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EXISTENCE OF POSITIVE SOLUTIONS FOR SOME NONLINEAR ELLIPTIC SYSTEMS ON THE HALF SPACE

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Abstract. We prove some existence of positive solutions to the semilinear elliptic system

$$\Delta u = \lambda p(x) g(v)$$
$$\Delta v = \mu q(x) f(u)$$

in the half space $\mathbb{R}^n_+$, $n \geq 2$, subject to some Dirichlet conditions, where λ and μ are nonnegative parameters. The functions f, g are nonnegative continuous monotone on $(0, \infty)$ and the potentials p, q are nonnegative and satisfy some hypotheses related to the Kato class $K^\infty(\mathbb{R}^n_+)$.

1. Introduction

The existence and nonexistence of solutions for semilinear elliptic systems have received much attention recently. Most of the studies are about existence and nonexistence of positive radial solutions [8, 12].

In [8], the authors consider the system

$$\begin{aligned} \Delta u &= p(x) g(v), \\ \Delta v &= q(x) f(u) \quad x \in \mathbb{R}^n, \end{aligned} \tag{1.1}$$

where f, g are positive and nondecreasing functions on $(0, \infty)$ and p, q are nonnegative locally holder and radially symmetric functions in $\mathbb{R}^n$, $n \geq 2$. They established the existence of positive entire solutions for (1.1) provided that $\lim_{t\to\infty} g(cf(t))/t = 0$ for all $c > 0$. Moreover, they proved that if

$$\int_0^\infty t p(t)\, dt = \int_0^\infty t q(t)\, dt = \infty,$$

then all positive entire radial solutions of (1.1) blow-up at infinity. However, if p and q satisfy the following condition

$$\int_0^\infty t[p(t) + q(t)]\, dt < \infty,$$

then all positive entire radial solutions of (1.1) are bounded.

In [12], the authors studied the system (1.1) when $f(u) = u^\beta$, $g(v) = v^\alpha$, $\alpha > 0$, $\beta > 0$ and p, q are nonnegative continuous and not necessarily radial. They showed

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that entire positive bounded solutions exist if p and q satisfy at infinity the following decay condition

$$p(x)+q(x)\le C|x|^{-(2+\delta)}$$

for some positive constant δ.

In [9], we were interested in the existence of positive bounded solution for (1.1) in some domains with compact boundary in the case where f and g are monotone on $(0,\infty)$ and p,q satisfy some hypotheses related to the Kato class associated to these domains. Our aim in this paper is to establish the existence of positive bounded and unbounded continuous solutions for a domain with non compact boundary which are parallel to those established in [9].

Throughout this paper, we denote

$$\mathbb{R}^n_+=\{x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n : x_n>0\},$$

where $n\ge 2$. By $\partial\mathbb{R}^n_+$ we denote the boundary of $\mathbb{R}^n_+$, by $B(\mathbb{R}^n_+)$ the set of Borel measurable functions in $\mathbb{R}^n_+$, and by $C_0(\mathbb{R}^n_+)$ the set of continuous functions vanishing at $\partial\mathbb{R}^n_+\cup\{\infty\}$. We fix some nonnegative constants a,b,α,β such that $a+\alpha>0$, $b+\beta>0$ and two nontrivial nonnegative bounded continuous functions φ and ψ on $\partial\mathbb{R}^n_+$ and we will deal with the existence of positive continuous bounded solutions (in the sense of distributions) for the system

$$\begin{gathered}\Delta u=\lambda p(x)g(v),\quad \text{in } \mathbb{R}^n_+\\ \Delta v=\mu q(x)f(u),\quad \text{in } \mathbb{R}^n_+\\ u\big|_{\partial\mathbb{R}^n_+}=a\varphi,\quad \lim_{x_n\to\infty}\frac{u(x)}{x_n}=\alpha,\\ v\big|_{\partial\mathbb{R}^n_+}=b\psi,\quad \lim_{x_n\to\infty}\frac{v(x)}{x_n}=\beta,\end{gathered}\tag{1.2}$$

where λ,μ are nonnegative constants, the functions $f,g:(0,\infty)\to[0,\infty)$ are continuous and the functions p, q are nonnegative in $B(\mathbb{R}^n_+)$ satisfying some hypotheses related to the Kato class $K^\infty(\mathbb{R}^n_+)$ introduced and studied in [3] for $n\ge 3$ and in [4] for $n=2$. More precisely, we will give two existence results for (1.2) as f and g are nondecreasing or nonincreasing. To this aim, we give in the sequel some notations and we recall some properties of the Kato class defined by means of the Green function $G(x,y)$ of the Dirichlet Laplacian in $\mathbb{R}^n_+$.

Definition 1.1 ([3, 4])**.** A Borel measurable function s in $\mathbb{R}^n_+$ belongs to the Kato class $K^\infty(\mathbb{R}^n_+)$ if

$$\lim_{\alpha\to 0}\sup_{x\in\mathbb{R}^n_+}\int_{\mathbb{R}^n_+\cap B(x,\alpha)}\frac{y_n}{x_n}G(x,y)|s(y)|dy=0,$$

$$\lim_{M\to\infty}\sup_{x\in\mathbb{R}^n_+}\int_{\mathbb{R}^n_+\cap\{|y|\ge M\}}\frac{y_n}{x_n}G(x,y)|s(y)|dy=0.$$

For any nonnegative function f in $B(\mathbb{R}^n_+)$, we denote the Green potential of f defined on $\mathbb{R}^n_+$ by

$$Vf(x):=\int_{\mathbb{R}^n_+}G(x,y)f(y)dy$$

and

$$\|f\|:=\sup_{x\in\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}\frac{y_n}{x_n}G(x,y)f(y)dy.$$

Next, we recall some properties of $K^\infty(\mathbb{R}^n_+)$.

Proposition 1.2. *Let q be a nonnegative function in $K^\infty(\mathbb{R}^n_+)$. Then we have*

(i) $\|q\| < \infty$.
(ii) $Vq \in C_0(\mathbb{R}^n_+)$.

The proof of the above propositions is found in [3, 4].

Theorem 1.3 (3G-Theorem)**.** *There exists a constant $C_0 > 0$ such that for all x, y and z in $\mathbb{R}^n_+$, we have*

$$\frac{G(x,z)G(y,z)}{G(x,y)} \le C_0\Big(\frac{z_n}{x_n}G(x,z) + \frac{z_n}{y_n}G(x,z)\Big).$$

The proof of the above Theorem is found in [3, 4].

Proposition 1.4. *Let q be a nonnegative function in $K^\infty(\mathbb{R}^n_+)$. Then we have*

(i) $\alpha_q := \sup_{x,y\in\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \frac{G(x,z)G(z,y)}{G(x,y)} q(z)dz < \infty$.
(ii) *For any nonnegative superharmonic function v in $\mathbb{R}^n_+$ and all $x \in \mathbb{R}^n_+$, we have*

$$\int_{\mathbb{R}^n_+} G(x,y)v(y)q(y)\,dy \le \alpha_q v(x).$$

(iii) *Let h_0 be a positive harmonic function in $\mathbb{R}^n_+$ which is continuous and bounded in $\overline{\mathbb{R}^n_+}$. Then the family of functions*

$$\Big\{\int_{\mathbb{R}^n_+} G(.,y)h_0(y)p(y)\,dy : |p| \le q\Big\}$$

is relatively compact in $C_0(\mathbb{R}^n_+)$.

Proof. (i) From the 3G-Theorem, we have $\alpha_q \le 2C_0\|q\|$. Which implies by Proposition 1.2 that $\alpha_q < \infty$.

(ii) Let v be a nonnegative superharmonic function in $\mathbb{R}^n_+$. Then by [13, theorem 2.1], there exists a sequence $(f_k)_{k\in\mathbb{N}}$ of nonnegative measurable functions in $\mathbb{R}^n_+$ such that the sequence $(v_k)_k$ defined on $\mathbb{R}^n_+$ by

$$v_k(y) := \int_{\mathbb{R}^n_+} G(y,z)f_k(z)dz$$

increases to v. Since for each $x \in \mathbb{R}^n_+$, we have

$$\int_{\mathbb{R}^n_+} G(x,y)v_k(y)q(y)\,dy \le \alpha_q v_k(x),$$

the result follows from the monotone convergence theorem.

(iii) This assertion was proved in [5, 4]. □

For any nonnegative bounded continuous function φ on $\partial\mathbb{R}^n_+$, we denote by $H\varphi$ the unique bounded harmonic function u in $\mathbb{R}^n_+$ with boundary value φ. As long of this work, we denote by θ the harmonic function defined on $\mathbb{R}^n_+$ by $\theta(x) = x_n$.

Let v and ω be two positive functions on a set S. We denote $v \sim \omega$, if there exists a constant $C > 0$ such that

$$\frac{1}{C}v(x) \le \omega(x) \le Cv(x), \quad \forall x \in S.$$

In this paper, by C we denote a positive generic constant whose value may vary from line to line.

2. First existence result

In this section we will give a first existence result for the system (1.2) in the case where f and g are nondecreasing. We assume the following hypotheses:

- (H1) The functions $f, g : [0,\infty) \to [0,\infty)$ are nondecreasing and continuous.
- (H2) The functions p, q are nonnegative in $\mathbb{R}^n_+$ such that for each positive constant c, the functions $x \mapsto p(x)g(c(x_n+1))$ and $x \mapsto q(x)f(c(x_n+1))$ belong to $K^\infty(\mathbb{R}^n_+)$.
- (H3)

$$\lambda_0 := \inf_{x\in\mathbb{R}^n_+} \frac{\alpha\theta(x) + aH\varphi(x)}{V\big(pg(\beta\theta + bH\psi)\big)(x)} > 0, \quad \mu_0 := \inf_{x\in\mathbb{R}^n_+} \frac{\beta\theta(x) + bH\psi(x)}{V\big(qf(\alpha\theta + aH\varphi)\big)(x)} > 0.$$

Next, we give our first existence result.

Theorem 2.1. *Assume* (H1)–(H3)*. Then for each $\lambda \in [0,\lambda_0)$ and each $\mu \in [0,\mu_0)$, problem* (1.2) *has a positive continuous solution (u,v) such that*

$$(1-\frac{\lambda}{\lambda_0})[\alpha\theta + aH\varphi] \le u \le \alpha\theta + aH\varphi,$$
$$(1-\frac{\mu}{\mu_0})[\beta\theta + bH\psi] \le v \le \beta\theta + bH\psi.$$

For the next Corollary, (H2) and (H3) are replaced by the following hypotheses:

- (H2') The functions p, q are nonnegative in $K^\infty(\mathbb{R}^n_+)$;
- (H3') $\lambda_0' := \inf_{x\in\mathbb{R}^n_+} \frac{H\varphi(x)}{V(pg(H\psi))(x)} > 0$ and $\mu_0' := \inf_{x\in\mathbb{R}^n_+} \frac{H\psi(x)}{V(qf(H\varphi))(x)} > 0$.

Corollary 2.2. *Assume* (H1), (H2'), (H3')*. Then for each $\lambda \in [0,\lambda_0')$ and each $\mu \in [0,\mu_0')$, problem* (1.2) *has a positive bounded continuous solution (u,v) such that*

$$(1-\frac{\lambda}{\lambda_0'})H\varphi \le u \le H\varphi,$$
$$(1-\frac{\mu}{\mu_0'})H\psi \le v \le H\psi.$$

Before proving Theorem 2.1, we give an example where the hypotheses (H2) and (H3) are satisfied.

Example. Let f, g be two continuous functions such that there exists $\eta > 0$ satisfying $0 \le f(t) \le \eta(t+1)$ and $0 \le g(t) \le \eta(t+1)$ for all $t > 0$. Let ψ be a nontrivial nonnegative bounded continuous function in $\partial\mathbb{R}^n_+$. Let $\alpha = 1$, $a = 0$, $\beta = 0$, $b = 1$ and p, q be two nonnegative measurable function in $\mathbb{R}^n_+$ such that

$$0 \le p(y) \le \frac{C}{y_n^\sigma(1+|y|)^{\gamma-\sigma}} \quad \text{with } \sigma < 1 < 3 < \gamma,$$
$$0 \le q(y) \le \frac{C}{y_n^r(1+|y|)^{s-r}} \quad \text{with } r < 1,\ n+2 < s.$$

For this choice of γ, σ and using [3, Proposition 5] we deduce that for each $c > 0$, the functions $y \to p(y)g(c(y_n+1))$; $y \to q(y)f(c(y_n+1))$ and $y \to p_0(y) = \frac{p(y)}{y_n}$ are in $K^\infty(\mathbb{R}^n_+)$. This implies that (H2) is satisfied. Moreover, using Proposition 1.4 we obtain

$$\frac{\theta(x)}{V(pg(H\psi))(x)} \ge C\frac{\theta(x)}{\|g(H\psi)\|_\infty V(p_0\theta)(x)} \ge C\frac{\theta(x)}{\alpha_{p_0}\theta(x)}.$$

Therefore, $\lambda_0 > 0$.

On the other hand taking into account this choice of q, we deduce from [3, Proposition 8] that

$$V(q(1+\theta))(x) \le C\frac{x_n}{(1+|x|)^n}.$$

This together with $H\psi(x) \ge C\frac{x_n}{(1+|x|)^n}$ imply that

$$\frac{H\psi(x)}{V(qf(\theta))(x)} \ge \frac{H\psi(x)}{\eta V(q(1+\theta))(x)} \ge C > 0.$$

Consequently $\mu_0 > 0$.

Proof of Theorem 2.1. Let $\lambda \in [0,\lambda_0)$ and $\mu \in [0,\mu_0)$, then for each $x \in \mathbb{R}^n_+$ we have

$$\lambda_0 V(pg(\beta\theta + bH\psi))(x) \le \alpha\theta(x) + aH\varphi(x),$$
$$\mu_0 V(qf(\alpha\theta + aH\varphi))(x) \le \beta\theta(x) + bH\psi(x).$$

We define the sequences $(u_k)_{k\ge 0}$ and $(v_k)_{k\ge 0}$ by

$$v_0 = \beta\theta + bH\psi,$$
$$u_k = \alpha\theta + aH\varphi - \lambda V(pg(v_k)),$$
$$v_{k+1} = \beta\theta + bH\psi - \mu V(qf(u_k)).$$

We intend to prove that for all $k \in \mathbb{N}$,

$$0 < (1-\frac{\lambda}{\lambda_0})(\alpha\theta + aH\varphi) \le u_k \le u_{k+1} \le \alpha\theta + aH\varphi,$$
$$0 < (1-\frac{\mu}{\mu_0})(\beta\theta + bH\psi) \le v_{k+1} \le v_k \le \beta\theta + bH\psi.$$

For all integer k, we have

$$\begin{aligned} u_k &\ge \alpha\theta + aH\varphi - \lambda V(pg(\beta\theta + bH\psi)) \\ &\ge \alpha\theta + aH\varphi - \frac{\lambda}{\lambda_0}(\alpha\theta + aH\varphi) \\ &\ge (1-\frac{\lambda}{\lambda_0})(\alpha\theta + aH\varphi) > 0. \end{aligned}$$

and

$$\begin{aligned} v_k &\ge \beta\theta + bH\psi - \mu V(qf(\alpha\theta + aH\varphi)) \\ &\ge \beta\theta + bH\psi - \frac{\mu}{\mu_0}(\beta\theta + bH\psi) \\ &\ge (1-\frac{\mu}{\mu_0})(\beta\theta + bH\psi) > 0. \end{aligned}$$

On the other hand, we have $v_1 - v_0 = -\mu V(qf(u_0)) \le 0$ and $u_1 - u_0 = \lambda V(p(g(v_0) - g(v_1)) \ge 0$. Since $u_1 \le \alpha\theta + aH\varphi$, we have

$$u_0 \le u_1 \le \alpha\theta + aH\varphi, \quad v_1 \le v_0 \le \beta\theta + bH\psi.$$

By induction, assume that $u_k \le u_{k+1} \le \alpha\theta + aH\varphi$ and $v_{k+1} \le v_k$. Then, we have

$$v_{k+2} - v_{k+1} = \mu V(q(f(u_k) - f(u_{k+1}))) \le 0,$$
$$u_{k+2} - u_{k+1} = \lambda V(p(g(v_{k+1}) - g(v_{k+2}))) \ge 0.$$

Since $v_{k+1} > 0$, we have,

$$u_{k+1} \leq u_{k+2} \leq \alpha\theta + aH\varphi, \quad v_{k+2} \leq v_{k+1} \leq \beta\theta + bH\psi.$$

Therefore, the sequences $(u_k)_{k\geq 0}$ and $(v_k)_{k\geq 0}$ converge to two functions u and v (respectively) satisfying

$$\begin{gathered}0 < (1 - \frac{\lambda}{\lambda_0})(\alpha\theta + aH\varphi) \leq u \leq \alpha\theta + aH\varphi,\\ 0 < (1 - \frac{\mu}{\mu_0})(\beta\theta + bH\psi) \leq v \leq \beta\theta + bH\psi.\end{gathered}$$

We prove now that (u, v) is a solution for the system (1.2). Since $(u_k)_k$ and $(v_k)_k$ are monotone and f, g are nondecreasing, then the sequences $(f(u_k))_k$ and $(g(v_k))_k$ are monotone. Hence it follows from hypothesis (H2), Proposition 1.2 and Lebesgue's theorem that (u, v) satisfies

$$\begin{gathered}u = \alpha\theta + aH\varphi - \lambda V(pg(v)),\\ v = \beta\theta + bH\psi - \mu V(qf(u)).\end{gathered} \tag{2.1}$$

So (u, v) is a positive continuous solution of (1.2). □

3. Second existence result

Let φ and ψ be two nontrivial nonnegative bounded continuous functions on $\partial\mathbb{R}^n_+$ and $\alpha, \beta \geq 0$. We fix ϕ a nontrivial nonnegative bounded continuous function on $\partial\mathbb{R}^n_+$ and we put $h_0 = H\phi$.

In this section, we aim at proving the existence of positive continuous solutions for the system

$$\begin{gathered}\Delta u = p(x)g(v), \quad \text{in } \mathbb{R}^n_+\\ \Delta v = q(x)f(u), \quad \text{in } \mathbb{R}^n_+\\ u\big|_{\partial\mathbb{R}^n_+} = \varphi, \quad \lim_{x_n\to\infty} \frac{u(x)}{x_n} = \alpha,\\ v\big|_{\partial\mathbb{R}^n_+} = \psi, \quad \lim_{x_n\to\infty} \frac{v(x)}{x_n} = \beta,\end{gathered} \tag{3.1}$$

where f and g are continuous and nonincreasing. We assume the following hypotheses:

- (H4) The functions $f, g : (0, \infty) \to [0, \infty)$ are non-increasing and continuous;
- (H5) the functions $\widetilde{p} := p\frac{f(h_0)}{h_0}$ and $\widetilde{q} := q\frac{g(h_0)}{h_0}$ belong to the Kato class $K^\infty(\mathbb{R}^n_+)$.

Our second existence result is the following.

Theorem 3.1. *Under assumptions* (H4) *and* (H5)*, there exists a constant* $c > 1$ *such that if* $\varphi \geq c\phi$ *and* $\psi \geq c\phi$ *on* $\partial\mathbb{R}^n_+$*, then problem* (3.1) *has a positive continuous solution* (u, v) *satisfying for each* $x \in \mathbb{R}^n_+$,

$$\begin{gathered}\alpha x_n + h_0(x) \leq u(x) \leq \alpha x_n + H\varphi(x),\\ \beta x_n + h_0(x) \leq v(x) \leq \beta x_n + H\psi(x).\end{gathered}$$

We note that this result generalizes those of Athreya [2] and Bachar, Mâagli and Zribi [5] stated for semilinear elliptic equations.

Proof of Theorem 2.1. Let $c = 1 + \alpha_{\widetilde{p}} + \alpha_{\widetilde{q}}$, where $\alpha_{\widetilde{p}}$ and $\alpha_{\widetilde{q}}$ are the constants defined in Proposition 1.4 associated to the functions $\widetilde{p}$ and $\widetilde{q}$ given in hypothesis (H5). Let us consider two nonnegative continuous functions φ and ψ on $\partial\mathbb{R}^n_+$ such that $\varphi \geq c\phi$ and $\psi \geq c\phi$. It follows from the maximum principle that for each $x \in \mathbb{R}^n_+$, we have

$$H\varphi(x) \geq ch_0(x), \quad H\psi(x) \geq ch_0(x).$$

Let $\alpha \geq 0$, $\beta \geq 0$ and Λ be the non-empty closed convex set given by

$$\Lambda = \{w \in C_b(\mathbb{R}^n_+) : h_0 \leq w \leq H\varphi\},$$

where $C_b(\mathbb{R}^n_+)$ denotes the set of continuous bounded functions in $\mathbb{R}^n_+$.

We define the operator T on Λ by

$$T(w) = H\varphi - V(pf[\beta\theta + H\psi - V(qg(w + \alpha\theta))]).$$

And we prove that T has a fixed point. Let $w \in \Lambda$. Since $w + \alpha\theta \geq h_0$, then we deduce from hypotheses (H4) that

$$V(qg(w + \alpha\theta)) \leq V(qg(h_0)).$$

Then

$$\begin{aligned} \beta\theta + H\psi - V(qg(w + \alpha\theta)) &\geq \beta\theta + H\psi - V(\widetilde{q}h_0) \\ &\geq \beta\theta + H\psi - \alpha_{\widetilde{q}}h_0 \\ &\geq \beta\theta + ch_0 - \alpha_{\widetilde{q}}h_0 \\ &= \beta\theta + (1 + \alpha_{\widetilde{p}})h_0 \\ &\geq h_0 > 0. \end{aligned}$$

Hence, $V(pf(\beta\theta+H\psi-V(qg(w+\alpha\theta)))) \leq V(pf(h_0)) = V(\widetilde{p}h_0)$. Using Proposition 1.4 we deduce that the family of functions

$$\big\{V(pf(\beta\theta + H\psi - V(qg(w + \alpha\theta)))) : w \in \Lambda\big\}$$

is relatively compact in $C_0(\mathbb{R}^n_+)$. Since $H\varphi \in C_b(\mathbb{R}^n_+)$, we deduce that the set $T\Lambda$ is relatively compact in $C_b(\mathbb{R}^n_+)$.

Next, we shall prove that T maps Λ into itself. Since $\beta\theta+H\psi-V(qg(w+\alpha\theta)) \geq h_0 > 0$, we have for all $w \in \Lambda$, $Tw(x) \leq H\varphi(x)$, for all $x \in \mathbb{R}^n_+$. Moreover,

$$V(pf(\beta\theta + H\psi - V(qg(w + \alpha\theta)))) \leq V(pf(h_0)) = V(\widetilde{p}h_0) \leq \alpha_{\widetilde{p}}h_0.$$

Then, we obtain $Tw(x) \geq H\varphi - \alpha_{\widetilde{q}}h_0 \geq h_0$, which proves that $T(\Lambda) \subset \Lambda$.

Now, we prove the continuity of the operator T in Λ in the supremum norm. Let $(w_k)_{k\in\mathbb{N}}$ be a sequence in Λ which converges uniformly to a function w in Λ. Then, for each $x \in \mathbb{R}^n_+$, we have

$$|Tw_k(x)-Tw(x)| \leq V[p|f(\beta\theta+H\psi-V(qg(w_k+\alpha\theta)))-f(\beta\theta+H\psi-V(qg(w+\alpha\theta)))|].$$

On the other hand we have

$$\begin{aligned} &p|f(\beta\theta + H\psi - V(qg(w_k + \alpha\theta))) - f(\beta\theta + H\psi - V(qg(w + \alpha\theta)))| \\ &\leq p[f(\beta\theta + H\psi - V(qg(w_k + \alpha\theta))) + f(\beta\theta + H\psi - V(qg(w + \alpha\theta)))] \\ &\leq 2pf(\beta\theta + H\psi - V(qg(h_0))) \\ &\leq 2pf(\beta\theta + H\psi - \alpha_{\widetilde{q}}h_0) \\ &\leq 2pf(h_0) \\ &\leq 2\|h_0\|_\infty\widetilde{p}. \end{aligned}$$

Since $\widetilde{p}$ belongs to $K^\infty(\mathbb{R}^n_+)$, $V\widetilde{p}$ is bounded, we conclude by the dominated convergence theorem that for all $x \in \mathbb{R}^n_+$,

$$Tw_k(x) \to Tw(x) \quad \text{as } k \to +\infty.$$

Consequently, as $T(\Lambda)$ is relatively compact in $C_b(\mathbb{R}^n_+)$, we deduce that the pointwise convergence implies the uniform convergence, namely,

$$\|Tw_k - Tw\|_\infty \to 0 ask \to +\infty.$$

Therefore, T is a continuous mapping from Λ into itself. So, since $T(\Lambda)$ is relatively compact in $C_b(\mathbb{R}^n_+)$, it follows that T is compact mapping on Λ. Finally, the Schauder fixed-point theorem implies the existence of a function $w \in \Lambda$ such that $w = Tw$. For $x \in \mathbb{R}^n_+$, put

$$u(x) = \alpha\theta(x) + w(x), \quad v(x) = \beta\theta(x) + H\psi(x) - V(qg(u)),.$$

Then (u, v) is a positive continuous solution of (3.1). □

Example. Let $\delta > 0$, $\gamma > 0$, $\lambda < 2 < \mu$ and $r < 2 < s$. Let p, q be two nonnegative functions such that

$$p(x) \le \frac{C}{(1+|x|)^{n(1+\delta)+\mu-\lambda} x_n^{\lambda-1-\delta}}, \quad q(x) \le \frac{C}{(1+|x|)^{n(1+\gamma)+s-r} x_n^{r-1-\gamma}}.$$

Let φ, ψ and ϕ be three nontrivial nonnegative bounded continuous functions on $\partial\mathbb{R}^n_+$. Then, for each $\alpha \ge 0$, $\beta \ge 0$, there exist a constant $c > 1$ such that if $\varphi \ge c\phi$ and $\psi \ge c\phi$, the problem

$$\begin{gathered} \Delta u = p(x) v^{-\gamma}, \quad \text{in } \mathbb{R}^n_+ \\ \Delta v = q(x) u^{-\delta}, \quad \text{in } \mathbb{R}^n_+ \\ u\big|_{\partial\mathbb{R}^n_+} = \varphi, \quad \lim_{x_n\to\infty} \frac{u(x)}{x_n} = \alpha, \\ v\big|_{\partial\mathbb{R}^n_+} = \psi, \quad \lim_{x_n\to\infty} \frac{v(x)}{x_n} = \beta, \end{gathered}$$

has a positive continuous solution (u, v) satisfying for each $x \in \mathbb{R}^n_+$,

$$\begin{gathered} \alpha x_n + H\phi(x) \le u(x) \le \alpha x_n + H\varphi(x), \\ \beta x_n + H\phi(x) \le v(x) \le \beta x_n + H\psi(x). \end{gathered}$$

References

[1] D. H. Armitage; S. J. Gardiner: *Classical Potential Theory*, Springer-Verlag (2001).
[2] S. Atherya: *On a singular Semilinear Elliptic Boundary Value Problem and the Boundary Harnack Principle*, Potential Anal. 17 (2002) 293-301.
[3] I. Bachar; H. Mâagli: *Estimates on the Green's Function and Existence of Positive Solutions of Nonlinear Singular Elliptic Equations in the Half Space*, Positivity. Vol.9, (2005) 153-192.
[4] I. Bachar; H. Mâagli; L. Mâatoug: *Positive solutions of nonlinear elliptic equations in a half space in* $\mathbb{R}^2$ Electronic Journal of Differential Equations, Vol. 2002 (2002), No. 41, pp. 1-24.
[5] I. Bachar ; H. Mâagli; M. Zribi: *Estimates on the Green Function and Existence of Positive Solutions for some Polyharmonic Nonlinear Equations in the Half Space*, Manuscripta. Math. Vol. 113, (2004), 269-291.
[6] S. Chen; G. Lu: *Existence and nonexistence of positive solutions for a class of semilinear elliptic systems*, Nonlinear Anal. 38. No.7 (1999) 919-932.
[7] K. L. Chung; Z. Zhao: *From Brownian Motion to Schrödinger's Equation*, Springer (1995).
[8] F. C. Cirstea; V. D. Radulescu: *Entire solutions blowing up at infinity for semilinear elliptic systems*, J. Math. Pures. Appl. 81 (2002), 827-846.

[9] A. Ghanmi; H. Mâagli; S. Turki and N. Zeddini: *Existence of Positive Bounded Solutions for Some Nonlinear Elliptic Systems*, J. Math. Anal. Appl. doi:10.1016/j.jmaa.2008.04.029
[10] M. Ghergu, V. Radulescu: *Explosive solutions of semilinear elliptic systems with gradient term*, RACSAM Revista Real Academia de Ciencias (Serie A, Matemáticas) 97 (2003), 437-445.
[11] A. Kristály; V. Radulescu; C. Varga; *Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Encyclopedia of Mathematics and its Applications, No. 136, Cambridge University Press, Cambridge, 2010.
[12] A. V. Lair; A. W. Wood: *Existence of entire large positive solutions of semilinear elliptic systems*, Journal of Differential Equations. 164. No.2 (2000), 380-394.
[13] S. C. Port and C. J. Stone; *Brownian Motion and Classical Potential Theory*, Academic Press, 1978.
[14] X. Wang; Aihua W. Wood: *Existence and nonexistence of entire positive solutions of semilinear elliptic systems*, J. Math. Anal. Appl. 267, (2002), 361-368.

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