$, and $x_h(s) = x(s) + h\mu(s)x^{\Delta}(s)$. As an application, we show that the superlinear dynamic equation

$$[r(t)x^{\Delta}(\rho(t))]^{\Delta} + p(t)f(x(t)) = 0,$$

is oscillatory, under certain conditions.

1. Introduction

Consider the second order nonlinear dynamic equation

$$(r(t)x^{\Delta}(\rho(t)))^{\Delta} + p(t)f(x(t)) = 0, \tag{1.1}$$

where $r(t), p(t) \in C(\mathbb{T}, R)$, $f(x) \in C(R, R)$, $r(t) > 0$ and $\int_{T_0}^{\infty}[r^{\sigma}(t)]^{-1}\Delta t = \infty$. We assume that $\lim_{t\to\infty}\int_{T_0}^{t} p(s)\Delta s$ exists and is finite;

$$xf(x) > 0, \text{ for } x \neq 0 \text{ and } f'(x) \geq 0; \tag{1.2}$$

$$\lim_{x\to\pm\infty} f(x) = \pm\infty. \tag{1.3}$$

When $\mathbb{T} = \mathbb{R}$, Equation (1.1) becomes the second-order nonlinear differential equations

$$x''(t) + p(t)f(x(t)) = 0. \tag{1.4}$$

Kwong and Wong [4] proved the following result.

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Theorem 1.1. *Suppose that f satisfies* (1.2) *and* (1.3) *and* $\lim_{t\to\infty}\int_{T_0}^t p(s)\Delta s$ *exists and finite. If $x(t)$ is a nonoscillatory solution of* (1.4) *on $[T_0,\infty)$, then the integral equation*

$$\frac{x'(t)}{f(x(t))} = P(t) + \int_t^\infty \frac{f'(x(s))[x'(s)]^2}{f^2(x(s))} ds$$

is satisfied for $t \ge T_0$, where $P(t) = \int_t^\infty p(s)\Delta s$.

We note that Theorem 1.1 has been used by Naito [6] in proving results on asymptotic behavior of nonoscillatory solution of equation (1.4).

In this article, we extend Theorem 1.1 to dynamic equations on time scales. As an application, we prove that the superlinear dynamic equation

$$[r(t)x^\Delta(\rho(t))]^\Delta + p(t)f(x(t)) = 0$$

is oscillatory, if

$$\limsup_{t\to\infty} \frac{1}{r^\sigma(t)} \int_{T_0}^t P^\sigma(s)\Delta s = \infty,$$

where $f(x)$ satisfies the superlinearity conditions

$$0 < \int_\epsilon^\infty \frac{dx}{f(x)}, \quad \int_{-\infty}^{-\epsilon} \frac{dx}{f(x)} < \infty, \quad \text{for all } \epsilon > 0. \tag{1.5}$$

It should be pointed out that our proof of the main theorem is different from the one in Kwong and Wong for differential equation in [4].

For completeness, we recall some basic results for dynamic equations and the calculus on time scales; see [1] and [2] for elementary results for the time scale calculus. Let $\mathbb{T}$ be a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup\mathbb{T} = \infty$. The forward jump operator is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\sup\emptyset = \inf\mathbb{T}$, where $\emptyset$ denotes the empty set. If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$ we say t is left-scattered. If $\sigma(t) = t$ we say t is right-dense, while if $\rho(t) = t$ and $t \ne \inf\mathbb{T}$ we say t is left-dense. Given a time scale interval $[c,d]_\mathbb{T} := \{t \in \mathbb{T} : c \le t \le d\}$ in $\mathbb{T}$ the notation $[c,d]_\mathbb{T}^\kappa$ denotes the interval $[c,d]_\mathbb{T}$ in case $\rho(d) = d$ and denotes the interval $[c,d)_\mathbb{T}$ in case $\rho(d) < d$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$. We say that $x : \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}$ provided

$$x^\Delta(t) := \lim_{s\to t} \frac{x(t) - x(s)}{t - s},$$

exists when $\sigma(t) = t$ (here by $s \to t$ it is understood that s approaches t in the time scale) and when x is continuous at t and $\sigma(t) > t$

$$x^\Delta(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)}.$$

Note that if $\mathbb{T} = \mathbb{R}$, then the delta derivative is just the standard derivative, and when $\mathbb{T} = \mathbb{Z}$ the delta derivative is just the forward difference operator. Hence our results contain the discrete and continuous cases as special cases and generalize these results to arbitrary time scales.

2. Lemmas

The following condition was introduced in [7].

Condition (H): We say that $\mathbb{T}$ satisfies Condition (H), provided one of the following holds:

(1) There exists a strictly increasing sequence $\{t_n\}_{n=0}^{\infty} \subset \mathbb{T}$ with $\lim_{n\to\infty} t_n = \infty$ and for each $n \geq 0$ either $\sigma(t_n) = t_{n+1}$ or the real interval $[t_n, t_{n+1}] \subset \mathbb{T}$; or

(2) $\mathbb{T} \bigcap \mathbb{R} = [T_0, \infty)$ for some $T_0 \in \mathbb{T}$.

We say $\mathbb{T}$ is a **regular time scale** provided it is a time scale with $\inf \mathbb{T} = T_0$, $\sup \mathbb{T} = \infty$ and $\mathbb{T}$ is either an isolated time scale (all points in $\mathbb{T}$ are isolated) or $\mathbb{T}$ is the real interval $[T_0, \infty)$. Note that in every regular time scale the backward jump operator is (delta) differentiable (which is used in the proof of the following theorem).

Remark 2.1. Time scales that satisfy Condition (H) include most of the important time scales, such as $\mathbb{R}$, $\mathbb{Z}$, $q^{\mathbb{N}_0}$, harmonic numbers $\{\sum_{k=1}^{n} \frac{1}{k},\ n \in \mathbb{N}\}$, etc.

We need the following lemmas.

Lemma 2.2. *Assume that $\mathbb{T}$ satisfies Condition (H) and the function $g(t) > 0$ for $t \in [T_0, \infty)$. Then we have for $t \in [T_0, \infty)_{\mathbb{T}}$,*

$$\int_{T_0}^{t} \frac{g^{\Delta}(s)}{g(s)} \Delta s \geq \ln \frac{g(t)}{g(T_0)}.$$

Proof. Assume that $t = t_{i-1} < t_i = \sigma(t)$. Then

$$\int_{t}^{\sigma(t)} \frac{g^{\Delta}(s)}{g(s)} \Delta s = \frac{g^{\Delta}(t)\mu(t)}{g(t)} = \frac{g(\sigma(t)) - g(t)}{g(t)}. \tag{2.1}$$

We consider the two possible cases: (i) $g(t) \leq g(\sigma(t))$ and (ii) $g(t) > g(\sigma(t))$. First, if $g(t) \leq g(\sigma(t))$ we have

$$\frac{g(\sigma(t)) - g(t)}{g(t)} \geq \int_{g(t)}^{g(\sigma(t))} \frac{1}{v} dv = \ln \frac{g(\sigma(t))}{g(t)}. \tag{2.2}$$

On the other hand, if $g(t) > g(\sigma(t))$, then

$$\frac{g(t) - g(\sigma(t))}{g(t)} \leq \int_{g(\sigma(t))}^{g(t)} \frac{1}{v} ds = \ln \frac{g(t)}{g(\sigma(t))},$$

which implies that

$$\frac{g(\sigma(t)) - g(t)}{g(t)} \geq \ln \frac{g(\sigma(t))}{g(t)}. \tag{2.3}$$

Hence, whenever $t_{i-1} = t < \sigma(t) = t_i$, we have from (2.1) and (2.2) in the first case and (2.1) and (2.3) in the second case, that

$$\int_{t_{i-1}}^{t_i} \frac{g^{\Delta}(s)}{g(s)} \Delta s \geq \ln \frac{g(\sigma(t))}{g(t)} = \ln \frac{g(t_i)}{g(t_{i-1})}. \tag{2.4}$$

If the real interval $[t_{i-1}, t_i] \subset \mathbb{T}$, then

$$\int_{t_{i-1}}^{t_i} \frac{g^{\Delta}(s)}{g(s)} \Delta s = \ln \frac{g(t_i)}{g(t_{i-1})}. \tag{2.5}$$

and so (2.4) also holds in this case.

Note that since $\mathbb{T}$ satisfies condition (H), we have from (2.4), (2.5) and the additivity of the integral that for $t \in [T_0, \infty)_{\mathbb{T}}$

$$\int_{T_0}^{t} \frac{g^{\Delta}(s)}{g(s)} \Delta s \geq \ln \frac{g(t)}{g(T_0)}. \tag{2.6}$$

□

Lemma 2.3. *Suppose that* $\mathbb{T}$ *satisfies Condition* (H). $x(t) > 0$ *is a solution of* (1.1). $f(x)$ *satisfies the superlinearity conditions*

$$0 < \int_{\epsilon}^{\infty} \frac{dx}{f(x)}, \quad \int_{-\infty}^{-\epsilon} \frac{dx}{f(x)} < \infty, \quad \textit{for all } \epsilon > 0. \tag{2.7}$$

Then

$$\int_{T}^{t} \frac{x^{\Delta}(s)}{f(x^{\sigma}(s))} \Delta s \leq F(x(T)) - F(x(t)) \leq F(x(T)),$$

where $F(x) = \int_x^{\infty} \frac{dv}{f(v)}$.

Proof. Assume that $t = t_{i-1} < t_i = \sigma(t)$. Then

$$\int_{t}^{\sigma(t)} \frac{x^{\Delta}(s)}{f(x(\sigma(s)))} \Delta s = \frac{x^{\Delta}(t)\mu(t)}{f(x(\sigma(t)))} = \frac{x(\sigma(t)) - x(t)}{f(x(\sigma(t)))}. \tag{2.8}$$

We consider the two possible cases: (i) $x(t) \leq x(\sigma(t))$ and (ii) $x(t) > x(\sigma(t))$. First, if $x(t) \leq x(\sigma(t))$ we have

$$\frac{x(\sigma(t)) - x(t)}{f(x(\sigma(t)))} \leq \int_{x(t)}^{x(\sigma(t))} \frac{1}{f(v)} dv = F(x(t)) - F(x(\sigma(t))), \tag{2.9}$$

since f is increasing. On the other hand, if $x(t) > x(\sigma(t))$, then

$$\frac{x(t) - x(\sigma(t))}{f(x(\sigma(t)))} \geq \int_{x(\sigma(t))}^{x(t)} \frac{1}{f(v)} ds = F(x(\sigma(t))) - F(x(t)),$$

which implies that

$$\frac{x(\sigma(t)) - x(t)}{f(x(\sigma(t)))} \leq F(x(t)) - F(x(\sigma(t))). \tag{2.10}$$

Hence, whenever $t_{i-1} = t < \sigma(t) = t_i$, we have from (2.8) and (2.9) in the first case and (2.8) and (2.10) in the second case, that

$$\int_{t_{i-1}}^{t_i} \frac{x^{\Delta}(s)}{f(x(\sigma(s)))} \Delta s \leq F(x(t_{i-1})) - F(x(t_i)). \tag{2.11}$$

If the real interval $[t_{i-1}, t_i] \subset \mathbb{T}$, then

$$\begin{aligned}\int_{t_{i-1}}^{t_i} \frac{x^{\Delta}(s)}{f(x(\sigma(s)))}\Delta &= \int_{t_{i-1}}^{t_i} \frac{x^{\Delta}(s)}{f(x(s))}\Delta s \\ &= \int_{x(t_{i-1})}^{x(t_i)} \frac{1}{f(v)}\,dv \\ &= F(x(t_{i-1})) - F(x(t_i)),\end{aligned} \tag{2.12}$$

and so (2.11) also holds in this case.

Note that since $\mathbb{T}$ satisfies condition (H), we have from (2.11), (2.12) and the additivity of the integral that for $t \in [T, \infty)_{\mathbb{T}}$,

$$\int_T^t \frac{x^{\Delta}(s)}{f(x(\sigma(s)))}\Delta s \le F(x(T)) - F(x(t)) \le F(x(T)). \tag{2.13}$$

□

3. Main results

Theorem 3.1. *Suppose that $\mathbb{T}$ is a regular time scale. $f(x)$ satisfies* (1.2) *and* (1.3)*. If $x(t)$ is a nonoscillatory solution of* (1.1) *on $[T_0, \infty)$. Then the integral equation*

$$\frac{r^{\sigma}(t)x^{\Delta}(t)}{f(x^{\sigma}(t))} = P^{\sigma}(t) + \int_{\sigma(t)}^{\infty} \frac{r^{\sigma}(s)[\int_0^1 f'(x_h(s))dh][x^{\Delta}(s)]^2}{f(x(s))f(x^{\sigma}(s))}\Delta s \tag{3.1}$$

is satisfied for $t \ge T_0$, where $P^{\sigma}(t) = \int_{\sigma(t)}^{\infty} p(s)\Delta s$, $x_h(s) = x(s) + h\mu(s)x^{\Delta}(s)$.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1) on $[T_0, \infty)$. Without loss of generality, assume that $x(t)$ is positive for $t \in [T_0, \infty)$.

In the first place, we will prove

$$\int_{T_0}^{\infty} \frac{r^{\sigma}(s)[\int_0^1 f'(x_h(s))dh][x^{\Delta}(s)]^2}{f(x(s))f(x^{\sigma}(s))}\Delta s < \infty, \tag{3.2}$$

where $x_h(t) = x(t) + h\mu(t)x^{\Delta}(t) = (1-h)x(t) + hx(\sigma(t)) > 0$. From (1.1), it is easy to see that

$$\begin{aligned}\Big(\frac{r(t)x^{\Delta}(\rho(t))}{f(x(t))}\Big)^{\Delta} &= [r(t)x^{\Delta}(\rho(t))]^{\Delta}\frac{1}{f(x(t))} + r^{\sigma}(t)x^{\Delta}(t)\Big(\frac{1}{f(x(t))}\Big)^{\Delta} \\ &= -p(t) - \frac{r^{\sigma}(t)[\int_0^1 f'(x_h(t))dh][x^{\Delta}(t)]^2}{f(x(t))f(x^{\sigma}(t))}\end{aligned}$$

Integrating from T_0 to t,

$$\begin{aligned}&\frac{r(t)x^{\Delta}(\rho(t))}{f(x(t))} - \frac{r(T_0)x^{\Delta}(\rho(T_0))}{f(x(T_0))} \\ &= -\int_{T_0}^t p(s)\Delta s - \int_{T_0}^t \frac{r^{\sigma}(s)[\int_0^1 f'(x_h(s))dh][x^{\Delta}(s)]^2}{f(x(s))f(x^{\sigma}(s))}\Delta s.\end{aligned} \tag{3.3}$$

If (3.2) fails to hold; that is,

$$\int_{T_0}^{\infty} \frac{r^{\sigma}(s)[\int_0^1 f'(x_h(s))dh][x^{\Delta}(s)]^2}{f(x(s))f(x^{\sigma}(s))}\Delta s = \infty. \tag{3.4}$$

From (3.3), we have

$$\lim_{t\to\infty}\frac{r(t)x^{\Delta}(\rho(t))}{f(x(t))}=-\infty. \tag{3.5}$$

We can assume that

$$\frac{r(T_0)x^{\Delta}(\rho(T_0))}{f(x(T_0))}-\int_{T_0}^{t}p(s)\Delta s<-1, \tag{3.6}$$

for $t\geq T_0$. Otherwise let $L=\sup_{t\geq T_0}|\int_{T_0}^t p(s)\Delta s|$. By (3.5), we can take a large $T_1>T_0$ such that $\frac{r(T_1)x^{\Delta}(\rho(T_1))}{f(x(T_1))}<-(2L+1)$. So we have

$$\begin{aligned}\frac{r(T_1)x^{\Delta}(\rho(T_1))}{f(x(T_1))}-\int_{T_1}^{t}p(s)\Delta s&<-(2L+1)-\Big[\int_{T_0}^{t}p(s)\Delta s-\int_{T_0}^{T_1}p(s)\Delta s\Big]\\&\leq-(2L+1)-[-2L]=-1.\end{aligned}$$

So we can replace T_0 by $T_1>T_0$ such that (3.6) still holds. From (3.3) and (3.6), we obtain, for $t\geq T_0$,

$$\frac{r(t)x^{\Delta}(\rho(t))}{f(x(t))}+\int_{T_0}^{t}\frac{r^{\sigma}(s)[\int_0^1 f'(x_h(s))dh][x^{\Delta}(s)]^2}{f(x(s))f(x^{\sigma}(s))}\Delta s<-1. \tag{3.7}$$

In particular,

$$x^{\Delta}(t)<0,\quad \text{for } t\geq T_0. \tag{3.8}$$

Therefore, $x(t)$ is strictly decreasing.

First assume that $\mathbb{T}$ is an isolated time scale; that is,

$$\mathbb{T}=\{t_0,t_1,t_2,\dots\},\quad t_0<t_1<t_2\dots,\lim_{k\to\infty}t_k=\infty.$$

If $t=t_{i-1}<t_i=\sigma(t)$, then $x(\sigma(t))<x(t)$, so

$$\begin{aligned}\int_0^1 f'(x_h(s))dh&=\int_0^1 f'((1-h)x(s)+h(x(\sigma(s))))dh\\&=\frac{f((1-h)x(s)+h(x(\sigma(s))))|_0^1}{x(\sigma(s))-x(s)}\\&=\frac{f(x(\sigma(s)))-f(x(s))}{x(\sigma(s))-x(s)}.\end{aligned} \tag{3.9}$$

On the other hand if the $\mathbb{T}$ is the real interval $[T_0,\infty)$, then

$$\int_0^1 f'(x_h(s))dh=f'(x(s)). \tag{3.10}$$

Let

$$y(t):=1+\int_{T_0}^{t}\frac{r^{\sigma}(s)\int_0^1 f'(x_h(s))dh[x^{\Delta}(s)]^2}{f(x(s))f(x^{\sigma}(s))}\Delta s. \tag{3.11}$$

Hence from (3.7), we obtain

$$-\frac{r(t)x^{\Delta}(\rho(t))}{f(x(t))}>y(t). \tag{3.12}$$

Replacing t by $\sigma(t)$ in (3.12) and using [1, Theorem1.75], we obtain

$$-\frac{r^{\sigma}(t)x^{\Delta}(t)}{f(x^{\sigma}(t))}>y^{\sigma}(t)=y(t)+\frac{r^{\sigma}(t)\int_0^1 f'(x_h(t))dh[x^{\Delta}(t)]^2\mu(t)}{f(x(t))f(x^{\sigma}(t))}. \tag{3.13}$$

Noting that
$$\frac{1}{f(x^\sigma(t))}=\frac{1}{f(x(t))}+\Big(\frac{1}{f(x(t))}\Big)^\Delta\mu(t). \tag{3.14}$$
Using (3.14) in the left side of (3.13) and noticing that
$$(\frac{1}{f(x(t))})^\Delta=-\frac{\int_0^1 f'(x_h(t))dhx^\Delta(t)}{f(x(t))f(x^\sigma(t))},$$
from (3.13), it is easy to see that
$$-\frac{r^\sigma(t)x^\Delta(t)}{f(x(t))}>y(t). \tag{3.15}$$
From (3.11) and (3.15), we obtain
$$\begin{aligned} y^\Delta(t)&=\frac{r^\sigma(t)\int_0^1 f'(x_h(t))dh[x^\Delta(t)]^2}{f(x(t))f(x^\sigma(t))}\\ &>y(t)\frac{\int_0^1 f'(x_h(t))dh[-x^\Delta(t)]}{f(x^\sigma(t))}. \end{aligned} \tag{3.16}$$
In the isolated time scale case from (3.16) and (3.9), we obtain
$$\frac{y(\sigma(t))-y(t)}{y(t)(\sigma(t)-t)}>\frac{f(x(\sigma(t)))-f(x(t))}{x(\sigma(t))-x(t)}\cdot\frac{x(t)-x(\sigma(t))}{f(x(\sigma(t)))[\sigma(t)-t]}.$$
So
$$\frac{y(\sigma(t))}{y(t)}>\frac{f(x(t))}{f(x(\sigma(t)))};$$
that is,
$$\frac{y(t_i)}{y(t_{i-1})}>\frac{f(x(t_{i-1}))}{f(x(t_i))}. \tag{3.17}$$
Let $T_0=t_{n_0}$ and $t=t_n, n>n_0$, then using (3.17) we have that
$$\frac{y(t_n)}{y(t_{n_0})}=\prod_{k=0}^{n-n_0-1}\frac{y(t_{n_0+k+1})}{y(t_{n_0+k})}>\prod_{k=0}^{n-n_0-1}\frac{f(x(t_{n_0+k}))}{f(x(t_{n_0+k+1}))}=\frac{f(x(t_{n_0}))}{f(x(t_n))};$$
that is,
$$\frac{y(t)}{y(t_{n_0})}>\frac{f(x(t_{n_0}))}{f(x(t))}, \tag{3.18}$$
for $t>T_0$. To obtain (3.18) in the case where $\mathbb{T}$ is the real interval $[T_0,\infty)$, from (3.10) and (3.16) we have
$$\frac{y'(t)}{y(t)}>\frac{f'(x(t))[-x'(t)]}{f(x(t))};$$
that is,
$$(\ln y(t))'>-(\ln f(x(t)))'.$$
Integrating from T_0 to t, we obtain
$$\frac{y(t)}{y(T_0)}>\frac{f(x(T_0))}{f(x(t))},\quad t>T_0. \tag{3.19}$$
Using (3.15) again, from (3.18) and (3.19), we obtain
$$-\frac{r^\sigma(t)x^\Delta(t)}{f(x(t))}>y(t)>\frac{y(T_0)f(x(T_0))}{f(x(t))}.$$

If we set $L_1 := y(T_0)f(x(T_0))$, we obtain

$$x^{\Delta}(t) < -\frac{L_1}{r^{\sigma}(t)}.$$

Integrating from T_0 to t, we obtain

$$x(t) - x(T_0) < -\int_{T_0}^{t} \frac{L_1}{r^{\sigma}(s)}\Delta s \to -\infty, \quad \text{as } t \to \infty.$$

which contradicts $x(t) > 0$.

In (3.3), letting $t \to \infty$ and replacing T_0 by $\sigma(\tau)$, denoting

$$\alpha = \lim_{t\to\infty} \frac{r(t)x^{\Delta}(\rho(t))}{f(x(t))} = \lim_{t\to\infty} \frac{r^{\sigma}(t)x^{\Delta}(t)}{f(x^{\sigma}(t))}, \tag{3.20}$$

we obtain

$$\alpha + \int_{\sigma(\tau)}^{\infty} p(s)\Delta s + \int_{\sigma(\tau)}^{\infty} \frac{r^{\sigma}(s)[\int_0^1 f'(x_h(s))dh][x^{\Delta}(s)]^2}{f(x(s))f(x^{\sigma}(s))}\Delta s = \frac{r^{\sigma}(\tau)x^{\Delta}(\tau)}{f(x^{\sigma}(\tau))}. \tag{3.21}$$

We claim that $\alpha = 0$. In the right side of (3.21), using

$$\frac{1}{f(x^{\sigma}(\tau))} = \frac{1}{f(x(\tau))} - \frac{\int_0^1 f'(x_h(\tau))dh x^{\Delta}(\tau)}{f(x(\tau))f(x^{\sigma}(\tau))}\mu(t)$$

and in the second integral term of left side of (3.21), noticing that

$$\begin{aligned}
&\int_{\sigma(\tau)}^{\infty} \frac{r^{\sigma}(s)[\int_0^1 f'(x_h(s))dh][x^{\Delta}(s)]^2}{f(x(s))f(x^{\sigma}(s))}\Delta s \\
&= \int_{\tau}^{\infty} \frac{r^{\sigma}(s)[\int_0^1 f'(x_h(s))dh][x^{\Delta}(s)]^2}{f(x(s))f(x^{\sigma}(s))}\Delta s - \int_{\tau}^{\sigma(\tau)} \frac{r^{\sigma}(s)[\int_0^1 f'(x_h(s))dh][x^{\Delta}(s)]^2}{f(x(s))f(x^{\sigma}(s))}\Delta s \\
&= \int_{\tau}^{\infty} \frac{r^{\sigma}(s)[\int_0^1 f'(x_h(s))dh][x^{\Delta}(s)]^2}{f(x(s))f(x^{\sigma}(s))}\Delta s - \frac{r^{\sigma}(\tau)[\int_0^1 f'(x_h(\tau))dh][x^{\Delta}(\tau)]^2}{f(x(\tau))f(x^{\sigma}(\tau))}\mu(\tau),
\end{aligned}$$

we obtain

$$\alpha + \int_{\sigma(\tau)}^{\infty} p(s)\Delta s + \int_{\tau}^{\infty} \frac{r^{\sigma}(s)[\int_0^1 f'(x_h(s))dh][x^{\Delta}(s)]^2}{f(x(s))f(x^{\sigma}(s))}\Delta s = \frac{r^{\sigma}(\tau)x^{\Delta}(\tau)}{f(x(\tau))}. \tag{3.22}$$

From (3.22), we have

$$\lim_{t\to\infty} \frac{r^{\sigma}(t)x^{\Delta}(t)}{f(x(t))} = \alpha. \tag{3.23}$$

Suppose that $\alpha < 0$. Then from (3.23) there exists a large T_1 such that for $t > T_1$, we have

$$\frac{r^{\sigma}(t)x^{\Delta}(t)}{f(x(t))} \le \frac{\alpha}{2}.$$

So

$$x^{\Delta}(t) \le \frac{\alpha}{2} \cdot \frac{f(x(t))}{r^{\sigma}(t)} < 0. \tag{3.24}$$

Thus

$$
\begin{aligned}
M(T_1) =: &\int_{T_1}^{\infty} \frac{r^\sigma(s)[\int_0^1 f'(x_h(s))dh][x^\Delta(s)]^2}{f(x(s))f(x^\sigma(s))}\Delta s \\
&\geq -\frac{\alpha}{2}\int_{T_1}^{\infty} \frac{[\int_0^1 f'(x_h(s))dh][-x^\Delta(s)]}{f(x^\sigma(s))}\Delta s.
\end{aligned}
\tag{3.25}
$$

Assume that $t = t_{i-1} < t_i = \sigma(t)$. From [1, Theorem 1.75], (3.9) and $x^\Delta(t) < 0$, we have

$$
\begin{aligned}
\int_t^{\sigma(t)} \frac{[\int_0^1 f'(x_h(s))dh][-x^\Delta(s)]}{f(x^\sigma(s))}\Delta s &= \frac{[\int_0^1 f'(x_h(t))dh][-x^\Delta(t)](\sigma(t)-t)}{f(x^\sigma(t))} \\
&= \frac{f(x(t)) - f(x^\sigma(t))}{f(x^\sigma(t))} \\
&\geq \int_{f(x^\sigma(t))}^{f(x(t))} \frac{1}{v}dv \\
&= \ln\frac{f(x(t))}{f(x^\sigma(t))} \\
&= \ln\frac{f(x(t_{i-1}))}{f(x(t_i))}.
\end{aligned}
\tag{3.26}
$$

In the isolated time scale, from (3.26), we obtain

$$
\int_{T_1}^{t} \frac{[\int_0^1 f'(x_h(s))dh][-x^\Delta(s)]}{f(x^\sigma(s))}\Delta s \geq \ln\frac{f(x(T_1))}{f(x(t))}. \tag{3.27}
$$

In the case where $\mathbb{T}$ is the real interval $[T_0, \infty)$, from (3.10), we have

$$
\begin{aligned}
\int_{T_1}^{t} \frac{[\int_0^1 f'(x_h(s))dh][-x^\Delta(s)]}{f(x^\sigma(s))}\Delta s &= \int_{T_1}^{t} \frac{[\int_0^1 f'(x_h(s))dh][-x'(s)]}{f(x(s))}ds \\
&= \ln\frac{f(x(T_1))}{f(x(t))}.
\end{aligned}
\tag{3.28}
$$

From (3.25), (3.27), (3.28) and the additivity of the integral, it is easy to see that

$$
M(T_1) \geq -\frac{\alpha}{2}\lim_{t\to\infty}\int_{T_1}^{t} \frac{[\int_0^1 f'(x_h(s))dh][-x^\Delta(s)]}{f(x^\sigma(s))}\Delta s \geq -\frac{\alpha}{2}\lim_{t\to\infty}\ln\frac{f(x(T_1))}{f(x(t))}.
$$

So there exists a large T_2 such that for $t \geq T_2$, we have that

$$
\ln\frac{f(x(T_1))}{f(x(t))} \leq -\frac{2M(T_1)}{\alpha} + 1.
$$

Thus for $t \geq T_2$,

$$
f(x(t)) \geq f(x(T_1))\exp\Big(\frac{2M(T_1)}{\alpha} - 1\Big).
$$

By (3.24) and noticing that $\alpha < 0$, we have that for $t \geq T_2$

$$
x^\Delta(t) \leq \frac{\alpha}{2}\cdot\frac{1}{r^\sigma(t)}\cdot f(x(T_1))\exp\Big(\frac{2M(T_1)}{\alpha} - 1\Big). \tag{3.29}
$$

Integrating (3.29) from T_2 to t, we obtain $x(t) \to -\infty$ as $t \to \infty$, which is a contradiction.

If $\alpha > 0$, from (3.20), we have that there exists T_3 such that for $t \ge T_3$,

$$\frac{r^\sigma(t)x^\Delta(t)}{f(x^\sigma(t))} \ge \frac{\alpha}{2}. \tag{3.30}$$

Therefore, from Lemma 2.2, we have

$$\begin{aligned}
\int_{T_3}^\infty \frac{r^\sigma(s)[\int_0^1 f'(x_h(s))dh][x^\Delta(s)]^2}{f(x(s))f(x^\sigma(s))}\Delta s &\ge \frac{\alpha}{2}\int_{T_3}^\infty \frac{[\int_0^1 f'(x_h(s))dh][x^\Delta(s)]}{f(x(s))}\Delta s\\
&= \frac{\alpha}{2}\lim_{t\to\infty}\int_{T_3}^t \frac{[f(x(s))]^\Delta}{f(x(s))}\Delta s\\
&\ge \frac{\alpha}{2}\lim_{t\to\infty}\ln\frac{f(x(t))}{f(x(T_3))}.
\end{aligned}$$

Due to condition (1.2) and (1.3), it is easy to know that $x(t)$ is bounded.

On the other hand, from (3.30) and the monotonicity of f, we obtain that in the isolated time scale case

$$\begin{gathered}
r^\sigma(T_3)x^\Delta(T_3) \ge \frac{\alpha}{2}f(x^\sigma(T_3)),\\
r^{\sigma^2}(T_3)x^\Delta(\sigma(T_3)) \ge \frac{\alpha}{2}f(x^{\sigma^2}(T_3)) \ge \frac{\alpha}{2}f(x^\sigma(T_3)).
\end{gathered}$$

By induction, it is easy to get that for $t \ge T_3$,

$$r^\sigma(t)x^\Delta(t) \ge \frac{\alpha}{2}f(x^\sigma(T_3));$$

that is,

$$x^\Delta(t) \ge \frac{\alpha}{2r^\sigma(t)}f(x^\sigma(T_3)). \tag{3.31}$$

Integrating (3.31) from T_3 to t, we obtain $x(t) \to +\infty$ as $t \to \infty$, which contradicts the boundedness of $x(t)$.

If $\mathbb{T}$ is the real interval $[T_3, \infty)$, then from (3.30) and the monotonicity of f, we have that for $t \ge T_3$,

$$x'(t) \ge \frac{\alpha}{2r(t)}f(x(t)) \ge \frac{\alpha}{2r(t)}f(x(T_2)). \tag{3.32}$$

Integrating (3.32) from T_3 to t, we obtain $x(t) \to +\infty$ as $t \to \infty$, which also contradicts the boundedness of $x(t)$. Therefore $\alpha = 0$.

From (3.21), we obtain (3.1). The proof is complete. $\square$

Theorem 3.2. *Suppose $\mathbb{T}$ is a regular time scale, $r(t) > 0$ with $\int_{T_0}^\infty [r^\sigma(t)]^{-1}\Delta t = \infty$ and suppose that $\lim_{t\to\infty}\int_{T_0}^t p(s)\Delta s$ exists and finite. Let $P(t) = \int_t^\infty p(s)\Delta s$. $f(x)$ satisfies the superlinearity conditions*

$$0 < \int_\epsilon^\infty \frac{dx}{f(x)}, \int_{-\infty}^{-\epsilon}\frac{dx}{f(x)} < \infty, \quad \text{for all } \epsilon > 0. \tag{3.33}$$

Then the superlinear dynamic equation

$$[r(t)x^\Delta(\rho(t))]^\Delta + p(t)f(x(t)) = 0, \tag{3.34}$$

is oscillatory, if

$$\limsup_{t\to\infty}\frac{1}{r^\sigma(t)}\int_{T_0}^t P^\sigma(s)\Delta s = \infty. \tag{3.35}$$

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1) on $[T_0,\infty)$. Without loss of generality, assume that $x(t)$ is positive for $t\in[T_0,\infty)$. From Theorem 3.1, $x(t)$ satisfies the integral equation (3.1). Dropping the last integral term in (3.1), we have the inequality

$$\frac{r^\sigma(t)x^\Delta(t)}{f(x^\sigma(t))}\ge P^\sigma(t). \tag{3.36}$$

Dividing (3.36) by $r^\sigma(t)$ and integrating from T_0 to t and using Lemma 2.3, we find

$$F(x(T_0))\ge\int_{T_0}^t\frac{x^\Delta(s)}{f(x^\sigma(s))}\Delta s\ge\frac{1}{r^\sigma(t)}\int_{T_0}^t P^\sigma(s)\Delta s.$$

This contradicts (3.35), so equation (3.34) is oscillatory. □

4. Examples

Example 4.1. Consider the superlinear difference equation

$$\Delta^2 x(n-1)+p(n)x^\gamma(n)=0,\quad \gamma>1, \tag{4.1}$$

where $P(n)=\frac{1}{n}+\frac{2(-1)^n}{\sqrt{n}}$,

$$p(n)=P(n)-P(n+1)=\frac{1}{n(n+1)}+\frac{2(-1)^n(\sqrt{n+1}+\sqrt{n})}{\sqrt{n(n+1)}}. \tag{4.2}$$

It is to see that $\sum_{n=1}^\infty P(n+1)=\infty$. So from Theorem 3.2, (4.1) is oscillatory.

In [8, Theorem 2.5], we proved that the equation $\Delta^2x(n-1)+q(n)x^\gamma(n)=0,\gamma>1$, is oscillatory, if $\sum_{n=1}^\infty nq(n)=\infty$. In the following, we will prove that

$$\sum_{n=1}^{2k+1} np(n)\to-\infty\quad\text{as}\quad k\to\infty. \tag{4.3}$$

So [8, Theorem 2.5] would not apply in (4.1).

We need the following lemmas. The first lemma may be regarded as a discrete version of L'Hopital's rule and can be found in [1, page 48].

Lemma 4.2 (Stolz-Cesáro Theorem)**.** *Let $\{a_n\}_{n\ge1}$ and $\{b_n\}_{n\ge1}$ be two sequences of real number. Assume b_n is positive, strictly increasing and unbounded and the following limit exists:*

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=l.$$

Then

$$\lim_{n\to\infty}\frac{a_n}{b_n}=l.$$

We will use Lemma 4.2 to prove the following result.

Lemma 4.3. *For each real number $d>0$, we have*

$$\lim_{m\to\infty}\frac{\sum_{i=1}^m i^d-\frac{m^{d+1}}{d+1}}{m^d}=\frac{1}{2}. \tag{4.4}$$

Proof. By Taylor's formula, we have

$$\big(1+\frac{1}{m}\big)^d = 1+\frac{d}{m}+\frac{d(d-1)}{2m^2}+o\big(\frac{1}{m^2}\big). \tag{4.5}$$

By (4.5) and the Stolz-Cesáro Theorem (Lemma 4.2), it is easy to see that

$$\begin{aligned} \lim_{m\to\infty}\frac{\sum_{i=1}^m i^d-\frac{m^{d+1}}{d+1}}{m^d} &= \lim_{m\to\infty}\frac{(m+1)^d-\frac{(m+1)^{d+1}}{d+1}+\frac{m^{d+1}}{d+1}}{(m+1)^d-m^d}\\ &= \lim_{m\to\infty}\frac{(1+\frac{1}{m})^d-\frac{m+1}{d+1}(1+\frac{1}{m})^d+\frac{m}{d+1}}{(1+\frac{1}{m})^d-1}. \end{aligned} \tag{4.6}$$

Using (4.5) in (4.6), it follows that (4.4) holds. □

So given $0<\epsilon<1$, for large m, we have the inequality

$$\frac{m^{d+1}}{d+1}+\frac{m^d}{2}-\epsilon m^d < \sum_{i=1}^m i^d < \frac{m^{d+1}}{d+1}+\frac{m^d}{2}+\epsilon m^d. \tag{4.7}$$

Set $d=1/2$. we have

$$\frac{2}{3}m^{3/2}+\frac{1}{2}m^{1/2}-\epsilon m^{1/2} < \sum_{i=1}^m i^{1/2} < \frac{2}{3}m^{3/2}+\frac{1}{2}m^{1/2}+\epsilon m^{1/2}. \tag{4.8}$$

From (4.2) and using Taylor's expansion, we have

$$\begin{aligned} np(n) &= \frac{1}{n+1}+2(-1)^n\sqrt{n}\big[1+\big(1+\frac{1}{n}\big)^{-1/2}\big]\\ &= \frac{1}{n+1}+2(-1)^n\sqrt{n}\big[2-\frac{1}{2n}+O\big(\frac{1}{n^2}\big)\big]\\ &= \frac{1}{n+1}+4(-1)^n\sqrt{n}-\frac{(-1)^n}{\sqrt{n}}+O\big(\frac{1}{n^{3/2}}\big). \end{aligned} \tag{4.9}$$

Using the Euler formula [5, page 205],

$$C=\lim_{n\to\infty}\Big(\sum_{k=1}^n\frac{1}{k}-\ln n\Big),$$

where C is called Euler constant, we obtain

$$\sum_{n=1}^{2k+1}\frac{1}{n+1}=o(1)+C-1+\ln(2k+2). \tag{4.10}$$

From (4.8) and using Taylor's expansion, we have

$$\begin{aligned} &\sum_{n=1}^{2k+1}(-1)^n\sqrt{n}\\ &= -\sum_{n=1}^{2k+1}\sqrt{n}+2\sqrt{2}\sum_{n=1}^{k}\sqrt{n}\\ &< -\big[\frac{2}{3}(2k+1)^{3/2}+(\frac{1}{2}-\epsilon)(2k+1)^{1/2}\big]+2\sqrt{2}\big[\frac{2}{3}k^{3/2}+(\frac{1}{2}+\epsilon)k^{1/2}\big]\\ &= \frac{4\sqrt{2}}{3}k^{3/2}\big[1-\big(1+\frac{1}{2k}\big)^{3/2}\big]+\sqrt{2}k^{1/2}\big[1-\frac{1}{2}\big(1+\frac{1}{2k}\big)^{1/2}\big] \end{aligned}$$

$$
\begin{aligned}
&+\epsilon\sqrt{2}k^{1/2}\big[2+\big(1+\frac{1}{2k}\big)^{1/2}\big] \\
&=\frac{4\sqrt{2}}{3}k^{3/2}\big[-\frac{3}{4k}+O\big(\frac{1}{k^2}\big)\big]+\sqrt{2}k^{1/2}\big[\frac{1}{2}+O(\frac{1}{k})\big] \\
&+\epsilon\sqrt{2}k^{1/2}[3+O(\frac{1}{k})] \qquad (4.11)\\
&=-(\frac{1}{2}-3\epsilon)\sqrt{2}k^{1/2}+O\big(\frac{1}{k^{1/2}}\big).
\end{aligned}
$$

Take $\epsilon < 1/6$. From (4.9), (4.10), (4.11) and noticing that $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ and $\sum_{n=1}^{\infty}O\big(\frac{1}{n^{3/2}}\big)$ are convergent, we obtain that

$$
\sum_{n=1}^{2k+1} np(n)\to-\infty, \quad \text{as } k\to\infty.
$$

So we complete the proof of (4.3).

Example 4.4. Consider the differential equation

$$
x''(t)+p(t)x^{\gamma}(t)=0, \quad \gamma>1, \tag{4.12}
$$

where $P(t)=(t^{1/4}\sin\sqrt{t})'$. So we have $\int_1^t P(s)ds=t^{1/4}\sin\sqrt{t}-\sin 1$ and

$$
\limsup_{t\to\infty}\int_1^t P(s)ds=\infty. \tag{4.13}
$$

It is easy to see that

$$
\begin{aligned}
p(t)=-(P(t))'&=\frac{3}{16}t^{-7/4}\sin\sqrt{t}-\frac{1}{8}t^{-5/4}\cos\sqrt{t}\\
&+\frac{1}{8}t^{-5/4}\cos\sqrt{t}-\frac{1}{4}t^{-3/4}\sin\sqrt{t}.
\end{aligned}
$$

When $\beta<-\frac{1}{2}$, $\int_1^{\infty}t^{\beta}\sin\sqrt{t}dt$ and $\int_1^{\infty}t^{\beta}\cos\sqrt{t}dt$ are convergent, we have that $\lim_{t\to\infty}\int_1^t p(s)ds$ exists and finite. From (4.13) and Theorem 3.2, (4.12) is oscillatory.

Example 4.5. Consider the q-difference equation

$$
(x(q^{-1}t))^{\Delta\Delta}+p(t)x^{\gamma}(t)=0, \quad \gamma>1, \tag{4.14}
$$

where $t=q^n$, $n\in\mathbb{N}_0$, $P(t)=\frac{1+2(-1)^n}{t}$. It is easy to see that

$$
p(t)=-P^{\Delta}(t)=\big[1+\frac{2(-1)^n(1+q)}{q-1}\big]\frac{1}{qt^2}.
$$

We have that $\int_1^{\infty}p(t)\Delta t$ is convergent and $\int_1^{\infty}P^{\sigma}(t)\Delta t=\infty$. From Theorem 3.2, (4.14) is oscillatory.

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