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# PERIODIC BOUNDARY-VALUE PROBLEMS AND THE DANCER-FUČÍK SPECTRUM UNDER CONDITIONS OF RESONANCE

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Abstract. We prove the existence of solutions to the nonlinear $2\pi$-periodic problem

$$\begin{gathered}u''(x)+\mu u^+(x)-\nu u^-(x)+g(x,u(x))=f(x)\,,\quad x\in(0,2\pi)\,,\\ u(0)-u(2\pi)=0\,,\quad u'(0)-u'(2\pi)=0,\end{gathered}$$

where the point $(\mu,\nu)$ lies in the Dancer-Fučík spectrum, with

$$0<\frac{4}{9}\mu\leqslant\nu<\mu\quad\text{and}\quad\mu<(m+1)^2,$$

for some natural number $m$, and the nonlinearity $g(x,\xi)$ is bounded with primitive, $G(x,\xi)$, satisfying a Landesman-Lazer type condition introduced by Tomiczek in 2005. We use variational methods based on the generalization of the Saddle Point Theorem of Rabinowitz.

## 1. Introduction

We consider the question of existence of solutions to the problem

$$\begin{gathered}-u''(x)=\mu u^+(x)-\nu u^-(x)+g(x,u(x))+f(x),\quad x\in(0,2\pi)\\ u(0)=u(2\pi),\quad u'(0)=u'(2\pi)\end{gathered}\tag{1.1}$$

where $u^\pm=\max\{\pm u,0\}$; $\mu,\nu\in\mathbb{R}$ and $\mu,\nu>0$; $g\colon[0,2\pi]\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying

$$|g(x,\xi)|\leqslant p(x)\quad\text{for a.e. } x\in[0,2\pi],\text{ and all }\xi\in\mathbb{R},\tag{1.2}$$

where $p\in L^1[0,2\pi]$; and $f$ is in $L^1([0,2\pi])$.

A function $g\colon[0,2\pi]\times\mathbb{R}\to\mathbb{R}$ is said to be a Carathéodory function if the map $x\mapsto g(x,\xi)$ is Lebesgue measurable for all $\xi\in\mathbb{R}$, and the map $\xi\mapsto g(x,\xi)$ is continuous for a.e. $x\in[0,2\pi]$.

For the case in which $g\equiv0$ and $f\equiv0$, problem (1.1) yields the homogeneous, piece-wise linear problem

$$\begin{gathered}-u''(x)=\mu u^+(x)-\nu u^-(x),\quad x\in(0,2\pi)\\ u(0)=u(2\pi),\quad u'(0)=u'(2\pi).\end{gathered}\tag{1.3}$$

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It is well known (see, for instance, [5]) that problem (1.3) has non-trivial solutions only when the pairs $(\mu, \nu)$ lie in the set of points made up of the curves

$$\Sigma_o = \{(\mu,\nu) \in \mathbb{R}^2 \mid \nu = 0\} \cup \{(\mu,\nu) \in \mathbb{R}^2 \mid \mu = 0\}, \tag{1.4}$$

$$\Sigma_m = \big\{(\mu,\nu) \in \mathbb{R}^2 \mid m\big(\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}}\big) = 2\big\}, \quad m = 1, 2, 3, \dots \tag{1.5}$$

The collection of all the curves, $\Sigma_m$, $m = 0, 1, 2, \dots$, defined in (1.4) and (1.5), is known in the literature as the Fučík spectrum, or Dancer-Fučík spectrum, associated with the boundary-value problem (1.3). We will denote this set by $\Sigma$, so that

$$\Sigma = \cup_{m=0}^{\infty} \Sigma_m. \tag{1.6}$$

The first three curves making up $\Sigma$ in (1.6) are pictured in Figure 1.

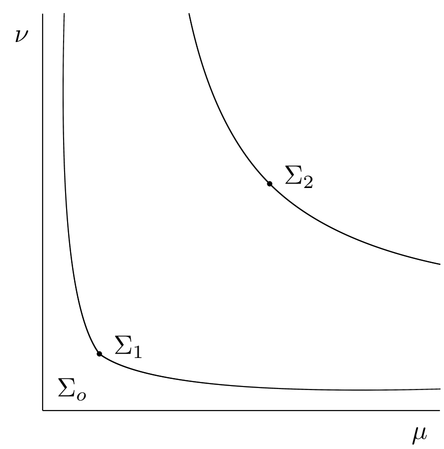


FIGURE 1. $\Sigma_0$, $\Sigma_1$ and $\Sigma_2$

We search for conditions under which the boundary-value problem in (1.1) has solutions for the case in which $(\mu, \nu) \in \Sigma$, $\mu > 0$, $\nu > 0$; i.e.,

$$m\big(\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}}\big) = 2, \tag{1.7}$$

for some $m = 1, 2, 3, \dots$, and $g$ is a Carathéodory function satisfying the bound in (1.2). We will assume further that

$$\frac{4}{9}\mu \leqslant \nu < \mu, \tag{1.8}$$

$$\mu < (m+1)^2. \tag{1.9}$$

Note that for the case in which $1 \leqslant m \leqslant 4$, the left-most inequality in (1.8) together with (1.7) implies the condition (1.9). For the rest of the cases, $m \geqslant 5$, we will assume the conjunction of the conditions in (1.8) and (1.9).

Figure 2 illustrates conditions (1.7) and (1.8) for $\Sigma_1$ and $\Sigma_2$. We are restricting $(\mu, \nu)$ to lie on portion of a branch of the Dancer-Fučík spectrum which lies below the line $\nu = \mu$ and above the line $\nu = \frac{4}{9}\mu$, for the case $1 \leqslant m \leqslant 4$. For the case, $m \geqslant 5$, we further restrict that portion to lie to the left of the line $\mu = (m+1)^2$, by condition (1.9).

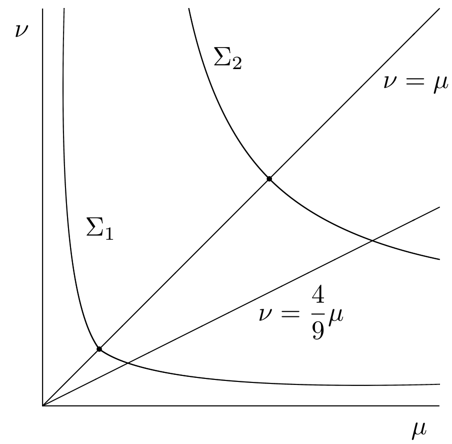


FIGURE 2. Conditions (1.7) and (1.8) for $\Sigma_1$ and $\Sigma_2$

We note for future reference that condition (1.7), together with the right-most inequality in (1.8), yields that

$$\mu > m^2 \quad \text{and} \quad \nu < m^2. \tag{1.10}$$

In addition, the left-most inequality in (1.8) in conjunction with (1.7), for the case $1 \leqslant m \leqslant 4$, or the inequality in (1.9) in conjunction with (1.7), for the case $m \geqslant 5$, can be used to obtain

$$\nu > (m-1)^2. \tag{1.11}$$

The assumption that $(\mu, \nu) \in \Sigma$ and the bound (1.2) on $g$ make problem (1.1) into a problem with asymptotics at resonance with respect to the Dancer-Fučík spectrum. In fact, writing

$$\widetilde{g}(x,\xi) = \mu\xi^+ - \nu\xi^- + g(x,\xi),$$

we see that

$$\lim_{\xi\to+\infty} \frac{\widetilde{g}(x,\xi)}{\xi} = \mu \quad \text{and} \quad \lim_{\xi\to-\infty} \frac{\widetilde{g}(x,\xi)}{\xi} = \nu$$

for a.e. $x \in [0, 2\pi]$, by the bound (1.2) on $g$; so that, asymptotically, the boundary-value problem (1.1) is related to the piece-wise linear problem

$$\begin{gathered} -u''(x) = \mu u^+(x) - \nu u^-(x) + f(x), \quad x \in (0, 2\pi); \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \end{gathered}$$

By analogy to what happens in the linear case, $\mu = \nu = \lambda$, where $\lambda$ is an eigenvalue of the linear boundary-value problem

$$\begin{gathered} -u''(x) = \lambda u(x), \quad x \in (0, 2\pi) \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{gathered}$$

in which the solvability of the boundary-value problem

$$\begin{gathered} -u''(x) = \lambda u(x) + f(x), \quad x \in (0, 2\pi); \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{gathered}$$

depends on conditions imposed on $f$ with the respect to the eigenspace corresponding to $\lambda$ (more specifically, requiring that $f$ be orthogonal to the eigenfunctions corresponding to $\lambda$), we expect the solvability of (1.1) to depend on conditions imposed on $f$ and $g$ in relation to the set of of nontrivial solutions to the homogeneous, piece-wise linear problem in (1.3). Solvability conditions for semilinear boundary-value problems for elliptic equations have been around since the pioneering work of Landesman and Lazer [7] in 1970. In this work, we will impose a condition on the primitive of $g$,

$$G(x,\xi) = \int_0^{\xi} g(x,t)\,\mathrm{d}t, \quad \text{for } x \in [0,2\pi] \text{ and } \xi \in \mathbb{R}, \tag{1.12}$$

and the function $f$ in connection with nontrivial solutions to the piecewise linear problem (1.3) corresponding to the pair $(\mu,\nu) \in \Sigma$. Specifically, we require that

$$\int_0^{2\pi} [G_+(x)\psi^+(x) - G_-(x)\psi^-(x) + f(x)\psi(x)]\,\mathrm{d}x > 0, \tag{1.13}$$

for any nontrivial solution, $\psi$, of (1.3) corresponding to $(\mu,\nu)$, where

$$G_+(x) = \liminf_{\xi\to+\infty} \frac{G(x,\xi)}{\xi} \quad \text{and} \quad G_-(x) = \limsup_{\xi\to-\infty} \frac{G(x,\xi)}{\xi}. \tag{1.14}$$

We shall refer to the condition in (1.13) and (1.14) as the Tomiczek condition. Tomiczek [11] introduced similar conditions in the study of two-point boundary-value problems with Dirichlet boundary conditions in 2001 as a generalization to the Landesman-Lazer condition.

The main result in this paper is the following theorem.

**Theorem 1.1.** *Let $f$ be a function in $L^1[0,2\pi]$, and suppose $\mu > 0$ and $\nu > 0$ satisfy* (1.7), (1.8) *and* (1.9) *for some $m \in \mathbb{N}$. Suppose also that $g$ is a Carathéodory function satisfying the bound in* (1.2)*, and whose primitive, $G$, defined in* (1.12)*, along with $f$, satisfy the Tomiczek condition in* (1.13) *and* (1.14) *for any nontrivial solution, $\psi$, of* (1.3) *corresponding to $(\mu,\nu) \in \Sigma_m$. Then, the boundary-value problem in* (1.1) *has at least one solution.*

By a solution of the boundary-value problem (1.1) we mean an absolutely continuous functions, $u\colon [0,2\pi] \to \mathbb{R}$, which can be extended to a $2\pi$-periodic function in $\mathbb{R}$, such that $u' \in L^2[0,2\pi]$, and for which

$$\int_0^{2\pi} u'v'\,\mathrm{d}x - \int_0^{2\pi} [\mu u^+(x) - \nu u^-(x) + g(x,u(x)) + f(x)]v(x)\,\mathrm{d}x = 0 \tag{1.15}$$

for all $C^1$, $2\pi$-periodic functions $v$.

We will denote by $H$ the space of absolutely continuous, $2\pi$-periodic functions with $L^2[0,2\pi]$ derivatives. In the proof of Theorem 1.1 we will apply a generalization of the Saddle Point Theorem of Rabinowitz [8], where the space $H$ is decomposed into cones instead of subspaces. In order to come up with the proper decomposition needed in the saddle point theorem, we need to learn more about the structure of the set of solutions of the piece-wise linear problem (1.3). Section 2 will be devoted to the study of that problem. In Section 3 we develop the variational setting needed for the application of the saddle point theorem.

**Remark 1.2.** The work in this paper was motivated by a result of Castro and Chang (see [1]), in which they proved the existence of a solution for the boundary-value problem

$$\begin{gathered} -u'' = \widetilde{g}(u) + f, \quad \text{in } (0, 2\pi) \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{gathered} \tag{1.16}$$

for any $f$ in $L^1[0, 2\pi]$, where the nonlinearity, $\widetilde{g}$, satisfies $\xi\widetilde{g}(\xi) > 0$ for $|\xi| \geqslant \xi_o$, for some $\xi_o > 0$, and the primitive of $\widetilde{g}$, $\widetilde{G}(u) = \int_0^u \widetilde{g}(t)\,\mathrm{d}t$, satisfies the conditions

$$\liminf_{u\to+\infty} \frac{2\widetilde{G}(u)}{u^2} = \mu \quad \text{and} \quad \lim_{u\to-\infty} \frac{2\widetilde{G}(u)}{u^2} = \nu,$$

or

$$\lim_{u\to+\infty} \frac{2\widetilde{G}(u)}{u^2} = \mu \quad \text{and} \quad \liminf_{u\to-\infty} \frac{2\widetilde{G}(u)}{u^2} = \nu,$$

where $\mu > 0$, $\nu > 0$ and

$$\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} > 2. \tag{1.17}$$

Observe that condition (1.17) puts the pair $(\mu, \nu)$ outside of the realm of the Fučík-Dancer spectrum. In fact, condition (1.17) implies that $(\mu, \nu)$ lies in the region bounded by the curves $\Sigma_o$ and $\Sigma_1$ of the Fučík-Dancer spectrum (see Figure 1). Thus, our result complements the result of Castro and Chang in [1] by shedding some light on what happens when $(\mu, \nu)$ lies in a portion of the curve $\Sigma_1$.

**Remark 1.3.** The result by Castro and Chang mentioned in the previous remark falls in the category of nonresonance problems. In the literature on semilinear boundary-value problems, of which problem (1.1) is an instance, nonresonance refers to the situation in which existence of solutions can be obtained for any forcing term, $f$, in the equation. This is to be contrasted with the situation in this paper in which existence is obtained for a certain class of functions, $f$, satisfying the Tomiczek condition in (1.13) and (1.14).

For other nonresonance results related to the work in this paper, the reader is referred to the articles by Gossez and Omari in [6] and Cuesta and Gossez in [2].

**Remark 1.4.** One of the earliest resonance results with respect to the Fučik-Dancer spectrum related to the problem presented in this paper goes back to the work of Fabry and Fonda [4] in 1998. The results in [4] applied to the boundary-value problem (1.1) yield the following Landesman-Lazer type sufficient condition for solvability:

$$\int_0^{2\pi} [g_+(x)\psi^+(x) - g_-(x)\psi^-(x) + f(x)\psi(x)]\,\mathrm{d}x > 0, \tag{1.18}$$

for any nontrivial solution, $\psi$, of (1.3) corresponding to $(\mu, \nu) \in \Sigma_m$, where

$$g_+(x) = \liminf_{\xi\to+\infty} g(x, \xi) \quad \text{and} \quad g_-(x) = \limsup_{\xi\to-\infty} g(x, \xi). \tag{1.19}$$

The condition in (1.18) and (1.19) is to be contrasted with the Tomiczek condition in (1.13) and (1.14), in that the latter is a condition on the primitive, $G$, of $g$, while the former is a condition on $g$.

Another contrast between the work of Fabry and Fonda in [4] and the current work is that Fabry and Fonda use degree theoretic techniques, while we use variational techniques.

## 2. A Structure Theorem

In this section we describe the structure of the set of solutions to the homogeneous, piece-wise linear problem (1.3),

$$\begin{gathered} -u''(x) = \mu u^+(x) - \nu u^-(x), \quad x \in (0, 2\pi) \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{gathered}$$

where the pair $(\mu, \nu)$ lies in the Dancer-Fučik spectrum, $\Sigma$, with $\mu > 0$ and $\nu > 0$. More specifically, suppose that $\mu > 0$, $\nu > 0$ and

$$m\Big(\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}}\Big) = 2, \quad \text{for some } m = 1, 2, 3, \dots \tag{2.1}$$

In other words, $(\mu, \nu) \in \Sigma_m$ for some $m \in \mathbb{N}$.

We will first prove that any nontrivial solution to the boundary-value problem (1.3) corresponding to $(\mu, \nu) \in \Sigma_m$ must be a translate, or phase shift, of a positive multiple of the function $\psi_{m,\mu,\nu} \colon \mathbb{R} \to \mathbb{R}$ given by

$$\psi_{m,\mu,\nu}(x) = \begin{cases} \frac{1}{\sqrt{\mu}} \sin(\sqrt{\mu} x) & \text{for } 0 \leqslant x < \tau_1; \\ -\frac{1}{\sqrt{\nu}} \sin(\sqrt{\nu}(x - \tau_1)) & \text{for } \tau_1 \leqslant x < \tau_2; \\ \dots \\ \frac{1}{\sqrt{\mu}} \sin(\sqrt{\mu}(x - \tau_{2m-2})) & \text{for } \tau_{2m-2} \leqslant x < \tau_{2m-1}; \\ -\frac{1}{\sqrt{\nu}} \sin(\sqrt{\nu}(x - \tau_{2m-1})) & \text{for } \tau_{2m-1} \leqslant x \leqslant 2\pi, \end{cases} \tag{2.2}$$

after it has been extended to be $2\pi$-periodic over all of $\mathbb{R}$, where

$$\tau_0 = 0, \quad \tau_{2m} = 2\pi, \tag{2.3}$$

and, for any odd $k$,

$$\tau_k - \tau_{k-1} = \frac{\pi}{\sqrt{\mu}}; \tag{2.4}$$

while, for any even $k$,

$$\tau_k - \tau_{k-1} = \frac{\pi}{\sqrt{\nu}}. \tag{2.5}$$

More precisely, we have the Structure Theorem for Solutions of (1.3):

**Theorem 2.1.** *Let $u$ be any non-trivial solution to* (1.3) *corresponding to positive real numbers $\mu$ and $\nu$. Then, $(\mu, \nu) \in \Sigma_m$ for some natural number $m$, and there exist numbers $\theta$ and $C$, with $C > 0$, such that*

$$u(x) = C\psi_{m,\mu,\nu}(x - \theta) \quad \textit{for } 0 \leqslant x \leqslant 2\pi.$$

**Remark 2.2.** The fact that the function $\psi_{m,\mu,\nu}$ defined in (2.2) solves the boundary-value problem (1.3) can be verified directly. We can also proceed as in [5, Chapter 42] and search for a solution, $u$, of (1.3) satisfying

$$u(0) = 0, \tag{2.6}$$
$$u'(0) = 1. \tag{2.7}$$

See, for instance, the proof of Lemma 42.2 on page 323 in [5].

**Remark 2.3.** Note that the solution, $\psi_{m,\mu,\nu}$, to problem (1.3) defined in (2.2) has $2m-1$ zeros in the interval $(0,2\pi)$; namely, $\tau_1,\tau_2,\dots,\tau_{2m-1}$, given by the conditions in (2.3)–(2.5). It then follows from those conditions that

$$\sum_{k=1}^{2m}\frac{\tau_k-\tau_{k-1}}{\pi}=\frac{m}{\sqrt{\mu}}+\frac{m}{\sqrt{\nu}}.$$

On the other hand

$$\sum_{k=1}^{2m}\frac{\tau_k-\tau_{k-1}}{\pi}=\frac{\tau_{2m}-\tau_0}{\pi}=2,$$

which yields equation (2.1).

**Remark 2.4.** Observe that, as a result of the condition in (2.1), the solution, $\psi_{m,\mu,\nu}$, to problem (1.3) defined in (2.2) is a periodic function of period $\tau_2=2\pi/m$. So that, $\psi_{m,\mu,\nu}$ can also be thought of the $\tau_2$-periodic extension of the function

$$\psi_{m,\mu,\nu}(x)=\begin{cases}\frac{1}{\sqrt{\mu}}\sin(\sqrt{\mu}x) & \text{for } 0\leqslant x<\tau_1;\\ -\frac{1}{\sqrt{\nu}}\sin(\sqrt{\nu}(x-\tau_1)) & \text{for } \tau_1\leqslant x<\tau_2.\end{cases}\tag{2.8}$$

*Proof of the Structure Theorem, Theorem 2.1.* Let $u$ denote a non-trivial solution to the boundary value problem (1.3) with positive $\mu$ and $\nu$. Then, $u$ can be extended to a $2\pi$-periodic function defined on $\mathbb{R}$ and satisfying the differential equation

$$-u''=\mu u^+-\nu u^- \quad \text{in } \mathbb{R}.\tag{2.9}$$

We claim that $u$ must have at least two zeroes in the interval $[0,2\pi]$; otherwise, $u$ would be either strictly positive or strictly negative on $[0,2\pi]$, and then $u$ would solve an equation of the type

$$-u''=\lambda u,\tag{2.10}$$

where $\lambda>0$. But any solution to (2.10) with $\lambda>0$ must oscillate through 0; this is a contradiction. One of the zeroes, call it $\theta$, of $u$ in $[0,2\pi]$ must have the property that $u'(\theta)>0$; this follows from the existence and uniqueness theorem for ordinary differential equations, since we are assuming the $u$ is non-trivial. Shift $u$ to the left by $\theta$ to obtain a $2\pi$-periodic function,

$$w(x)=u(x+\theta),\tag{2.11}$$

satisfying the differential equation in (2.9) and the initial conditions

$$w(0)=0 \quad\text{and}\quad w'(0)>0.$$

Put

$$v(x)=\frac{1}{w'(0)}w(x).\tag{2.12}$$

Then, $v$ is non-trivial, $2\pi$-periodic and satisfies the differential equation in (2.9). Furthermore, $v$ satisfies the initial conditions

$$v(0)=0,\tag{2.13}$$

$$v'(0)=1.\tag{2.14}$$

Observe also that $v$ must have an odd number, $2m-1$, of zeroes in $(0,2\pi)$, for some positive integer $m$; otherwise, we would have $v'(2\pi)<0$, by (2.13); however, $v'(2\pi)=v'(0)=1>0$, by (2.14) and the fact the $v$ is $2\pi$-periodic. This is a contradiction.

By the existence and uniqueness theorem for ordinary differential equations applied to the differential equation in (2.9) and the initial conditions in (2.13) and (2.14), we obtain

$$v(x) = \psi_{m,\mu,\nu}(x) \quad \text{for all } x \in \Re,$$

in light of (2.6) and (2.7) in Remark 2.2. It then follows from the calculations in Remark 2.3 that

$$m\Big(\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}}\Big) = 2,$$

which shows that $(\mu, \nu) \in \Sigma_m$. Furthermore, in view of (2.11) and (2.12) we see that

$$u(x) = u'(\theta)\ \psi_{m,\mu,\nu}(x - \theta),$$

which completes the proof of Theorem 2.1. □

**Remark 2.5.** A closer examination of the proof of Theorem 2.1 reveals that the value of $\theta$ given in the theorem may be chosen so that

$$-\frac{\pi}{\sqrt{\mu}} \leqslant \theta \leqslant \frac{\pi}{\sqrt{\nu}}.$$

Set

$$\theta_1 = -\frac{\pi}{\sqrt{\mu}} \quad \text{and} \quad \theta_2 = \frac{\pi}{\sqrt{\nu}}. \tag{2.15}$$

We will therefore restrict the values of $\theta$ in Theorem 2.1 so that $\theta \in [\theta_1, \theta_2]$.

Let $u$ be a nontrivial solution of the boundary-value problem (1.3) for positive $\mu$ and $\nu$. In view of the structure theorem (Theorem 2.1) and the remark following the proof of that theorem,

$$u(x) = C\psi_{m,\mu,\nu}(x - \theta) \quad \text{for all } x,$$

for some positive constant $C$ and $\theta \in [\theta_1, \theta_2]$, where $\theta_1$ and $\theta_2$ are given in (2.15). We then have that

$$u(x) = \begin{cases} \frac{C}{\sqrt{\mu}} \sin\big(\sqrt{\mu}(x - \theta)\big) & \text{for } 0 \leqslant x < \tau_1; \\ -\frac{C}{\sqrt{\nu}} \sin(\sqrt{\nu}(x - \theta - \tau_1)) & \text{for } \tau_1 \leqslant x < \tau_2; \\ \dots \\ \frac{C}{\sqrt{\mu}} \sin(\sqrt{\mu}(x - \theta - \tau_{2m-2})) & \text{for } \tau_{2m-2} \leqslant x < \tau_{2m-1}; \\ -\frac{C}{\sqrt{\nu}} \sin(\sqrt{\nu}(x - \theta - \tau_{2m-1})) & \text{for } \tau_{2m-1} \leqslant x \leqslant 2\pi, \end{cases} \tag{2.16}$$

where $\tau_o, \tau_1, \dots, \tau_{2m}$ are given by the conditions in (2.3)–(2.5).

From (2.16) we obtain the expansion for $u$:

$$u = C\cos(\sqrt{\mu}\theta)\varphi_{m,1} + C\sin(\sqrt{\mu}\theta)\varphi_{m,2} + C\cos(\sqrt{\nu}\theta)\varphi_{m,3} + C\sin(\sqrt{\nu}\theta)\varphi_{m,4},$$

in terms of the functions

$$\varphi_{m,1}(x) = \begin{cases} \frac{1}{\sqrt{\mu}} \sin(\sqrt{\mu}x) & \text{for } 0 \leqslant x < \tau_1; \\ 0 & \text{for } \tau_1 \leqslant x < \tau_2; \\ \frac{1}{\sqrt{\mu}} \sin(\sqrt{\mu}(x - \tau_2)) & \text{for } \tau_2 \leqslant x < \tau_3; \\ \dots \\ \frac{1}{\sqrt{\mu}} \sin(\sqrt{\mu}(x - \tau_{2m-2})) & \text{for } \tau_{2m-2} \leqslant x < \tau_{2m-1}; \\ 0 & \text{for } \tau_{2m-1} \leqslant x \leqslant 2\pi; \end{cases} \tag{2.17}$$

$$\varphi_{m,2}(x) = \begin{cases} -\frac{1}{\sqrt{\mu}}\cos(\sqrt{\mu}x) & \text{for } 0 \leqslant x < \tau_1; \\ 0 & \text{for } \tau_1 \leqslant x < \tau_2; \\ -\frac{1}{\sqrt{\mu}}\cos(\sqrt{\mu}(x-\tau_2)) & \text{for } \tau_2 \leqslant x < \tau_3; \\ \dots \\ -\frac{1}{\sqrt{\mu}}\cos\left(\sqrt{\mu}(x-\tau_{2m-2})\right) & \text{for } \tau_{2m-2} \leqslant x < \tau_{2m-1}; \\ 0 & \text{for } \tau_{2m-1} \leqslant x \leqslant 2\pi; \end{cases} \tag{2.18}$$

$$\varphi_{m,3}(x) = \begin{cases} 0 & \text{for } 0 \leqslant x < \tau_1; \\ -\frac{1}{\sqrt{\nu}}\sin\left(\sqrt{\nu}(x-\tau_1)\right) & \text{for } \tau_1 \leqslant x < \tau_2; \\ 0 & \text{for } \tau_2 \leqslant x < \tau_3; \\ \dots \\ 0 & \text{for } \tau_{2m-2} \leqslant x < \tau_{2m-1}; \\ -\frac{1}{\sqrt{\nu}}\sin(\sqrt{\nu}(x-\tau_{2m-1})) & \text{for } \tau_{2m-1} \leqslant x \leqslant 2\pi; \end{cases} \tag{2.19}$$

and

$$\varphi_{m,4}(x) = \begin{cases} 0 & \text{for } 0 \leqslant x < \tau_1; \\ \frac{1}{\sqrt{\nu}}\cos(\sqrt{\nu}(x-\tau_1)) & \text{for } \tau_1 \leqslant x < \tau_2; \\ 0 & \text{for } \tau_2 \leqslant x < \tau_3; \\ \dots \\ 0 & \text{for } \tau_{2m-2} \leqslant x < \tau_{2m-1}; \\ \frac{1}{\sqrt{\nu}}\cos(\sqrt{\nu}(x-\tau_{2m-1})) & \text{for } \tau_{2m-1} \leqslant x \leqslant 2\pi. \end{cases} \tag{2.20}$$

Observe that the functions $\varphi_{m,1}, \varphi_{m,2}, \varphi_{m,3}, \varphi_{m,4}$ are mutually orthogonal; that is,

$$\int_0^{2\pi} \varphi_{m,i}(x)\varphi_{m,j}(x)\,\mathrm{d}x = 0 \quad \text{for } i \neq j,$$

in view of the conditions (2.3)–(2.5) defining $\tau_i$, $i = 0, 1, \dots, 2m$. Hence, the set $\{\varphi_{m,1}, \varphi_{m,2}, \varphi_{m,3}, \varphi_{m,4}\}$ is linearly independent and it, therefore, forms a basis for a four-dimensional subspace of $H$.

For parameters $\mu > 0$, $\nu > 0$ and $m \in \mathbb{N}$ satisfying (1.7), we define the following subset, $K_{m,\mu,\nu}$, of the linear span of $\{\varphi_{m,1}, \varphi_{m,2}, \varphi_{m,3}, \varphi_{m,4}\}$.

**Definition 2.6** (Definition of $K_{m,\mu,\nu}$)**.** Put

$$z_\theta = \cos(\sqrt{\mu}\theta)\varphi_{m,1} + \sin(\sqrt{\mu}\theta)\varphi_{m,2} + \cos(\sqrt{\nu}\theta)\varphi_{m,3} + \sin(\sqrt{\nu}\theta)\varphi_{m,4}, \tag{2.21}$$

for $\theta \in [\theta_1, \theta_2]$, where the functions $\varphi_{m,1}$, $\varphi_{m,2}$, $\varphi_{m,3}$ and $\varphi_{m,4}$ are given in (2.17)–(2.20), and $\theta_1$ and $\theta_2$ are given in (2.15). Thus, $z_\theta$ is the phase shift by $\theta$, or the horizontal translation of the of the $\tau_2$-periodic function given in (2.8). Define

$$K_{m,\mu,\nu} = \{\rho z_\theta \mid \rho \geqslant 0,\ \theta \in [\theta_1, \theta_2]\}. \tag{2.22}$$

Theorem 2.1 then says that $K_{m,\mu,\nu}$ is precisely the set of solutions of the boundary-value problem (1.3) corresponding to the pair $(\mu, \nu) \in \Sigma_m$.

For future reference, we end this section with the results of calculations involving the $L^2$ norms of the functions, $\varphi_{m,1}, \varphi_{m,2}, \varphi_{m,3}, \varphi_{m,4}$, defined in (2.17)–(2.20), and their $L^2$ inner products with $\sin mx$ and $\cos mx$. These are summarized in Table 1 on page 10, where $\langle\cdot,\cdot\rangle$ denotes the $L^2[0,2\pi]$ inner product; that is, given functions $v, w \in L^2[0,2\pi]$,

$$\langle v, w\rangle = \int_0^{2\pi} v(x)w(x)\,\mathrm{d}x.$$

TABLE 1. $L^2$ Inner Products and Norms

| $i$ | $\langle\varphi_{m,i}, \cos mx\rangle$ | $\langle\varphi_{m,i}, \sin mx\rangle$ | $\Vert\varphi_{m,i}\Vert_2^2$ | $\Vert\varphi'_{m,i}\Vert_2^2$ |
|---|---|---|---|---|
| 1 | $\frac{m(\cos m\tau_1 + 1)}{\mu - m^2}$ | $\frac{m\sin m\tau_1}{\mu - m^2}$ | $\frac{m\pi}{2\mu\sqrt{\mu}}$ | $\frac{m\pi}{2\sqrt{\mu}}$ |
| 2 | $-\frac{m^2\sin m\tau_1}{\sqrt{\mu}(\mu - m^2)}$ | $\frac{m^2(\cos m\tau_1 + 1)}{\sqrt{\mu}(\mu - m^2)}$ | $\frac{m\pi}{2\mu\sqrt{\mu}}$ | $\frac{m\pi}{2\sqrt{\mu}}$ |
| 3 | $-\frac{m(\cos m\tau_1 + 1)}{\nu - m^2}$ | $-\frac{m\sin m\tau_1}{\nu - m^2}$ | $\frac{m\pi}{2\nu\sqrt{\nu}}$ | $\frac{m\pi}{2\sqrt{\nu}}$ |
| 4 | $\frac{m^2\sin m\tau_1}{\sqrt{\nu}(\nu - m^2)}$ | $-\frac{m^2(\cos m\tau_1 + 1)}{\sqrt{\nu}(\nu - m^2)}$ | $\frac{m\pi}{2\nu\sqrt{\nu}}$ | $\frac{m\pi}{2\sqrt{\nu}}$ |

The last column in Table 1 contains the $L^2[0,2\pi]$ norms of the derivatives, $\varphi'_{m,i}$, for $i = 1, 2, 3, 4$.

## 3. Variational Setting

Let $H$ denote the Sobolev space of absolutely continuous functions on $[0, 2\pi]$, which can be extended to $2\pi$-periodic functions in $\mathbb{R}$, and whose derivatives are in $L^2[0, 2\pi]$. The space $H$ is endowed with the norm, $\Vert\cdot\Vert$, defined by

$$\Vert v\Vert = \Big(\int_0^{2\pi} v^2\,\mathrm{d}x + \int_0^{2\pi} (v')^2\,\mathrm{d}x\Big)^{1/2}, \quad \text{for all } v \in H, \tag{3.1}$$

or $\Vert v\Vert^2 = \Vert v\Vert_2^2 + \Vert v'\Vert_2^2$, where $\Vert\cdot\Vert_2$ denotes the norm in $L^2[0, 2\pi]$.

At times, it will be convenient to use the norm, $\Vert\cdot\Vert_H$, in $H$ defined by

$$\Vert v\Vert_H^2 = 2\pi(\overline{v})^2 + \Vert v'\Vert_2^2, \quad \text{for all } v \in H, \tag{3.2}$$

where $\overline{v} = \frac{1}{2\pi}\int_0^{2\pi} v(x)\,\mathrm{d}x$, the average value of $v$ over $[0, 2\pi]$. The norms $\Vert\cdot\Vert$ and $\Vert\cdot\Vert_H$ are equivalent. In fact, it is possible to prove that

$$\frac{1}{2}\Vert v\Vert^2 \leqslant \Vert v\Vert_H^2 \leqslant \Vert v\Vert^2 \quad \text{for all } v \in H. \tag{3.3}$$

We will prove Theorem 1.1 by means of variational methods. Specifically, we will show that the functional $I : H \to \mathbb{R}$, defined by

$$I(u) = \frac{1}{2}\int_0^{2\pi} \big[(u')^2 - \mu(u^+)^2 - \nu(u^-)^2\big]\,\mathrm{d}x - \int_0^{2\pi} \big[G(x,u) + fu\big]\,\mathrm{d}x, \tag{3.4}$$

for all $u \in H$, has at least one critical point.

The assumption that $g$ is a Carathéodory function satisfying the bound in (1.2) can be used to prove that $I \in C^1(H, \mathbb{R})$ with Fréchet derivative at each $u \in H$, $I'(u)\colon H \to \mathbb{R}$, given by

$$I'(u)v = \int_0^{2\pi} u'v'\,\mathrm{d}x - \int_0^{2\pi} [\mu u^+(x) - \nu u^-(x) + g(x,u(x)) + f(x)]v(x)\,\mathrm{d}x, \tag{3.5}$$

for all $v \in H$. We then see, in view of (1.15), that a critical point, $u$, of $I$ in $H$ will be a solution to problem (1.1).

We will obtain a critical point of $I$ in (3.4) by means of a variant of the Saddle Point Theorem of Rabinowitz (see [8]) which is proved in Struwe [9, Theorem 8.4, pg. 118].

**Theorem 3.1.** *Let $H$ denote a Banach space. Assume $H^+$ is a closed subset in $H$ and $Q$ is a bounded subset in $H$ with boundary $\partial Q$. Let*

$$\Gamma = \{\gamma \in C(H, H) \mid \gamma(u) = u \text{ on } \partial Q\}. \tag{3.6}$$

*If $I \in C^1(H, \mathbb{R})$ and*

(i) $H^+ \cap \partial Q = \emptyset$,
(ii) $H^+ \cap \gamma(Q) \neq \emptyset$, *for every* $\gamma \in \Gamma$,
(iii) *there are constants $c_o$ and $c_1$ such that*
$$c_o = \inf_{u \in H^+} I(u) > \sup_{u \in \partial Q} I(u) = c_1,$$
(iv) *$I$ satisfies Palais-Smale condition;*

*then, the number*

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in Q} I(\gamma(u))$$

*defines a critical value $c > c_1$ of $I$.*

$H^+$ and $\partial Q$ are said to *link* if they satisfy conditions (i) and (ii) of the theorem above.

Our strategy to prove Theorem 1.1 will be to find sets $H^+$ and $Q$ for which conditions (i)–(ii) in the Saddle Point Theorem (Theorem 3.1) hold. We then show the functional $I$ defined in (3.4) satisfies (iii) and the Palais-Smale condition; that is, every sequence $(u_n)$ for which $(I(u_n))$ is bounded and $I'(u_n) \to 0$ (as $n \to \infty$) possesses a convergent subsequence in $H$. One of the difficulties in the argument presented here lies in the construction of the set $Q$ needed in the application of the Saddle Point Theorem. In order to construct $Q$, we will use the set $K_{m,\mu,\nu}$ whose construction has been described in Section 2 and presented as Definition 2.6 on page 9.

Decompose the space $H$ into

$$H = H^- \oplus E_m \oplus H^+, \tag{3.7}$$

where

$$H^- = \operatorname{span}\{1, \cos(x), \sin(x), \dots, \cos(m-1)x, \sin(m-1)x\};$$
$$E_m = \operatorname{span}\{\sin mx, \cos mx\};$$

and $H^+$ is the orthogonal complement of $H^- \oplus E_m$ in $H$. Observe that the decomposition of $H$ given in (3.7) is a consequence of Fourier's Theorem (see [10, Theorem 2 on page 119]).

Next, for $K, L > 0$, define the set $Q_{K,L}$ as follows:

**Definition 3.2** (Definition of $Q_{K,L}$)**.** $v \in Q_{K,L}$ if and only if $v = w + z$, where $w \in H^-$ with $\|w\| \leqslant L$, and

$$z = \rho z_\theta \quad \text{for } 0 \leqslant \rho \leqslant K, \ \theta \in [\theta_1, \theta_2],$$

where $\theta_1$ and $\theta_2$ are defined in (2.15), and $z_\theta$ is defined in (2.21).

Observe that $Q_{K,L}$ is bounded. In fact, a calculation involving the $L^2$ norms of $\varphi_{m,i}$, for $i = 1, 2, 3, 4$, and their derivatives, listed in Table 1 on page 10, shows that, if $u \in Q_{K,L}$, then

$$\|u\|^2 \leqslant L^2 + \frac{m\pi}{2} K^2 \big(\frac{1}{\mu\sqrt{\mu}} + \frac{1}{\nu\sqrt{\nu}} + \frac{2}{m}\big).$$

Next, suppose that $u \in H^+ \cap \partial Q_{K,L}$. Then,

$$\langle u, \cos mx \rangle = 0 \quad \text{and} \quad \langle u, \sin mx \rangle = 0, \tag{3.8}$$

since $u$ is in $H^+$, and is, therefore, orthogonal to $E_m$. Furthermore, $u = w + z$, where $w \in H^-$ and $z \in K_{m,\mu,\nu}$; that is,

$$z = \rho \left( \cos\sqrt{\mu}\theta \ \varphi_{m,1} + \sin\sqrt{\mu}\theta \ \varphi_{m,2} + \cos\sqrt{\nu}\theta \ \varphi_{m,3} + \sin\sqrt{\nu}\theta \varphi_{m,4} \right),$$

for some real numbers $\rho$ and $\theta$ with $0 \leqslant \rho \leqslant K$ and $\theta \in [\theta_1, \theta_2]$.

We first argue that $\rho > 0$; for if $\rho = 0$, it follows that $u = w \in H^-$ with $\|w\| = L > 0$. However, $u \in H^+$ as well. This is impossible since $H^- \cap H^+ = \{0\}$. It then follows from (3.8) and the fact that $w \in H^-$ that

$$\begin{aligned} &\cos(\sqrt{\mu}\theta)\langle \varphi_{m,1}, \cos mx\rangle + \sin(\sqrt{\mu}\theta)\langle \varphi_{m,2}, \cos mx\rangle \\ &+ \cos(\sqrt{\nu}\theta)\langle \varphi_{m,3}, \cos mx\rangle + \sin(\sqrt{\nu}\theta)\langle \varphi_{m,4}, \cos mx\rangle = 0, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &\cos(\sqrt{\mu}\theta)\langle \varphi_{m,1}, \sin mx\rangle + \sin(\sqrt{\mu}\theta)\langle \varphi_{m,2}, \sin mx\rangle \\ &+ \cos(\sqrt{\nu}\theta)\langle \varphi_{m,3}, \sin mx\rangle + \sin(\sqrt{\nu}\theta)\langle \varphi_{m,4}, \sin mx\rangle = 0. \end{aligned} \tag{3.10}$$

Using the inner-product formulas recorded in Table 1, the system in (3.9)–(3.10) can be re-written in the form

$$\begin{aligned} &[\frac{\cos(\sqrt{\mu}\theta)}{\mu - m^2} - \frac{\cos(\sqrt{\nu}\theta)}{\nu - m^2}](\cos m\tau_1 + 1) \\ &+ \frac{m}{\sqrt{\mu\nu}}[-\frac{\sqrt{\nu}\sin(\sqrt{\mu}\theta)}{\mu - m^2} + \frac{\sqrt{\mu}\sin(\sqrt{\nu}\theta)}{\nu - m^2}] \sin m\tau_1 = 0, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\frac{m}{\sqrt{\mu\nu}}[\frac{\sqrt{\nu}\sin(\sqrt{\mu}\theta)}{\mu - m^2} - \frac{\sqrt{\mu}\sin(\sqrt{\nu}\theta)}{\nu - m^2}](\cos m\tau_1 + 1) \\ &+ [\frac{\cos(\sqrt{\mu}\theta)}{\mu - m^2} - \frac{\cos(\sqrt{\nu}\theta)}{\nu - m^2}] \sin m\tau_1 = 0. \end{aligned} \tag{3.12}$$

The system in (3.11)–(3.12) can in turn be written in matrix form as

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \cos m\tau_1 + 1 \\ \sin m\tau_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.13}$$

where

$$a = \frac{\cos(\sqrt{\mu}\theta)}{\mu - m^2} - \frac{\cos(\sqrt{\nu}\theta)}{\nu - m^2}, \tag{3.14}$$

$$b = m[\frac{\sin(\sqrt{\mu}\theta)}{\sqrt{\mu}(\mu - m^2)} - \frac{\sin(\sqrt{\nu}\theta)}{\sqrt{\nu}(\nu - m^2)}]. \tag{3.15}$$

Observe that the determinant of the $2 \times 2$ matrix on the left-hand side of (3.13) is $a^2 + b^2$. Observe also that, given that $\tau_1 = \pi/\sqrt{\mu}$,

$$0 < m\tau_1 < \pi, \tag{3.16}$$

as $m < \sqrt{\mu}$ by (1.10). It then follows that

$$\sin m\tau_1 > 0 \quad \text{and} \quad \cos m\tau_1 + 1 > 0. \tag{3.17}$$

Consequently, the determinant of the $2 \times 2$ matrix on the left-hand side of (3.13) must be zero. Hence, $a = b = 0$, where $a$ and $b$ are given by (3.14) and (3.15). We therefore have the equations

$$\frac{\cos(\sqrt{\mu}\theta)}{\mu - m^2} - \frac{\cos(\sqrt{\nu}\theta)}{\nu - m^2} = 0 \tag{3.18}$$

and

$$\frac{\sin(\sqrt{\mu}\theta)}{\sqrt{\mu}(\mu - m^2)} - \frac{\sin(\sqrt{\nu}\theta)}{\sqrt{\nu}(\nu - m^2)} = 0. \tag{3.19}$$

Observe that (3.18) is impossible for $\theta = 0$, since

$$\frac{1}{\mu - m^2} + \frac{1}{m^2 - \nu} > 0$$

by the inequalities in (1.10).

We next show that both (3.18) and (3.19) are impossible for all $\theta \neq 0$ in $[\theta_1, \theta_2]$.

For values of $\theta \neq 0$ such that $\cos(\sqrt{\nu}\theta) \neq 0$ and $\cos(\sqrt{\mu}\theta) \neq 0$, we obtain from (3.18) and (3.19) that

$$\frac{\tan(\sqrt{\mu}\theta)}{\sqrt{\mu}\theta} = \frac{\tan(\sqrt{\nu}\theta)}{\sqrt{\nu}\theta}. \tag{3.20}$$

An examination of the function, $F\colon D \to \mathbb{R}$, where

$$D = \{t \in \mathbb{R} \mid \cos t \neq 0\}, \tag{3.21}$$

defined by

$$F(t) = \begin{cases} \frac{\tan(t)}{t} & \text{for } t \neq 0 \text{ and } \cos t \neq 0, \\ 1 & \text{for } t = 0 \text{ or } \cos t = 0, \end{cases} \tag{3.22}$$

shows that $F$ is strictly increasing in $t$ for values of $t$ with $0 < t < \pi/2$. Consequently,

$$F(\sqrt{\mu}\theta) > F(\sqrt{\nu}\theta),$$

which shows that (3.20) is impossible for

$$0 < \sqrt{\mu}\theta < \frac{\pi}{2}.$$

Next, suppose that $\sqrt{\mu}\theta = \pi/2$ so that $0 < \sqrt{\nu}\theta < \pi/2$. We then obtain from (3.14) that

$$a = \frac{\cos(\sqrt{\nu}\theta)}{m^2 - \nu} > 0,$$

by (1.10), which shows that (3.18) is impossible for the case $\sqrt{\mu}\theta = \pi/2$. Next, suppose that

$$0 < \sqrt{\nu}\theta \leqslant \frac{\pi}{2} \quad \text{and} \quad \frac{\pi}{2} < \sqrt{\mu}\theta \leqslant \pi.$$

Then, we obtain from (3.15) that $b > 0$, which shows that (3.19) is impossible in this case.

Next, observe that the case

$$0 < \sqrt{\nu}\theta \leqslant \frac{\pi}{2} \quad \text{and} \quad \pi < \sqrt{\mu}\theta < \frac{3\pi}{2}$$

is precluded by the left-most inequality in condition (1.8); namely, $4\mu/9 \leqslant \nu$. So, we now proceed to look at the case in which

$$\frac{\pi}{2} < \sqrt{\nu}\theta < \sqrt{\mu}\theta < \frac{3\pi}{2}.$$

For this case, examine again the function $F\colon D \to \mathbb{R}$, where $D$ is as defined in (3.21), given in (3.22) to see that

$$F(\sqrt{\mu}\theta) > F(\sqrt{\nu}\theta),$$

which shows (3.20) is impossible in this case.

Next, look at the case

$$\frac{\pi}{2} < \sqrt{\nu}\theta < \frac{3\pi}{2} \quad \text{and} \quad \sqrt{\mu}\theta = \frac{3\pi}{2}.$$

We then obtain from (3.14) that

$$a = \frac{\cos(\sqrt{\nu}\theta)}{m^2 - \nu} < 0,$$

by (1.10), which shows that (3.18) is impossible.

Consequently, (3.18) and (3.19) are impossible for

$$0 \leqslant \theta \leqslant \theta_1 = \frac{\pi}{\sqrt{\nu}}.$$

Next, consider negative values of $\theta$ with $-\pi \leqslant \sqrt{\mu}\theta < \sqrt{\nu}\theta < 0$. It this case, from (3.15), we obtain

$$b \leqslant \frac{m}{\sqrt{\nu}} \frac{\sin(\sqrt{\nu}\theta)}{m^2 - \nu} < 0,$$

which is shows that (3.19) is impossible.

Therefore, we have proved that it is impossible for the intersection of $H^+$ and $\partial Q_{K,L}$ to contain an element. Consequently,

$$H^+ \cap \partial Q_{K,L} = \emptyset \quad \text{for all } K > 0 \text{ and } L > 0. \tag{3.23}$$

Following the outline of the argument of Tomiczek in [12, pg. 7], define the set $V \subseteq H$ as follows:

**Definition 3.3** (Definition of $V$)**.** $v \in V$ if and only if $v = h + w$ where $h \in H^-$ and $w = \rho z_\theta$ for $\rho \geqslant 0$, $\theta \in [\theta_1, \theta_2]$, where $\theta_1$ and $\theta_2$ are defined in (2.15), and $z_\theta$ is defined in (2.21).

**Remark 3.4.** For future reference, we note that, since

$$z_\theta \in \operatorname{span}\{\varphi_{m,1}, \varphi_{m,2}, \varphi_{m,3}, \varphi_{m,4}\},$$

where the functions $\varphi_{m,1}, \varphi_{m,2}, \varphi_{m,3}, \varphi_{m,4}$ are defined in (2.17)–(2.20), it follows that $V$ is a subset of

$$H^- + \operatorname{span}\{\varphi_{m,1}, \varphi_{m,2}, \varphi_{m,3}, \varphi_{m,4}\},$$

and therefore $V$ lives in a finite-dimensional subspace of $H$.

**Remark 3.5.** Observe that, according to Definitions 3.3 and 3.2, $Q_{K,L} \subseteq V$.

Next, we show that

$$H = V \oplus H^+. \tag{3.24}$$

To prove (3.24), we first need to show that an arbitrary element, $u$, of $H$ can be expressed in the form

$$u = v + h, \quad \text{where } h \in H^+ \text{ and } v \in V. \tag{3.25}$$

To establish (3.25), first observe that, by (3.7), every $u \in H$ can be written in the form

$$u(x) = u_1(x) + a_m \cos mx + b_m \sin mx + u_2(x), \quad \text{for all } x \in [0, 2\pi], \tag{3.26}$$

and some constants $a_m$ and $b_m$, where $u_1 \in H^-$ and $u_2 \in H^+$. We want to show that we can also write $u$ in the form

$$u = \widetilde{u}_1 + \rho z_\theta + \widetilde{u}_2, \tag{3.27}$$

for some constants $\rho \geqslant 0$ and $\theta \in [\theta_1, \theta_2]$, where $\widetilde{u}_1 \in H^-$ and $\widetilde{u}_2 \in H^+$.

Taking inner products with $\cos mx$ and $\sin mx$ in (3.26) and (3.27) gives rise to the system

$$\begin{aligned} \rho\langle z_\theta, \cos mx\rangle &= \pi a_m \\ \rho\langle z_\theta, \sin mx\rangle &= \pi b_m. \end{aligned} \tag{3.28}$$

Using the definition of $z_\theta$ in (2.21) and the inner products recorded in Table 1, the system in (3.28) leads to the system

$$\begin{aligned} \rho[(\cos(m\tau_1) + 1)a(\theta) - \sin(m\tau_1)b(\theta)] &= \pi a_m \\ \rho[\sin(m\tau_1)a(\theta) + (\cos(m\tau_1) + 1)b(\theta)] &= \pi b_m, \end{aligned} \tag{3.29}$$

where

$$a(\theta) = m\left(\frac{\cos(\sqrt{\mu}\theta)}{\mu - m^2} - \frac{\cos(\sqrt{\nu}\theta)}{\nu - m^2}\right), \tag{3.30}$$

$$b(\theta) = m^2\left(\frac{\sin(\sqrt{\mu}\theta)}{\sqrt{\mu}(\mu - m^2)} - \frac{\sin(\sqrt{\nu}\theta)}{\sqrt{\nu}(\nu - m^2)}\right). \tag{3.31}$$

Note that the system in (3.29) can be written in matrix form as

$$\rho A \begin{pmatrix} a(\theta) \\ b(\theta) \end{pmatrix} = \begin{pmatrix} \pi a_m \\ \pi b_m \end{pmatrix}, \tag{3.32}$$

where $A$ is the $2 \times 2$ matrix given by

$$A = \begin{pmatrix} \cos(m\tau_1) + 1 & -\sin(m\tau_1) \\ \sin(m\tau_1) & \cos(m\tau_1) + 1 \end{pmatrix}. \tag{3.33}$$

The determinant of the matrix $A$ in (3.33) is

$$\det A = 2(\cos(m\tau_1) + 1),$$

which is positive by (3.17). Consequently, $A$ is invertible and, therefore, the system in (3.32) can be solved for $\begin{pmatrix} \rho a(\theta) \\ \rho b(\theta) \end{pmatrix}$ to yield the system

$$\begin{aligned} \rho a(\theta) &= \widetilde{a}_m \\ \rho b(\theta) &= \widetilde{b}_m, \end{aligned} \tag{3.34}$$

where

$$\begin{pmatrix} \widetilde{a}_m \\ \widetilde{b}_m \end{pmatrix} = A^{-1} \begin{pmatrix} \pi a_m \\ \pi b_m \end{pmatrix}.$$

We now show that the system in (3.34) can always be solved for $\rho$ and $\theta$, given any inputs $\widetilde{a}_m$ and $\widetilde{b}_m$. In order to see this, first observe that

$$[a(\theta)]^2 + [b(\theta)]^2 \neq 0, \quad \text{for all } \theta \in [\theta_1, \theta_2]. \tag{3.35}$$

To prove that (3.35) is true, argue by contradiction to obtain the equations in (3.18) and (3.19). An analysis using the function, $F : D \to \mathbb{R}$ defined in (3.21) and (3.22) can then be used to arrive at a contradiction.

Using (3.35) we then see that $\rho = 0$ if and only if $\widetilde{a}_m^2 + \widetilde{b}_m^2 = 0$, which, in turn, is equivalent to $a_m^2 + b_m^2 = 0$, by the invertibility of the matrix $A$ defined in (3.33). We therefore assume that

$$\widetilde{a}_m^2 + \widetilde{b}_m^2 \neq 0 \tag{3.36}$$

for the system in (3.34), and therefore

$$\rho \neq 0 \tag{3.37}$$

Assume first that $\widetilde{a}_m \neq 0$ in (3.34). We then obtain from (3.34) and (3.37) the equation

$$\frac{b(\theta)}{a(\theta)} = \frac{\widetilde{b}_m}{\widetilde{a}_m}. \tag{3.38}$$

We will show that, given any $\widetilde{a}_m$ and $\widetilde{b}_m$, with $\widetilde{a}_m \neq 0$, there exists $\theta \in (\theta_1, \theta_2)$ which solves the equation in (3.38). In order to prove this claim, first observe that there there exist

$$\theta_3 \in \big(\frac{\theta_1}{2}, 0\big) \quad \text{and} \quad \theta_4 \in \big(0, \frac{\theta_2}{2}\big) \tag{3.39}$$

such that

$$a(\theta_3) = a(\theta_4) = 0 \quad \text{and} \quad a(\theta) > 0 \quad \text{for all } \theta \in (\theta_3, \theta_4). \tag{3.40}$$

To see why (3.40) holds for $\theta_3$ and $\theta_4$ given in (3.39), note that, by the definition of $\theta_2$ in (2.15),

$$a\big(\frac{\theta_2}{2}\big) = \frac{m}{\mu - m^2} \cos\big(\sqrt{\mu}\frac{\theta_2}{2}\big) < 0, \tag{3.41}$$

since

$$\sqrt{\mu}\frac{\theta_2}{2} = \frac{\sqrt{\mu}}{\sqrt{\nu}}\frac{\pi}{2},$$

so that

$$\frac{\pi}{2} < \sqrt{\mu}\frac{\theta_2}{2} \leqslant \frac{3}{4}\pi,$$

by the inequalities in (1.8). It then follows from (3.41) and the Intermediate Value Theorem that there exists $a_4 \in (0, \theta_2/2)$ such that $a(\theta_4) = 0$, given that

$$a(0) = m\big(\frac{1}{\mu - m^2} + \frac{1}{m^2 - \nu}\big) > 0. \tag{3.42}$$

Similarly, since

$$a\big(\frac{3\theta_1}{4}\big) = \frac{m}{m^2 - \nu} \cos\big(\sqrt{\nu}\frac{3\theta_1}{4}\big) < 0, \tag{3.43}$$

we obtain from (3.43), (3.42) and the Intermediate Value Theorem that there exists $a_3 \in (3\theta_1/4, 0)$ such that $a(\theta_3) = 0$. Furthermore, $\theta_3$ and $\theta_4$ may be chosen so that the inequality in (3.40) holds.

Next, use the definition of $b(\theta)$ in (3.31) to obtain that

$$b(\theta) = m^2 \left( \frac{\sin(\sqrt{\mu}\theta)}{\sqrt{\mu}(\mu - m^2)} + \frac{\sin(\sqrt{\nu}\theta)}{\sqrt{\nu}(m^2 - \nu)} \right) > 0 \tag{3.44}$$

for all

$$\theta \in \big(0, \frac{\theta_2}{2}\big], \tag{3.45}$$

since

$$0 < \sqrt{\nu}\theta < \sqrt{\mu}\theta \leqslant \frac{3\pi}{4} < \pi,$$

for values $\theta$ in the range specified in (3.45), where we have used the left-most inequality in (1.8). Similarly, we can show that

$$b(\theta) = m^2 \left( \frac{\sin(\sqrt{\mu}\theta)}{\sqrt{\mu}(\mu - m^2)} + \frac{\sin(\sqrt{\nu}\theta)}{\sqrt{\nu}(m^2 - \nu)} \right) < 0 \tag{3.46}$$

for all

$$\theta \in \big[\frac{\theta_1}{2}, 0\big). \tag{3.47}$$

We can now put together the results in (3.39), (3.40), (3.44), (3.45), (3.46) and (3.47) to conclude that

$$\lim_{\theta \to \theta_3^+} \frac{b(\theta)}{a(\theta)} = -\infty \quad \text{and} \quad \lim_{\theta \to \theta_4^-} \frac{b(\theta)}{a(\theta)} = +\infty.$$

Consequently, it follows from the Intermediate Value Theorem that (3.38) can be solved for $\theta$, given any $\widetilde{a}_m$ and $\widetilde{b_m}$, with $\widetilde{a}_m \neq 0$. With that value of $\theta$, we let $\rho > 0$ be the positive solution to the equation

$$\rho^2 = \frac{\widetilde{a}_m^2 + \widetilde{b}_m^2}{[a(\theta)]^2 + [b(\theta)]^2}, \tag{3.48}$$

which is well-defined by (3.35). We have therefore shown that we can solve the system in (3.34) for $\rho$ and $\theta$ for any $\widetilde{a}_m$ and $\widetilde{b}_m$ with $\widetilde{a}_m \neq 0$.

In the case $\widetilde{a}_m = 0$, take $\theta = \theta_4$ if $\widetilde{b}_m > 0$, or $\theta = \theta_3$ if $\widetilde{b}_m < 0$; this choice is possible by (3.36). Then, let $\rho$ be the corresponding positive solution of (3.48). Thus, in the case $\widetilde{a}_m = 0$, the system in in (3.34) can also be solved.

Hence, the system in (3.34) is solvable for any $\widetilde{a}_m$ and $\widetilde{b}_m$ in $\mathbb{R}$. This implies that the system in (3.28) is solvable for any given $a_m$ and $b_m$ by the invertibility of the matrix $A$ defined in (3.33).

Now, given $u \in H$, let $a_m$ and $b_m$ be the Fourier coefficients of $u$ corresponding to $\cos mx$ and $\sin mx$, respectively. By what we have just proved, there exist $\rho_m \geqslant 0$ and $\theta_m \in [\theta_1, \theta_2]$ such that

$$\begin{aligned} \langle \rho_m z_{\theta_m}, \cos mx \rangle &= \pi a_m \\ \langle \rho_m z_{\theta_m}, \sin mx \rangle &= \pi b_m. \end{aligned} \tag{3.49}$$

It then follows that the Fourier series expansion of $\rho_m z_{\theta_m}$ yields

$$\rho_m z_{\theta_m}(x) = h_1(x) + a_m \cos mx + b_m \sin mx + h_2(x), \quad \text{for all } x \in [0, 2\pi], \tag{3.50}$$

for $h_1 \in H^-$ and $h_2 \in H^+$, where we have used the Fourier coefficient formulas given in (3.49).

Next, solve for $a_m \cos mx + b_m \sin mx$ in (3.50) and substitute into the expansion for $u$ in (3.26) to obtain the representation in (3.27), where

$$\widetilde{u}_1 = u_1 - h_1 \quad \text{and} \quad \widetilde{u}_2 = u_2 - h_2.$$

We have therefore proved that $H = V + H^+$.

To complete the proof of (3.24), we need to show that

$$V \cap H^+ = \{0\}. \tag{3.51}$$

Arguing by contradiction, suppose that

$$u \in V \cap H^+ \quad \text{and} \quad \|u\| \neq 0. \tag{3.52}$$

We then have that $u \in H^+$ and

$$u = h_1 + w, \tag{3.53}$$

where $h_1 \in H^-$, and

$$w = \rho z_\theta, \tag{3.54}$$

with $\rho \geqslant 0$, and $z_\theta$ as defined in (2.21).

Consequently,

$$\langle \rho z_\theta, \cos mx \rangle = 0 \quad \text{and} \quad \langle \rho z_\theta, \sin mx \rangle = 0. \tag{3.55}$$

First, observe that

$$\rho \neq 0. \tag{3.56}$$

Otherwise, it would follow from (3.54) and (3.53) that $u \in H^-$; we would then have $u = 0$, since $H^- \cap H^+ = \{0\}$; but, we are assuming that $u \neq 0$ in (3.52).

The conditions in (3.55) and (3.56), and the inner products recorded in Table 1 on page 10 lead to the system of equations

$$\begin{aligned} &[\frac{\cos(\sqrt{\mu}\theta)}{\mu - m^2} - \frac{\cos(\sqrt{\nu}\theta)}{\nu - m^2}](\cos m\tau_1 + 1) \\ &+ m[-\frac{\sin(\sqrt{\mu}\theta)}{\sqrt{\mu}(\mu - m^2)} + \frac{\sin(\sqrt{\nu}\theta)}{\sqrt{\nu}(\nu - m^2)}] \sin m\tau_1 = 0, \end{aligned} \tag{3.57}$$

and

$$\begin{aligned} &m[\frac{\sin(\sqrt{\mu}\theta)}{\sqrt{\mu}(\mu - m^2)} - \frac{\sin(\sqrt{\nu}\theta)}{\sqrt{\nu}(\nu - m^2)}](\cos m\tau_1 + 1) \\ &+ [\frac{\cos(\sqrt{\mu}\theta)}{\mu - m^2} - \frac{\sin(\sqrt{\nu}\theta)}{\nu - m^2}] \sin m\tau_1 = 0. \end{aligned} \tag{3.58}$$

The system in (3.57) and (3.58) can, in turn, be written in matrix form as

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \cos m\tau_1 + 1 \\ \sin m\tau_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.59}$$

where

$$a = \frac{\cos(\sqrt{\mu}\theta)}{\mu - m^2} - \frac{\cos(\sqrt{\nu}\theta)}{\nu - m^2}, \tag{3.60}$$

$$b = m\big[\frac{\sin(\sqrt{\mu}\theta)}{\sqrt{\mu}(\mu - m^2)} - \frac{\sin(\sqrt{\nu}\theta)}{\sqrt{\nu}(\nu - m^2)}\big]. \tag{3.61}$$

We deduce from the matrix equation in (3.59) that

$$a^2 + b^2 = 0, \tag{3.62}$$

in view of (3.17).

It then follows from (3.60), (3.61) and (3.62) that

$$\begin{aligned} \frac{\cos(\sqrt{\mu}\theta)}{\mu - m^2} - \frac{\cos(\sqrt{\nu}\theta)}{\nu - m^2} &= 0 \\ \frac{\sin(\sqrt{\mu}\theta)}{\sqrt{\mu}(\mu - m^2)} - \frac{\sin(\sqrt{\nu}\theta)}{\sqrt{\nu}(\nu - m^2)} &= 0. \end{aligned} \tag{3.63}$$

An analysis using the function, $F\colon D \to \mathbb{R}$ defined in (3.21) and (3.22) leads to the conclusion that the equations in (3.63) are impossible. This contradiction shows that (3.52) is impossible. Hence, $V \cap H^+ = \{0\}$.

We have therefore established (3.51) and the proof of (3.24) is now complete; that is $H = V \oplus H^+$. We can therefore define a continuous projection $P\colon H \to V$.

**Remark 3.6.** We observe for future reference that the set $V$ is precisely the set made up of elements of the form $h+w$, for $h \in H^-$ and $w \in K_{m,\mu,\nu}$, where $K_{m,\mu,\nu}$ is the set defined in (2.22); that is, the set of solutions of the boundary-value problem (1.3) corresponding to the pair $(\mu,\nu) \in \Sigma_m$. Thus,

$$V = H^- + K_{m,\mu,\nu}. \tag{3.64}$$

Proceeding with the outline of the argument in Tomiczek [12], we now show that

$$\gamma(Q_{K,L}) \cap H^+ \neq \emptyset, \tag{3.65}$$

for all $\gamma \in \Gamma$, where $\Gamma$ is defined in (3.6); that is,

$$\gamma\colon H \to H$$

is continuous and

$$\gamma(u) = u \quad \text{for all } u \in \partial Q_{K,L}. \tag{3.66}$$

We will establish (3.65) by proving that

$$0 \in P \circ \gamma(Q_{K,L}); \tag{3.67}$$

for, if (3.67) is true, there is $v \in \gamma(Q_{K,L})$ such that $v = v - Pv \in H^+$.

To prove (3.67), we show that the equation

$$P \circ \gamma(v) = 0 \tag{3.68}$$

has a solution in $Q_{K,L}$. We will prove this claim by means of the Brouwer degree. It follows from (3.66) and the definition of $P\colon H \to V$ that

$$P \circ \gamma = id \text{ on } \partial Q_{K,L}.$$

Thus, since $0 \notin \partial Q_{K,L}$, $P\circ\gamma$ is an admissible function for the degree, $\deg(\cdot, Q_{K,L}, 0)$. By the same token, the homotopy $\gamma_t\colon Q_{K,L} \to H$, for $0 \leqslant t \leqslant 1$, defined by

$$\gamma_t(u) = tP(\gamma(u)) + (1-t)u, \quad \text{for all } u \in Q_{K,L},$$

is also an acceptable homotopy. Therefore, by the homotopy invariance of the Brouwer degree,

$$\deg(\gamma_1, Q_{K,L}, 0) = \deg(\gamma_0, Q_{K,L}, 0) = \deg(id, Q_{K,L}, 0) = 1.$$

It then follows that

$$\deg(P \circ \gamma, Q_{K,L}, 0) \neq 0,$$

and, therefore, the equation (3.68) is solvable in $Q_{K,L}$. Hence, we have established (3.65).

## 4. Proof of Main Result

In the previous section we showed that $Q_{K,L}$, for $K > 0$ and $L > 0$, given in Definition 3.2, and $H^+$ link. In other words, conditions (i) and (ii) in the Saddle Point Theorem of Rabinowitz (Theorem 3.1) are satisfied for $Q = Q_{K,L}$, with $K > 0$ and $L > 0$. In this section we show that the functional, $I : H \to \mathbb{R}$, defined in (3.4) and $Q_{K,L}$, for some appropriately chosen $K$ and $L$, satisfy the remaining conditions in the Saddle Point Theorem. This will complete the proof of Theorem 1.1.

In the process of building the proof of Theorem 1.1, we will need to establish a few lemmas. Before proceeding any further, we present an estimate that will be needed for the application of Theorem 3.1, and whose proof is very similar to that of [12, Lemma 2.2, p. 6].

Set

$$J(u) = \int_0^{2\pi} \left[(u')^2 - \mu(u^+)^2 - \nu(u^-)^2\right] dx \quad \text{for all } u \in H. \tag{4.1}$$

**Lemma 4.1.** *Assume that $z$ is a solution of* (1.3) *corresponding to $(\mu,\nu) \in \Sigma$, with $\mu \geqslant \nu$. Set $u = cz + w$ for $c \geqslant 0$ and $w \in H$; then, we have the following inequalities:*

$$\int_0^{2\pi} \left[(w')^2 - \mu w^2\right] dx \leqslant J(u) \leqslant \int_0^{2\pi} \left[(w')^2 - \nu w^2\right] dx\,. \tag{4.2}$$

For a proof to the above lemma, see [12, Lemma 2.2].

**Lemma 4.2.** *Let $f \in L^1[0,2\pi]$ and $g(x,\xi)$ be a Carathéodory function satisfying* (1.2) *and with primitive $G(x,\xi)$, defined in* (1.12)*. Then*

$$\lim_{\|u\|\to\infty} \int_0^{2\pi} \frac{G(x,u) + fu}{\|u\|^2}\, \mathrm{d}x = 0. \tag{4.3}$$

*Proof.* Using (1.2) we can show that

$$|G(x,\xi)| \leqslant p(x)|\xi| \quad \text{for a.e. } x \in [0,2\pi], \text{ and all } \xi \in \mathbb{R}. \tag{4.4}$$

It then follows that, for any $u \in H$,

$$\Big|\int_0^{2\pi} G(x,u(x))\, \mathrm{d}x\Big| \leqslant \|p\|_{L^1} \cdot \max_{0\leqslant x\leqslant 2\pi} |u(x)|, \tag{4.5}$$

where

$$\|p\|_{L^1} = \int_0^{2\pi} |p(x)|\, \mathrm{d}x.$$

Next, use the estimate

$$|u(x) - u(y)| \leqslant \sqrt{|x-y|}\ \|u'\|_2, \quad \text{for all } x,y \in [0,2\pi],$$

to obtain that there exist positive constants $c_2$ and $c_3$ such that

$$\max_{0\leqslant x\leqslant 2\pi} |u(x)| \leqslant c_2 + c_3\|u'\|_2,$$

so that

$$\max_{0\leqslant x\leqslant 2\pi} |u(x)| \leqslant c_2 + c_3\|u\|. \tag{4.6}$$

We therefore get from (4.5) and (4.6) that

$$\lim_{\|u\|\to\infty} \int_0^{2\pi} \frac{G(x,u)}{\|u\|^2}\, \mathrm{d}x = 0. \tag{4.7}$$

Next, since $f \in L^1[0, 2\pi]$,

$$\Big| \int_0^{2\pi} fu \,\mathrm{d}x \Big| \leqslant \|f\|_{L^1} \cdot \max_{0 \leqslant x \leqslant 2\pi} |u(x)|,$$

so that, using (4.6),

$$\Big| \int_0^{2\pi} fu \,\mathrm{d}x \Big| \leqslant \|f\|_{L^1}(c_2 + c_3\|u\|),$$

from which we obtain

$$\lim_{\|u\| \to \infty} \int_0^{2\pi} \frac{fu}{\|u\|^2} \,\mathrm{d}x = 0. \tag{4.8}$$

The expressions in (4.7) and (4.8), taken together, yield (4.3). □

Let $m \in \mathbb{N}$. Recall that $H^+$ is the orthogonal complement of

$$H^- \oplus E_m = \operatorname{span}\{1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos mx, \sin mx\}.$$

It then follows that the Fourier series expansion for a function $u \in H^+$ is of the form

$$u(x) = \sum_{k=m+1}^{\infty} [a_k \cos kx + b_k \sin kx],$$

for all $x \in [0, 2\pi]$. Thus,

$$u'(x) = \sum_{k=m+1}^{\infty} k[-a_k \sin kx + b_k \cos kx],$$

for a.e. $x \in [0, 2\pi]$. We therefore have that

$$\int_0^{2\pi} |u'(x)|^2 \,\mathrm{d}x = \pi \sum_{k=m+1}^{\infty} k^2[a_k^2 + b_k^2] \geqslant \pi \ (m+1)^2 \sum_{k=m+1}^{\infty} [a_k^2 + b_k^2],$$

from which we obtain

$$\int_0^{2\pi} |u'(x)|^2 \,\mathrm{d}x \geqslant (m+1)^2 \int_0^{2\pi} |u(x)|^2 \,\mathrm{d}x, \quad \text{for } u \in H^+, \tag{4.9}$$

by Parseval's Theorem for Fourier series (see [10, Theorem 1]).

**Lemma 4.3.** *Assume that* (1.7), (1.8) *and* (1.9) *hold for some* $m \in \mathbb{N}$ *and let* $I\colon H \to \mathbb{R}$ *be as defined in* (3.4). *Then*

$$\lim_{\|u\| \to \infty,\ u \in H^+} I(u) = +\infty \tag{4.10}$$

*Proof.* We argue by contradiction. Suppose that (4.10) does not hold true. Then, there exists a constant $C$, such that for each $n \in \mathbb{N}$, there exists $u_n \in H^+$ with

$$\|u_n\| > n \quad \text{and} \quad I(u_n) \leqslant C. \tag{4.11}$$

It follows from (4.11) that

$$\lim_{n \to \infty} \|u_n\| = +\infty, \tag{4.12}$$

$$\limsup_{n \to \infty} \frac{I(u_n)}{\|u_n\|^2} \leqslant 0. \tag{4.13}$$

Now, use the definition of $I$ in (3.4) and the estimate in (4.9) to obtain that

$$\begin{aligned} I(u_n) \geqslant \frac{(m+1)^2-\mu}{2} \int_0^{2\pi} (u_n^+)^2 \,\mathrm{d}x + \frac{(m+1)^2-\nu}{2} \int_0^{2\pi} (u_n^-)^2 \,\mathrm{d}x \\ - \int_0^{2\pi} \big[G(x,u_n) + f u_n\big] \,\mathrm{d}x, \end{aligned} \tag{4.14}$$

for all $n \in \mathbb{N}$. Thus, dividing the expression in (4.14) by $\|u_n\|^2$ and letting $n \to \infty$, we obtain

$$0 \geqslant \limsup_{n\to\infty} \big[\frac{(m+1)^2-\mu}{2} \frac{\|u_n^+\|_2^2}{\|u_n\|^2} + \frac{(m+1)^2-\nu}{2} \frac{\|u_n^-\|_2^2}{\|u_n\|^2}\big] \tag{4.15}$$

where we have used (4.12), (4.13) and the result of Lemma 4.2 in (4.3). Hence, using the inequalities in (1.9) and (1.11), we obtain from (4.15) that

$$\lim_{n\to\infty} \frac{\|u_n^+\|_2^2}{\|u_n\|^2} = \lim_{n\to\infty} \frac{\|u_n^-\|_2^2}{\|u_n\|^2} = 0. \tag{4.16}$$

Consequently,

$$\lim_{n\to\infty} \frac{\|u_n\|_2^2}{\|u_n\|^2} = 0,$$

and therefore

$$\lim_{n\to\infty} \frac{\|u_n'\|_2^2}{\|u_n\|^2} = 1. \tag{4.17}$$

Next, use the definition of $I$ in (3.4) to get

$$\frac{I(u_n)}{\|u_n\|^2} = \frac{1}{2} \frac{\|u_n'\|_2^2}{\|u_n\|^2} - \frac{\mu}{2} \frac{\|u_n^+\|_2^2}{\|u_n\|^2} - \frac{\nu}{2} \frac{\|u_n^-\|_2^2}{\|u_n\|^2} - \int_0^{2\pi} \frac{G(x,u_n) + f u_n}{\|u_n\|^2} \,\mathrm{d}x.$$

Thus, letting $n \to \infty$ and using (4.16), (4.17) and Lemma 4.2,

$$\lim_{n\to\infty} \frac{I(u_n)}{\|u_n\|^2} = \frac{1}{2},$$

which contradicts (4.13). We have therefore established (4.10). $\square$

It follows from Lemma 4.2 and the fact that $H$ is compactly embedded in $C[0,2\pi]$ that there exists a real number, $c_o$, such that

$$I(u) \geqslant c_o \quad \text{for all } u \in H^+;$$

in fact, we may take $c_o$ to be defined by

$$c_o = \inf_{u\in H^+} I(u). \tag{4.18}$$

We will next show that we can pick $K > 0$ and $L > 0$ such that $\sup_{v\in\partial Q_{K,L}} I(v) < c_o$, where $Q_{K,L}$ is given in Definition 3.2; that is, $v \in Q_{K,L}$ if and only if $v = w + z$, where $w \in H^-$, with $\|w\| \leqslant L$, and

$$z = \rho(\cos(\sqrt{\mu}\theta)\varphi_{m,1} + \sin(\sqrt{\mu}\theta)\varphi_{m,2} + \cos(\sqrt{\nu}\theta)\varphi_{m,3} + \sin(\sqrt{\nu}\theta)\varphi_{m,4}), \tag{4.19}$$

with $0 \leqslant \rho \leqslant K$ and $\theta \in [\theta_1, \theta_2]$, where $\theta_1$ and $\theta_2$ are defined in (2.15).

Observe that $v \in \partial Q_{K,L}$ if and only if

(i) $v = w + z$, where $w \in H^-$ with $\|w\| = L$, and $z$ given by (4.19) with $0 \leqslant \rho \leqslant K$ and $\theta \in [\theta_1, \theta_2]$; or
(ii) $v = w + z$, where $w \in H^-$ and $z$ given by (4.19) with $\rho = K$ and $\theta \in [\theta_1, \theta_2]$.

**Lemma 4.4.** *Assume that* (1.7), (1.8) *and* (1.9) *hold for some* $m \in \mathbb{N}$. *Let* $I\colon H \to \mathbb{R}$ *be as defined in* (3.4), *and assume that* $f$ *and* $G$ *satisfy the Tomiczek condition in* (1.13) *and* (1.14). *Then*

$$\lim_{\min\{K,L\}\to\infty} \sup\{I(v) \mid v \in \partial Q_{K,L}\} = -\infty, \tag{4.20}$$

*where* $Q_{K,L}$ *is given in Definition 3.2.*

*Proof.* Recall that, for $\theta \in [\theta_1, \theta_2]$,

$$z_\theta = \cos(\sqrt{\mu}\theta)\varphi_{m,1} + \sin(\sqrt{\mu}\theta)\varphi_{m,2} + \cos(\sqrt{\nu}\theta)\varphi_{m,3} + \sin(\sqrt{\nu}\theta)\varphi_{m,4}; \tag{4.21}$$

(see the definition of $z_\theta$ in (2.21)). Using the formulas in Table 1 we can compute

$$\|z_\theta\|^2 = \frac{m\pi}{2}\Big(\frac{1}{\mu\sqrt{\mu}} + \frac{1}{\nu\sqrt{\nu}} + \frac{2}{m}\Big), \quad \text{for all } \theta. \tag{4.22}$$

We will denote the positive square root of the expression on the right-hand side of the equation in (4.22) by $r$, so that

$$\|z_\theta\| = r, \quad \text{for all } \theta. \tag{4.23}$$

To prove (4.20), we will argue by contradiction. If (4.20) does not hold, then there there exists a real number, $C_1$; sequences of positive positive real numbers $(K_n)$ and $(L_n)$, with

$$\min\{K_n, L_n\} \geqslant n \quad \text{for all } n \in \mathbb{N}; \tag{4.24}$$

sequences of real numbers $(\rho_n)$ and $(\theta_n)$, with $\rho_n \geqslant 0$ and $\theta_n \in [\theta_1, \theta_2]$; and a sequence of functions, $(w_n)$, in $H^-$, satisfying

$$w_n + \rho_n z_{\theta_n} \in \partial Q_{K_n,L_n}, \tag{4.25}$$

and

$$I(w_n + \rho_n z_{\theta_n}) \geqslant C_1. \tag{4.26}$$

Put

$$v_n = w_n + \rho_n z_{\theta_n}, \quad \text{for all } n \in \mathbb{N}. \tag{4.27}$$

It follows from (4.24) and (4.25) that

$$\|v_n\| \to \infty \quad \text{as } n \to \infty. \tag{4.28}$$

To see why (4.28) holds true, first note that $w_n + \rho_n z_{\theta_n} \in \partial Q_{K_n,L_n}$ implies that either

- (i) $\|w_n\| = L_n$ and $0 \leqslant \rho_n \leqslant K_n$, or
- (ii) $\|w_n\| \leqslant L_n$ and $\rho_n = K_n$.

Thus, either

- (i) there is a subsequence, $(v_{n_k})$, with $v_{n_k} = L_{n_k}\widehat{w}_{n_k} + \rho_{n_k} z_{\theta_{n_k}}$, where $\|\widehat{w}_{n_k}\| = 1$ for all $k$; or
- (ii) $v_{n_k} = w_{n_k} + K_{n_k} z_{\theta_{n_k}}$.

In case (i), we show that, passing to a further subsequence if necessary,

$$\frac{1}{L_{n_k} + K_{n_k}} v_{n_k} \to \widehat{w} + \rho z_\theta, \tag{4.29}$$

for some $\widehat{w} \in \partial B_1 \cap H^-$, $0 \leqslant \rho \leqslant 1$ and $\theta \in [\theta_1, \theta_2]$, where we have used (4.24).

We therefore obtain from (4.26) and (4.28) that

$$\liminf_{n\to\infty} \frac{I(v_n)}{\|v_n\|^2} \geqslant 0. \tag{4.30}$$

Next, put

$$\widehat{v}_n = \frac{v_n}{\|v_n\|}, \quad \text{for all } n \in \mathbb{N}. \tag{4.31}$$

Then,

$$\widehat{v}_n \in \partial B \cap V, \quad \text{for all } n \in \mathbb{N}, \tag{4.32}$$

where $B$ denotes the closed unit ball in $H$, and $V$ is as defined in Definition 3.3; so that $\partial B \cap V$ lives in a finite dimensional subspace of $H$ (see Remark 3.4 on page 14). We also have, as a consequence of (4.27) and (4.31), that

$$\widehat{v}_n = \widehat{w}_n + \widehat{\rho}_n z_{\theta_n}, \quad \text{for all } n \in \mathbb{N}, \tag{4.33}$$

where

$$\widehat{w}_n \in B \cap H^-, \quad \text{for all } n \in \mathbb{N}, \tag{4.34}$$

$$\widehat{\rho}_n \in [0, 1/r], \quad \text{for all } n \in \mathbb{N}, \tag{4.35}$$

where $r$ is as given by (4.23). Using the compactness of $B \cap H^-$ and the closed intervals $[0, 1/r]$ and $[\theta_1, \theta_2]$, we may assume, as a consequence of (4.33), (4.34) and (4.35), that

$$\widehat{v}_n \to v_o \text{ (strongly) as } n \to \infty, \tag{4.36}$$

where

$$v_o = w_o + \rho_o z_{\theta_o}, \tag{4.37}$$

with

$$w_o \in B \cap H^-, \tag{4.38}$$

$\rho_o \in [0, 1/r]$, and $\theta_o \in [\theta_1, \theta_2]$.

Next, use the definition of the functional $I$ in (3.4), to compute

$$\frac{I(v_n)}{\|v_n\|^2} = \frac{1}{2}\int_0^{2\pi} \left[(\widehat{v}_n')^2 - \mu(\widehat{v}_n^+)^2 - \nu(\widehat{v}_n^-)^2\right] \mathrm{d}x - \int_0^{2\pi} \frac{G(x, v_n) + f v_n}{\|v_n\|^2} \mathrm{d}x \tag{4.39}$$

Thus, letting $n \to \infty$, while using (4.28) in conjunction with Lemma 4.2 and (4.36), we obtain from (4.30) and (4.39) that

$$\int_0^{2\pi} \left[(v_o')^2 - \mu(v_o^+)^2 - \nu(v_o^-)^2\right] \mathrm{d}x \geqslant 0,$$

or

$$J(v_o) \geqslant 0, \tag{4.40}$$

where $J$ is as defined in (4.1). Now, it follows from Lemma 4.1 on page 20 and (4.37) that

$$J(v_o) \leqslant \int_0^{2\pi} [(w_o')^2 - \nu w_o^2] \, \mathrm{d}x. \tag{4.41}$$

Since $w_o \in H^-$, by (4.38) and the Fourier series expansion

$$w_o(x) = \overline{w}_o + \sum_{k=1}^{m-1} [a_k \cos kx + b_k \sin kx],$$

where $\overline{w}_o$ denotes the average value of $w_o$ over $[0, 2\pi]$, we have that

$$\int_0^{2\pi} (w_o')^2 \, \mathrm{d}x = \sum_{k=1}^{m-1} \pi k^2 [a_k^2 + b_k^2],$$

from which we obtain

$$\int_0^{2\pi} (w_o')^2 \,\mathrm{d}x \leqslant (m-1)^2 \sum_{k=1}^{m-1} \pi[a_k^2 + b_k^2],$$

so that

$$\int_0^{2\pi} (w_o')^2 \,\mathrm{d}x \leqslant (m-1)^2 \Big[2\pi(\overline{w}_o)^2 + \sum_{k=1}^{m-1} \pi[a_k^2 + b_k^2]\Big]. \tag{4.42}$$

Thus, by Parseval's Theorem for Fourier series (see Theorem 1 on page 119 in [10]), we obtain from (4.42) that

$$\int_0^{2\pi} (w_o')^2 \,\mathrm{d}x \leqslant (m-1)^2 \int_0^{2\pi} w_o^2 \,\mathrm{d}x. \tag{4.43}$$

We then get from (4.41) and (4.43) that

$$J(v_o) \leqslant [(m-1)^2 - \nu] \int_0^{2\pi} w_o^2 \,\mathrm{d}x. \tag{4.44}$$

Combining (4.40) with (4.44) and the condition in (1.11) we deduce that

$$\int_0^{2\pi} w_o^2 \,\mathrm{d}x = 0,$$

from which we deduce that $w_o \equiv 0$. We can therefore conclude from (4.36) and (4.37) that

$$\frac{v_n}{\|v_n\|} \to \rho_o z_{\theta_o}, \quad \text{as } n \to \infty, \tag{4.45}$$

in $H$ and in $C[0, 2\pi]$, where

$$\rho_o = \frac{1}{r} > 0, \tag{4.46}$$

by (4.23). Denoting $\rho_o z_{\theta_o}$ by $\varphi_o$, we then have from (4.45) and (4.46) that

$$\frac{v_n(x)}{\|v_n\|} \to \varphi_o(x), \quad \text{as } n \to \infty, \quad \text{for all } x \in [0, 2\pi], \tag{4.47}$$

where $\varphi_o$ is a nontrivial solution to the homogeneous boundary-value problem (1.3) corresponding to $(\mu, \nu) \in \Sigma_m$.

Next, use (4.26), (4.27) and (4.28) to deduce that

$$\liminf_{n\to\infty} \frac{I(v_n)}{\|v_n\|} \geqslant 0, \tag{4.48}$$

where

$$\frac{I(v_n)}{\|v_n\|} = \frac{1}{2}\frac{J(v_n)}{\|v_n\|} - \int_0^{2\pi} \frac{G(x, v_n(x)) + f(x)v_n(x)}{\|v_n\|} \,\mathrm{d}x, \tag{4.49}$$

by (3.4) and (4.1).

Applying Lemma 4.1 we obtain

$$J(v_n) \leqslant -[\nu - (m-1)^2] \int_0^{2\pi} w_n^2 \,\mathrm{d}x, \quad \text{for all } n \in \mathbb{N};$$

thus, by (1.11), $J(v_n) \leqslant 0$, for all $n \in \mathbb{N}$. We therefore get that

$$\limsup_{n\to\infty} \frac{J(v_n)}{2\|v_n\|} \leqslant 0. \tag{4.50}$$

Next, solve for the last term in equation (4.49) to get

$$\int_0^{2\pi} \frac{G(x, v_n(x)) + f(x)v_n(x)}{\|v_n\|}\,\mathrm{d}x = \frac{1}{2}\frac{J(v_n)}{\|v_n\|} - \frac{I(v_n)}{\|v_n\|}. \tag{4.51}$$

It then follows from (4.51), (4.50) and (4.48) that

$$\limsup_{n\to\infty} \int_0^{2\pi} \frac{G(x, v_n(x)) + f(x)v_n(x)}{\|v_n\|}\,\mathrm{d}x \leqslant 0,$$

from which we obtain

$$\liminf_{n\to\infty} \int_0^{2\pi} \frac{G(x, v_n(x)) + f(x)v_n(x)}{\|v_n\|}\,\mathrm{d}x \leqslant 0.$$

Consequently, using Fatou's Lemma,

$$\int_0^{2\pi} \liminf_{n\to\infty} \frac{G(x, v_n(x)) + f(x)v_n(x)}{\|v_n\|}\,\mathrm{d}x \leqslant 0. \tag{4.52}$$

Write

$$\frac{G(x, v_n(x)) + f(x)v_n(x)}{\|v_n\|} = \frac{G(x, v_n(x))}{v_n(x)}\ \frac{v_n(x)}{\|v_n\|} + f(x)\frac{v_n(x)}{\|v_n\|},$$

and take the limit inferior as $n \to \infty$ on both sides to get

$$\liminf_{n\to\infty} \frac{G(x, v_n(x)) + f(x)v_n(x)}{\|v_n\|} \geqslant \liminf_{n\to\infty} \frac{G(x, v_n(x))}{v_n(x)}\frac{v_n(x)}{\|v_n\|} + f(x)\varphi_o(x), \tag{4.53}$$

for a.e. $x \in [0, 2\pi]$, by (4.47). Next, use (4.47) again to get that

$$\liminf_{n\to\infty} \frac{G(x, v_n(x))}{v_n(x)}\ \frac{v_n(x)}{\|v_n\|} \geqslant G_+(x)\varphi_o^+(x) - G_-(x)\varphi_o^-(x), \tag{4.54}$$

for a.e. $x \in [0, 2\pi]$. It then follows from (4.53) and (4.54) that

$$\liminf_{n\to\infty} \frac{G(x, v_n(x)) + f(x)v_n(x)}{\|v_n\|} \geqslant G_+(x)\varphi_o^+(x) - G_-(x)\varphi_o^-(x) + f(x)\varphi_o(x), \tag{4.55}$$

for a.e. $x \in [0, 2\pi]$. The inequalities in (4.55) and (4.52) then imply that

$$\int_0^{2\pi} [G_+(x)\varphi_o^+(x) - G_-(x)\varphi_o^-(x) + f(x)\varphi_o(x)]\,\mathrm{d}x \leqslant 0,$$

which is in direct contradiction with the Tomiczek condition in (1.13) and (1.14). This contradiction establishes (4.20) and completes the proof of Lemma 4.4. $\square$

By Lemma 4.4, it is possible to find $K > 0$ and $L > 0$ such that

$$\sup_{v\in\partial Q_{K,L}} I(v) < c_o,$$

where $c_o$ is given in (4.18). It then follows that, under the assumptions in (1.7), (1.8), (1.13) and (1.14), the functional, $I$, defined in (3.4), satisfies condition (iii) in the Saddle Point Theorem (Theorem 3.1). It remains to prove that, under the same conditions, $I$ satisfies the Palais-Smale condition.

**Lemma 4.5.** *Suppose that* (1.7), (1.8) *and* (1.9) *hold for some* $m \in \mathbb{N}$. *Suppose also that* $f \in L^1[0, 2\pi]$, *and* $g$ *is a Carathéodory function satisfying* (1.2) *with primitive,* $G$, *given in* (1.12). *Let* $I\colon H \to \mathbb{R}$ *be as defined in* (3.4), *and assume that* $f$ *and* $G$ *satisfy the Tomiczek condition in* (1.13) *and* (1.14). *Then,* $I$ *satisfies the Palais-Smale condition.*

*Proof.* Let $(u_n)$ be a sequence in $H$ satisfying

$$|I(u_n)| \leqslant C, \quad \text{for all } n \in \mathbb{N}, \tag{4.56}$$

and some constant $C$; and

$$\lim_{n\to\infty} \|I'(u_n)\| = 0. \tag{4.57}$$

We claim that $(u_n)$ has a subsequence which converges in $H$. By the compact embedding of $H$ into $C[0, 2\pi]$, to prove this claim, it suffices to prove that $(u_n)$ is bounded.

Arguing by contradiction, suppose that $(u_n)$ is not bounded. We may then assume, passing to a subsequence if necessary, that

$$\|u_n\| \to \infty \quad \text{as } n \to \infty. \tag{4.58}$$

Put

$$\widehat{u}_n = \frac{u_n}{\|u_n\|} \quad \text{for all } n \in \mathbb{N}, \tag{4.59}$$

so that

$$\|\widehat{u}_n\| = 1 \quad \text{for all } n \in \mathbb{N}. \tag{4.60}$$

Then, the sequence $(\widehat{u}_n)$ is bounded in $H$, and so, by the compact embedding of $H$ into $C[0, 2\pi]$, we may assume, passing to subsequences if necessary, that there exists $u_o \in H$ such that

$$\widehat{u}_n \rightharpoonup u_o \quad \text{(weakly) in } H, \text{ as } n \to \infty; \tag{4.61}$$

$$\widehat{u}_n \to u_o \quad \text{(uniformly) in } C[0, 2\pi], \text{ as } n \to \infty. \tag{4.62}$$

Consequently, we also have that

$$\widehat{u}_n \to u_o \quad \text{in } L^2[0, 2\pi], \text{ as } n \to \infty. \tag{4.63}$$

Let $\varepsilon > 0$ be arbitrary. Then, there exists $n_1 \in \mathbb{N}$ such that

$$\Big| \int_0^{2\pi} u_n' v' \,\mathrm{d}x - \int_0^{2\pi} [\mu u_n^+ - \nu u_n^- + g(\cdot, u_n) + f] v \,\mathrm{d}x \Big| < \varepsilon \|v\|, \tag{4.64}$$

for $n \geqslant n_1$ and all $v \in H$, where we have used (4.57) and the definition of $I'$ in (3.5). Dividing the inequality in (4.64) by $\|u_n\|$ and letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} \Big| \int_0^{2\pi} \widehat{u}_n' v' \,\mathrm{d}x - \int_0^{2\pi} \big[\mu \widehat{u}_n^+ - \nu \widehat{u}_n^- + \frac{g(\cdot, u_n) + f}{\|u_n\|}\big] v \,\mathrm{d}x \Big| = 0, \tag{4.65}$$

for all $v \in H$, by (4.58).

Next, using (1.2), we see that

$$\Big| \int_0^{2\pi} [g(\cdot, u_n) + f] v \Big| \leqslant [\|p\|_{L^1} + \|f\|_{L^1}]\ \|v\|_{C[0,2\pi]}, \quad \text{for all } n \in N,$$

where

$$\|v\|_{C[0,2\pi]} = \max_{x\in[0,2\pi]} |v(x)|.$$

It then follows from (4.58) that

$$\lim_{n\to\infty} \int_0^{2\pi} \frac{g(\cdot, u_n) + f}{\|u_n\|} v \,\mathrm{d}x = 0 \tag{4.66}$$

for all $v \in H$. We can therefore conclude from (4.61), (4.62), (4.65) and (4.66) that

$$\int_0^{2\pi} u_o' v' \,\mathrm{d}x - \int_0^{2\pi} [\mu u_o^+ - \nu u_o^-] v \,\mathrm{d}x = 0 \quad \text{for all } v \in H. \tag{4.67}$$

Consequently, $u_o$ is a solution to the homogeneous, piece-wise linear problem (1.3).

We next see that $u_o$ is non-trivial. Arguing by contradiction, suppose that $u_o \equiv 0$. It then follows from (4.60) that

$$\lim_{n\to\infty} \int_0^{2\pi} (\widehat{u}_n')^2 \,\mathrm{d}x = 1, \tag{4.68}$$

where we have also used (4.63). On the other hand, dividing the inequality in (4.56) by $\|u_n\|^2$ and letting $n \to \infty$, while using (4.58), (4.59), (4.63) and Lemma 4.2, we obtain that

$$\lim_{n\to\infty} \frac{1}{2} \int_0^{2\pi} (\widehat{u}_n')^2 \,\mathrm{d}x = \frac{\mu}{2} \int_0^{2\pi} (u_o^+)^2 \,\mathrm{d}x + \frac{\nu}{2} \int_0^{2\pi} (u_o^-)^2 \,\mathrm{d}x = 0,$$

which is in contradiction with (4.68). Consequently, $u_o$ is a nontrivial solution of (1.3) corresponding to $(\mu, \nu) \in \Sigma_m$.

Next, we use (3.4) and (4.1) to write

$$\frac{I(u_n)}{\|u_n\|} = \frac{J(u_n)}{2\|u_n\|} - \int_0^{2\pi} \frac{G(\cdot, u_n) + f u_n}{\|u_n\|} \,\mathrm{d}x,$$

from which we obtain

$$\frac{I(u_n)}{\|u_n\|} = \frac{J(u_n)}{2\|u_n\|} - \int_0^{2\pi} [\frac{G(\cdot, u_n)}{u_n}\ \widehat{u}_n + f\widehat{u}_n] \,\mathrm{d}x. \tag{4.69}$$

We claim that

$$\lim_{n\to\infty} \frac{J(u_n)}{2\|u_n\|} = 0. \tag{4.70}$$

To prove the claim in (4.70), we proceed as in Tomiczek [12] by first decomposing $u_n$ into

$$u_n = \rho_n z_{\theta_n} + w_n, \quad \text{for } n \in N, \tag{4.71}$$

where $\rho_n \geqslant 0$, $\theta_n \in [\theta_1, \theta_2]$, $w_n \in H^- \oplus H^+$, for each $n \in \mathbb{N}$, where $z_\theta$ is as given in (4.21) and $\theta_1$ and $\theta_2$ in (2.15). The decomposition in (4.71) is possible because of (3.64) in Remark 3.6.

Observe that the Fréchet derivative of $I$ in (3.5) can be written as

$$I'(u)v = \frac{1}{2} J'(u)v - \int_0^{2\pi} [g(\cdot, u) + f] v \,\mathrm{d}x, \quad \text{for } u, v \in H, \tag{4.72}$$

where $J$ is the functional defined in (4.1).

We presently compute the first term on the right-hand side of the equation in (4.72), where $u = u_n$ as given in (4.71):

$$\begin{aligned} \frac{1}{2} J'(u_n)v &= \int_0^{2\pi} [\rho_n z_{\theta_n} + w_n]' v' - \int_0^{2\pi} [\mu u_n^+ - \nu u_n^-] v \\ &= \int_0^{2\pi} \rho_n z_{\theta_n}' v' + \int_0^{2\pi} + w_n' v' - \int_0^{2\pi} [(\mu - \nu) u_n^+ + \nu u_n] v, \end{aligned}$$

where we have used $u_n = u_n^+ - u_n^-$. Next, use the fact that $\rho_n z_{\theta_n}$ solves the homogeneous, piece-wise linear boundary-value problem (1.3) to write

$$\begin{aligned}
&\frac{1}{2}J'(u_n)v \\
&= \int_0^{2\pi}[\mu\rho_n z_{\theta_n}^+ - \nu\rho_n z_{\theta_n}^-]v + \int_0^{2\pi} +w_n'v' - \int_0^{2\pi}[(\mu-\nu)u_n^+ + \nu u_n]v \\
&= \int_0^{2\pi}[(\mu-\nu)\rho_n z_{\theta_n}^+ + \nu\rho_n z_{\theta_n}]v + \int_0^{2\pi} +w_n'v' - \int_0^{2\pi}[(\mu-\nu)u_n^+ + \nu u_n]v,
\end{aligned}$$

so that

$$\frac{1}{2}J'(u_n)v = \int_0^{2\pi}(\mu-\nu)[\rho_n z_{\theta_n}^+ - u_n^+]v + \int_0^{2\pi}\nu[\rho_n z_{\theta_n} - u_n]v + \int_0^{2\pi} w_n'v',$$

or

$$\frac{1}{2}J'(u_n)v = \int_0^{2\pi}[(\mu-\nu)[\rho_n z_{\theta_n}^+ - u_n^+]v - \nu w_n v + w_n'v'], \tag{4.73}$$

where we have used the decomposition in (4.71). A similar calculation leads to

$$\frac{1}{2}J'(u_n)v = \int_0^{2\pi}[(\mu-\nu)[\rho_n z_{\theta_n}^- - u_n^-]v - \mu w_n v + w_n'v']. \tag{4.74}$$

Adding (4.73) and (4.74) yields

$$J'(u_n)v = \int_0^{2\pi}[(\mu-\nu)[|\rho_n z_{\theta_n}| - |u_n|]v - (\mu+\nu)w_n v + 2w_n'v'],$$

where we have used the fact that $|u_n| = u_n^+ + u_n^-$. We therefore obtain the estimate

$$J'(u_n)v \leqslant \int_0^{2\pi}\left[(\mu-\nu)\ |w_n|\ |v| - (\mu+\nu)w_n v + 2w_n'v'\right]. \tag{4.75}$$

Write $w_n = w_{n,1} + w_{n,2}$, $w_{n,1} \in H^-$, $w_{n,2} \in H^+$, for all $n \in \mathbb{N}$, and put $v_n = w_{n,1} - w_{n,2}$, for all $n \in \mathbb{N}$. It then follows that

$$\|v_n\| = \|w_n\|, \quad \text{for all } n \in \mathbb{N}. \tag{4.76}$$

Substituting $v_n$ for $v$ in the estimate in (4.75) then yields

$$\begin{aligned}
J'(u_n)v_n \leqslant (\mu-\nu)\int_0^{2\pi} &|w_{n,1} + w_{n,2}|\ |w_{n,1} - w_{n,2}| \\
&- (\mu+\nu)\int_0^{2\pi}[w_{n,1}^2 - w_{n,1}^2] + 2\int_0^{2\pi}[(w_{n,1}')^2 - (w_{n,2}')^2].
\end{aligned}$$

Consequently,

$$\begin{aligned}
J'(u_n)v_n \leqslant (\mu-\nu)\int_0^{2\pi} &|w_{n,1}^2 - w_{n,2}^2| - (\mu+\nu)[\|w_{n,1}\|_2^2 - \|w_{n,2}\|_2^2] \\
&+ 2[\|w_{n,1}'\|_2^2 - \|w_{n,2}'\|_2^2].
\end{aligned}$$

It then follows that

$$\begin{aligned}
J'(u_n)v_n \leqslant (\mu-\nu)\int_0^{2\pi} &|w_{n,1}^2 - w_{n,2}^2| - (\mu+\nu)[\|w_{n,1}\|_2^2 - \|w_{n,2}\|_2^2] \\
&+ 2[\|w_{n,1}'\|_2^2 - \|w_{n,2}'\|_2^2],
\end{aligned}$$

or

$$\begin{aligned} J'(u_n)v_n \leqslant (\mu-\nu)\int_0^{2\pi} |w_{n,1}^2 - w_{n,2}^2| + 2\|w_{n,1}'\|_2^2 - (\mu+\nu)\|w_{n,1}\|_2^2 \\ + (\mu+\nu)\|w_{n,2}\|_2^2 - 2\|w_{n,2}'\|_2^2. \end{aligned} \tag{4.77}$$

Next, use the inequality $|x^2 - y^2| \leqslant x^2 + y^2$ for all real numbers $x$ and $y$, to obtain from (4.77) that

$$\begin{aligned} J'(u_n)v_n \leqslant (\mu-\nu)(\|w_{n,1}\|_2^2 + \|w_{n,2}\|_2^2) + 2\|w_{n,1}'\|_2^2 - (\mu+\nu)\|w_{n,1}\|_2^2 \\ + (\mu+\nu)\|w_{n,2}\|_2^2 - 2\|w_{n,2}'\|_2^2, \end{aligned}$$

from which we obtain

$$\frac{1}{2}J'(u_n)v_n \leqslant \|w_{n,1}'\|_2^2 - \nu\|w_{n,1}\|_2^2 + \mu\|w_{n,2}\|_2^2 - \|w_{n,2}'\|_2^2. \tag{4.78}$$

Now, using the definition of the norm $\|\cdot\|_H$ in (3.2) we see that

$$\|w_{n,1}\|_H^2 = 2\pi(\overline{w}_{n,1})^2 + \|w_{n,1}'\|_2^2,$$

from which we obtain that

$$\|w_{n,1}\|_H^2 \geqslant \|w_{n,1}'\|_2^2. \tag{4.79}$$

Similarly, from

$$\|w_{n,2}\|_H^2 = 2\pi(\overline{w}_{n,2})^2 + \|w_{n,2}'\|_2^2$$

we obtain

$$\|w_{n,2}\|_H^2 = \|w_{n,2}'\|_2^2, \tag{4.80}$$

since $\overline{w}_{n,2} = 0$ as $w_{n,2} \in H^+$. Hence, combining the estimate in (4.79) and the expression in (4.80) with the inequality in (4.78), we obtain from (4.78) that

$$\frac{1}{2}J'(u_n)v_n \leqslant \|w_{n,1}\|_H^2 - \nu\|w_{n,1}\|_2^2 + \mu\|w_{n,2}\|_2^2 - \|w_{n,2}\|_H^2. \tag{4.81}$$

Next, use the definition of $\|\cdot\|_H$ in (3.2) and the Fourier series expansion

$$w_{n,2}(x) = \sum_{k=m+1}^{\infty} [a_k \cos kx + b_k \sin kx],$$

to compute

$$\|w_{n,2}\|_H^2 = \|w_{n,2}'\|_2^2 = \sum_{k=m+1}^{\infty} \pi k^2 [a_k^2 + b_k^2], \tag{4.82}$$

where we have used Parseval's Theorem for Fourier series (see [10, Theorem 1]). It then follows from (4.82) and Parseval's Theorem for Fourier series that

$$\|w_{n,2}\|_H^2 \geqslant (m+1)^2\|w_{n,2}\|_2^2, \quad \text{for all } n \in \mathbb{N}. \tag{4.83}$$

Similarly, for the case $m \geqslant 2$, we obtain that

$$\|w_{n,1}\|_H^2 \leqslant (m-1)^2\|w_{n,1}\|_2^2, \quad \text{for all } n \in \mathbb{N}, \tag{4.84}$$

since

$$\begin{aligned} \|w_{n,1}\|_H^2 &= 2\pi(\overline{w}_{n,1})^2 + \|w_{n,1}'\|_2^2 \\ &= 2\pi(\overline{w}_{n,1})^2 + \sum_{k=1}^{m-1} \pi k^2 [a_k^2 + b_k^2] \\ &\leqslant 2\pi(\overline{w}_{n,1})^2 + (m-1)^2 \sum_{k=1}^{m-1} \pi [a_k^2 + b_k^2], \end{aligned} \tag{4.85}$$

where $\overline{w}_{n,1}$ denotes the average value of $w_{n,1}$ over $[0, 2\pi]$. We then get from the inequality derived in (4.85) that

$$\|w_{n,1}\|_H^2 \leqslant (m-1)^2 \big[2\pi(\overline{w}_{n,1})^2 + \sum_{k=1}^{m-1} \pi [a_k^2 + b_k^2]\big],$$

for $m \geqslant 2$, which yields (4.84) by Parseval's Theorem for Fourier series.

It follows from (4.83) and (4.84), by inequalities in (1.9) and (1.11), that

$$\|w_{n,1}\|_H^2 - \nu\|w_{n,1}\|_2^2 \leqslant 0 \quad \text{and} \quad \mu\|w_{n,2}\|_2^2 - \|w_{n,2}\|_H^2 \leqslant 0, \quad \text{for } m \geqslant 2.$$

It then follows from (4.81) that

$$\frac{1}{2} J'(u_n)v_n \leqslant \min\{\|w_{n,1}\|_H^2 - \nu\|w_{n,1}\|_2^2,\ \mu\|w_{n,2}\|_2^2 - \|w_{n,2}\|_H^2\}. \tag{4.86}$$

for the case $m \geqslant 2$.

Next, use (4.84) to obtain, for $m \geqslant 2$,

$$\|w_{n,1}\|_H^2 - \nu\|w_{n,1}\|_2^2 \leqslant -\big(\frac{\nu}{(m-1)^2} - 1\big)\|w_{n,1}\|_H^2, \quad \text{for all } n \in \mathbb{N}. \tag{4.87}$$

Similarly, using (4.83),

$$\mu\|w_{n,2}\|_2^2 - \|w_{n,2}\|_H^2 \leqslant -\big(1 - \frac{\mu}{(m+1)^2}\big)\|w_{n,2}\|_H^2, \quad \text{for all } n \in \mathbb{N}. \tag{4.88}$$

Let

$$\delta = \min\big\{\frac{\nu}{(m-1)^2} - 1,\ 1 - \frac{\mu}{(m+1)^2}\big\}; \tag{4.89}$$

then $\delta > 0$ by inequalities in (1.9) and (1.11). Furthermore, it follows from (4.87), (4.88) and (4.89) that

$$\|w_{n,1}\|_H^2 - \nu\|w_{n,1}\|_2^2 \leqslant -\delta\|w_{n,1}\|_H^2, \quad \text{for all } n \in \mathbb{N}, \tag{4.90}$$

$$\mu\|w_{n,2}\|_2^2 - \|w_{n,2}\|_H^2 \leqslant -\delta\|w_{n,2}\|_H^2, \quad \text{for all } n \in \mathbb{N}. \tag{4.91}$$

It then follows from (4.86), (4.90) and (4.91) that

$$\frac{1}{2} J'(u_n)v_n \leqslant -\delta \max\{\|w_{n,1}\|_H^2,\ \|w_{n,2}\|_H^2\}, \quad \text{for all } n \in \mathbb{N}, \tag{4.92}$$

where $\delta > 0$ is given in (4.89), in the case $m \geqslant 2$.

For the case $m = 1$, observe from (4.78) that

$$\frac{1}{2} J'(u_n)v_n \leqslant -2\pi\nu(\overline{w}_n)^2 + \mu\|w_{n,2}\|_2^2 - \|w_{n,2}'\|_2^2, \tag{4.93}$$

where $\overline{w}_n$ is the average value of $w_n$ over $[0, 2\pi]$. Next, use the definition of $\|\cdot\|_H$ in (3.2) to obtain from (4.93) that

$$\frac{1}{2} J'(u_n)v_n \leqslant -\nu\|w_{n,1}\|_H^2 + \mu\|w_{n,2}\|_2^2 - \|w_{n,2}\|_H^2,$$

We are therefore led to

$$\frac{1}{2}J'(u_n)v_n \leqslant \min\{-\nu\|w_{n,1}\|_H^2,\ \mu\|w_{n,2}\|_2^2 - \|w_{n,2}\|_H^2\}. \tag{4.94}$$

for the case $m = 1$. Thus, setting

$$\delta = \min\{\nu,\ 1 - \frac{\mu}{4}\}, \tag{4.95}$$

we deduce from (4.94) that

$$\frac{1}{2}J'(u_n)v_n \leqslant -\delta\ \max\{\|w_{n,1}\|_H^2,\ \|w_{n,2}\|_H^2\}, \quad \text{for all } n \in \mathbb{N}, \tag{4.96}$$

where $\delta > 0$ is given in (4.95) for the case $m = 1$; observe that, in this case, $\mu < 4$ by the inequality in (1.9). Hence, combining (4.92) and (4.96), we see that there exists $\delta > 0$ such that

$$\frac{1}{2}J'(u_n)v_n \leqslant -\delta\ \max\{\|w_{n,1}\|_H^2,\ \|w_{n,2}\|_H^2\}, \quad \text{for all } n \in \mathbb{N}. \tag{4.97}$$

Next, observe that it follows from (4.64) and (4.76) that

$$-\varepsilon\|w_n\| < \frac{1}{2}J'(u_n)v_n - \int_0^{2\pi} [g(\cdot, u_n) + f]v_n \, \mathrm{d}x, \quad \text{for } n \geqslant n_1. \tag{4.98}$$

We claim that the sequence $(w_n)$ defined by (4.71) is bounded in $H$. Suppose, by way of contradiction, that

$$\|w_n\| \to \infty \quad \text{as } n \to \infty. \tag{4.99}$$

Divide (4.98) by $\|w_n\|^2$ and let $n \to \infty$ to obtain that

$$0 \leqslant \liminf_{n\to\infty} \Big\{\frac{1}{2\|w_n\|^2}J'(u_n)v_n - \int_0^{2\pi} \frac{[g(\cdot, u_n) + f]v_n}{\|w_n\|^2}\, \mathrm{d}x\Big\} \tag{4.100}$$

Next, use (1.2), (4.76), (4.99), and the assumptions that $p, f \in L^1[0, 2\pi]$, to conclude that

$$\lim_{n\to\infty} \int_0^{2\pi} \frac{[g(\cdot, u_n) + f]v_n}{\|w_n\|^2}\, \mathrm{d}x = 0.$$

It then follows from (4.100) that

$$0 \leqslant \liminf_{n\to\infty} \frac{1}{2\|w_n\|^2}J'(u_n)v_n. \tag{4.101}$$

On the other hand, dividing (4.97) by $\|w_n\|^2$ and letting $n \to \infty$,

$$\liminf_{n\to\infty} \frac{J'(u_n)v_n}{2\|w_n\|^2} \leqslant -\delta \limsup_{n\to\infty} \frac{\max\{\|w_{n,1}\|_H^2, \|w_{n,2}\|_H^2\}}{\|w_n\|^2}. \tag{4.102}$$

Now,

$$2\max\{\|w_{n,1}\|_H^2, \|w_{n,2}\|_H^2\} \geqslant \|w_n\|_H^2 \geqslant \frac{1}{2}\|w_n\|^2, \quad \text{for all } n \in \mathbb{N},$$

where we have used the left-most inequality in (3.3). Thus,

$$\frac{\max\{\|w_{n,1}\|_H^2, \|w_{n,2}\|_H^2\}}{\|w_n\|^2} \geqslant \frac{1}{4}, \quad \text{for all } n \in \mathbb{N}. \tag{4.103}$$

Combining the estimates in (4.102) and (4.103) then yields that

$$\liminf_{n\to\infty} \frac{J'(u_n)v_n}{2\|w_n\|^2} \leqslant -\frac{\delta}{4},$$

which is in direct contradiction with (4.101). This contradiction proves that the sequence $(w_n)$ defined in (4.71) is bounded in $H$.

Next, use the estimate in Lemma 4.1 and (4.71) to obtain

$$\int_0^{2\pi} \left[(w_n')^2 - \mu w_n^2\right] dx \leqslant J(u_n) \leqslant \int_0^{2\pi} \left[(w_n')^2 - \nu w_n^2\right] dx\,, \quad \text{for all } n. \tag{4.104}$$

Consequently, using the fact that $(w_n)$ is bounded in $H$, we see that (4.70) follows from (4.104) after dividing the inequalities in (4.104) by $\|u_n\|$ and letting $n \to \infty$, by the assumption in (4.58). Therefore, it follows from (4.69), (4.70), (4.58) and (4.56) that

$$\lim_{n\to\infty} \int_0^{2\pi} [\frac{G(\cdot, u_n)}{u_n} \widehat{u}_n + f\widehat{u}_n]\,\mathrm{d}x = 0. \tag{4.105}$$

Now, it follows from (4.105) and Fatou's Lemma that

$$\int_0^{2\pi} \liminf_{n\to\infty} [\frac{G(\cdot, u_n)}{u_n} \widehat{u}_n + f\widehat{u}_n]\,\mathrm{d}x \leqslant 0. \tag{4.106}$$

Next, use (4.62) to obtain

$$\liminf_{n\to\infty} [\frac{G(x, u_n(x))}{u_n(x)} \widehat{u}_n(x) + f(x)\widehat{u}_n(x)] = G_+(x)u_o^+(x) - G_-(x)u_o^-(x) + f(x)u_o(x),$$

since

$$u_n(x) = \|u_n\|\widehat{u}_n(x) \to \begin{cases} +\infty & \text{if } u_o(x) > 0; \\ -\infty & \text{if } u_o(x) < 0, \end{cases}$$

by (4.58). It then follows from (4.106) that

$$\int_0^{2\pi} [G_+(x)u_o^+(x) - G_-(x)u_o^-(x) + f(x)u_o(x)]\,\mathrm{d}x \leqslant 0,$$

where $u_o$ is a nontrivial solution of (1.3) corresponding to $(\mu, \nu) \in \Sigma_m$. This is in direct contradiction with the Tomiczek condition in (1.13) and (1.14). This contradiction shows that the sequence $(u_n)$ must be bounded in $H$.

Standard arguments involving the use of the compact embedding of $H$ into $C[0, 2\pi]$ can now be used to show that $(u_n)$ has a subsequence that converges in $H$. Hence, the functional $I\colon H \to \mathbb{R}$ defined in (3.4) satisfies the Palais-Smale condition. $\square$

*Proof of Theorem 1.1.* We use Theorem 3.1 to show that the functional $I : H \to \mathbb{R}$ defined in (3.4) has a critical point. By Lemma 4.5, the functional $I$ satisfies the Palais-Smale condition. Use the result of Lemma 4.3 to deduce that $I$ is bounded from below in $H^+$; thus, there exists $c_o \in \mathbb{R}$ such that

$$I(v) \geqslant c_o \quad \text{for all } v \in H^+. \tag{4.107}$$

Next, apply Lemma 4.4 to obtain $K$ and $L$, both positive, such that

$$\sup_{v\in\partial Q_{K,L}} I(v) < c_o, \tag{4.108}$$

where $Q_{K,L}$ is as defined in Definitions 3.2. It follows from (4.107) and (4.108) that $I$ and $Q_{K,L}$ satisfy (iii) in the hypotheses of Theorem 3.1. Finally, the results in Section 3 show that $Q_{K,L}$ and $H^+$ link. Hence, all the hypotheses for the Saddle Point Theorem of Rabinowitz (Theorem 3.1) hold true. Consequently, the functional $I$ must have a critical point, and the proof of Theorem 1.1 is concluded. □

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