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NON-EXISTENCE OF LIMIT CYCLES VIA INVERSE INTEGRATING FACTORS

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Abstract. It is known that if a planar differential systems has an inverse integrating factor, then all the limit cycles contained in the domain of definition of the inverse integrating factor are contained in the zero set of this function. Using this fact we give some criteria to rule out the existence of limit cycles. We also present some applications and examples that illustrate our results.

1. Introduction and statement of results

Many problems in qualitative theory of differential equations in the plane are related to limit cycles; this fact motivates their study. We consider the system of differential equations

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2), \end{aligned} \tag{1.1}$$

where $f_i : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, $1 \le i \le 2$ are functions of class C^1 and U is a simply connected open set. Consider the vector field $F := f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2}$, then system (1.1) can be rewritten in the form

$$\dot{x} = F(x), \quad x := (x_1, x_2) \in U. \tag{1.2}$$

Its divergence is $\operatorname{div}(F) := \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$.

Definition 1.1. A function $\vartheta : U \subset \mathbb{R}$, $\vartheta \in C^1(U, \mathbb{R})$, is said to be an inverse integrating factor of (1.1) if it is not locally null and satisfies the partial differential equation

$$f_1 \frac{\partial \vartheta}{\partial x_1} + f_2 \frac{\partial \vartheta}{\partial x_2} = \big(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\big)\vartheta. \tag{1.3}$$

In short notation, an inverse integrating factor is a solution of the equation $F\vartheta = \operatorname{div}(F)\vartheta$. It is well known that inverse integrating factor is an important tool in the qualitative study of differential equations, but their determination is a difficult problem (see [3] and references therein). In particular, in [1] has been established that are also a very useful tool for investigation of limit cycles.

The aim of this article is to use inverse integrating factors to produce in a systematic way, criteria for non-existence of limit cycles in planar differential equations. In

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particular, we obtain an alternative proof to the nonexistence of limit cycles for homogeneous polynomial equations. We give some examples to illustrate applications of these results.

We rewrite (1.1) in its Pfaffian form

$$\omega := -f_2(x_1,x_2)dx_1 + f_1(x_1,x_2)dx_2 = 0, \quad (x_1,x_2) \in U. \tag{1.4}$$

Note that the above equation is just the differential equation of the orbits of system (1.1). Recall that an integrating factor for $\omega = 0$ is a C^1 function $\mu : U \to \mathbb{R}$, which makes $\mu\omega$ an exact form. In the case that U is simply connected; this is equivalent to

$$\frac{\partial(-\mu f_2)}{\partial x_2} = \frac{\partial(\mu f_1)}{\partial x_1} \tag{1.5}$$

It is clear that μ is an integrating factor for (1.4) if and only if, $\vartheta = \frac{1}{\mu}$ is an inverse integrating factor of (1.1), in the appropriate domain.

Before establishing our results we recall the following result.

Theorem 1.2 ([1, Theorem 9])**.** *Let $\vartheta : U \to \mathbb{R}$ be an inverse integrating factor of* (1.1)*. If $\gamma \subset U$ is a limit cycle of* (1.1)*, then γ is contained in the set $\vartheta^{-1}(0) := \{(x_1,x_2) \in U : \vartheta(x_1,x_2) = 0\}$.*

Recall that it is possible to impose certain conditions on (1.5), to determine special cases of integrating factors. Our first result is an observation that these techniques can be adapted to exclude existence of limit cycles. We start with the following result.

Proposition 1.3. *Let U be a simply connected open set. Suppose a vector field*

$$F = f_1\frac{\partial}{\partial x_1} + f_2\frac{\partial}{\partial x_2} \in C^1(U,\mathbb{R}^2).$$

If any of the following two conditions holds, then (1.1) *does not have limit cycles in U:*

(i) *The function $\alpha_i := \operatorname{div}(F)/f_i$ depends only on x_i, for some $i \in \{1,2\}$ and is continuous;*
(ii) *The function $\beta := \operatorname{div}(F)/\big(f_1x_2 + f_2x_1\big)$ depends on $z := x_1x_2$ and is continuous.*

Proof. We consider the case (i) with α_1 depending only on x_1. We seek an inverse integrating factor, using the associated equation

$$f_1\frac{\partial\vartheta}{\partial x_1} + f_2\frac{\partial\vartheta}{\partial x_2} = \Big(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\Big)\vartheta.$$

Assume that ϑ depends only on x_1. Thus the previous equation reduces to

$$f_1\frac{\partial\vartheta}{\partial x_1} = \Big(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\Big)\vartheta,$$

which is rewritten as

$$\frac{\partial\log\vartheta}{\partial x_1} = \frac{1}{f_1}\Big(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\Big) = \alpha_1.$$

From our hypothesis $\vartheta = \exp(\int^{x_1}\alpha_1(s)ds)$ is an inverse integrating factor and $\vartheta^{-1}(0) = \emptyset$, therefore by Theorem 1.2 system (1.1) has no limit cycles. The proof is complete. $\square$

Example 1.4. Consider the system

$$\dot{x_1} = -x_2 + ax_1^2 + bx_2^3\cos(x_2),$$
$$\dot{x_2} = x_1.$$

We have that $\frac{\operatorname{div}(F)}{f_2} = 2a$ is a function of x_2 so by Proposition 1.3(i). Then the system contains no limit cycles.

Example 1.5. Consider the system

$$\dot{x_1} = 2x_1x_2,$$
$$\dot{x_2} = x_1^3x_2^2 - x_2^2 - 1.$$

We have that $\frac{\operatorname{div}(F)}{f_1} = x_1^2$, by Proposition 1.3, this system contains no limit cycles.

We also have the following immediate result (well known in the literature [6, page 18]).

Corollary 1.6. *Let F be a C^1 vector field on U. If $\operatorname{div}(F) = 0$, then (1.1) does not have limit cycles in U.*

Now we use Proposition 1.3 to study some special systems. Consider the equation

$$\begin{aligned} \dot{x}_1 &= r_1(x_1)r_2(x_2), \\ \dot{x}_2 &= s_1(x_1). \end{aligned} \tag{1.6}$$

We establish the following result.

Corollary 1.7. *If $r_1(x_1) > 0 (< 0)$, then (1.6) does not have limit cycles in U.*

Proof. Indeed, the expression

$$\frac{\operatorname{div}(F)}{f_1} = \frac{r_1'(x_1)}{r_1(x_1)},$$

is continuous and depends only on x_1, hence the result follows from Proposition 1.3(i). □

Example 1.8. Consider the system

$$\dot{x_1} = (2 + \sin(x_1))(x_2^3 - x_2^2 + x_2),$$
$$\dot{x_2} = x_1^4 + 5x_1.$$

It contains no limit cycles.

Recall that the phase portrait of differential equation is essentially unchanged if we multiply the vector field by a nonzero function.

Lemma 1.9. *Suppose that system (1.1) has a limit cycle α and $B : U \to \mathbb{R}$ is a positive (negative) real valued function. Then α is a limit cycle of the system*

$$\begin{aligned} \dot{x}_1 &= Bf_1, \\ \dot{x}_2 &= Bf_2. \end{aligned} \tag{1.7}$$

Now using the above lemma, we obtain slightly general versions of our results.

Proposition 1.10. *Let U be a simply connected open set. Suppose that $B : U \to \mathbb{R}$ is a C^1 positive (negative) function such that $\operatorname{div}(BF)/Bf_i$ depends only on x_i, for some $i \in \{1, 2\}$ and is continuous. Then (1.1) does not have limit cycles in U.*

In particular from the preceding proposition or from Corollary 1.6, we have the following result.

Corollary 1.11. *Let U be a simply connected open set. Suppose that $B : U \to \mathbb{R}$ is a C^1 positive (negative) function such that* $\operatorname{div}(BF) = 0$*, then* (1.1) *does not have limit cycles in U.*

2. Polynomial vector fields

In this section we are mainly interested in studying polynomial vector fields. We start presenting some basic concepts. Let $\mathbb{R}[x_1, x_2]$ be the polynomial ring over $\mathbb{R}$ in two variables. Given $f \in \mathbb{R}[x_1, x_2]$, define its zero set by

$$V(f) := \{(x_1, x_2) \in \mathbb{R}^2 : f(x) = 0\}.$$

If $S \subset \mathbb{R}[x_1, x_2]$, we let $V(S)$ be the set of common zeros

$$V(S) = \cap_{f \in S} V(f).$$

A set of this form is called algebraic, in particular $V(f)$ is known as algebraic curve.

Lemma 2.1. *Let $P \in \mathbb{R}[x_1, x_2]$ be a non-zero homogeneous polynomial, then $V(P)$ contains no subset homeomorphic to $\mathbb{S}^1$.*

Proof. If $P := c \neq 0$, then $V(P) = \emptyset$ and the result is valid, so consider P a homogeneous polynomial of degree ≥ 1, then note that $0 \in V(P)$.

Suppose $V(P)$ contains a subset α homeomorphic to $\mathbb{S}^1$. If it happens that $0 \in \operatorname{int}(\alpha)$ (the region bounded by α), then $P \equiv 0$ which is a contradiction.

On the other hand, if we have $0 \notin \operatorname{int}(\alpha)$, then we would have the cone $C(\alpha) := \{\lambda x : \lambda \geq 0,\ x \in \alpha\} \subset V(P)$ which is a contradiction to Bézout's theorem [4], because there are lines without common components with $V(P)$, but an infinite number of points of intersection. This completes the proof. ☐

Based on this lemma we can prove the following result.

Theorem 2.2. *If a non-zero homogeneous polynomial is an inverse integrating factor of the differential equation* (1.1)*, then it has no limit cycles.*

Proof. Let ϑ be an inverse integrating factor of system (1.1) and is homogeneous polynomial. Suppose the system (1.1) has a limit cycle α, then $\alpha \subset \vartheta^{-1}(0)$ which contradicts Lemma 2.1. This concludes the proof. ☐

We have an alternative proof of the following result, which is proven in [5].

Corollary 2.3. *If f_1, f_2 are homogeneous polynomials of same degree, then* (1.1) *has no limit cycles.*

Proof. It is easy to check that

$$\vartheta(x_1, x_2) := x_1 f_2(x_1, x_2) - x_2 f_1(x_1, x_2),$$

is an inverse integrating factor of (1.1). It is clear that ϑ is a homogeneous polynomial, the result follows from Theorem 2.2. ☐

Example 2.4. Consider the system

$$\begin{aligned} \dot{x_1} &= 3x_2^5 - x_1^3 x_2^2 + 6x_1 x_2^4, \\ \dot{x_2} &= x_1^2 x_2^3 - 2x_1^5. \end{aligned}$$

This is a homogeneous vector field, by the above corollary, it contains no limit cycles.

In particular, we have the following well known result.

Corollary 2.5. *The linear differential equation*

$$\begin{aligned} \dot{x}_1 &= ax_1 + bx_2, \\ \dot{x}_2 &= cx_1 + dx_2, \end{aligned} \tag{2.1}$$

contains no limit cycles.

Another application of Theorem 2.2 is based on one of the main results in [2].

Theorem 2.6 ([2])**.** *Consider the polynomial system*

$$\begin{aligned} \dot{x}_1 &= P_n(x_1, x_2) + x_1 A_{d-1}(x_1, x_2), \\ \dot{x}_2 &= Q_n(x_1, x_2) + x_2 A_{d-1}(x_1, x_2), \end{aligned} \tag{2.2}$$

where $P_n, Q_n, A_{d-1} \in \mathbb{R}[x_1, x_2]$ are homogeneous and their degrees satisfy $d > n \geq 1$. Assume that $H(x_1, x_2)$ is a p-degree homogeneous first integral of the system

$$\begin{aligned} \dot{x}_1 &= P_n(x_1, x_2), \\ \dot{x}_2 &= Q_n(x_1, x_2). \end{aligned} \tag{2.3}$$

Then, the function

$$\vartheta(x_1, x_2) := (x_2 Q_n(x_1, x_2) - x_1 P_n(x_1, x_2)) H(x_1, x_2)^{(d-n)/p} \tag{2.4}$$

is an inverse integrating factor of (2.2)*.*

Now combining Theorem 2.2 and 2.6, we have the following consequence.

Corollary 2.7. *Under the hypotheses of Theorem 2.6, if $\frac{d-n}{p} \in \mathbb{N}$, then* (2.2) *is free of limit cycles.*

Proof. From the assumptions in Theorem 2.6 and $(d-n)/p \in \mathbb{N}$, it follows that $H(x_1, x_2)^{(d-n)/p}$ is homogeneous; therefore,

$$\vartheta(x_1, x_2) := (x_2 Q_n(x_1, x_2) - x_1 P_n(x_1, x_2)) H(x_1, x_2)^{(d-n)/p}$$

is a homogeneous polynomial, so the result follows from Theorem 2.2. □

Now we consider the differential equation

$$\begin{aligned} \dot{x}_1 &= r_1(x_1) r_2(x_2), \\ \dot{x}_2 &= s_1(x_1) s_2(x_2). \end{aligned} \tag{2.5}$$

Proposition 2.8. *If r_1 and s_2 are polynomial functions then* (2.5) *has no limit cycles.*

Proof. A calculation gives that the function

$$\vartheta(x_1, x_2) := r_1(x_1) s_2(x_2)$$

is an inverse integrating factor of system (2.5). Now $V(\vartheta) = V(r_1) \cup V(s_2)$. Taking the factorization in irreducible polynomials, one has that $V(r_1)$ consists of a finite union of vertical lines, similarly $V(s_2)$ is a finite union of horizontal lines. So every subset of $V(\vartheta)$ homeomorphic to $\mathbb{S}^1$ must contain points in $V(r_1) \cap V(s_2)$; i.e., critical points therefore can not be limit cycles. □

Example 2.9. Consider the predator-prey equation

$$\begin{aligned}\dot{x_1} &= x_1(a - bx_2)\\ \dot{x_2} &= (cx_1 - d)x_2,\end{aligned}$$

where a, b, c, d are positive constants. By Proposition 2.8 this system contains no limit cycles.

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