Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 100, pp. 1–54.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

STABILITY AND APPROXIMATIONS OF EIGENVALUES AND EIGENFUNCTIONS OF THE NEUMANN LAPLACIAN, PART 3

MICHAEL M. H. PANG

Abstract. This article is a sequel to two earlier articles (one of them written jointly with R. Banuelos) on stability results for the Neumann eigenvalues and eigenfunctions of domains in $\mathbb{R}^2$ with a snowflake type fractal boundary. In particular we want our results to be applicable to the Koch snowflake domain. In the two earlier papers we assumed that a domain $\Omega \subseteq \mathbb{R}^2$ which has a snowflake type boundary is approximated by a family of subdomains and that the Neumann heat kernel of Ω and those of its approximating subdomains satisfy a uniform bound for all sufficiently small $t > 0$. The purpose of this paper is to extend the results in the two earlier papers to the situations where the approximating domains are not necessarily subdomains of Ω. We then apply our results to the Koch snowflake domain when it is approximated from outside by a decreasing sequence of polygons.

1. Introduction

This paper is a sequel to the papers [1, 7]. The goal of these three papers is to prove stability results for the Neumann eigenvalues and eigenfunctions of domains in $\mathbb{R}^2$ with a snowflake type fractal boundary. In particular we want our results to be applicable to the Koch snowflake domain. In [1] and [7] we assumed that a domain $\Omega \subseteq \mathbb{R}^2$ which has a snowflake type boundary is approximated by a family of subdomains and that the Neumann heat kernel of Ω and those of its approximating subdomains satisfy a uniform bound for all sufficiently small $t > 0$ (see Hypothesis 1.1 of [1] and [7]). The referee of [1] asked whether stability results similar to those in [1] and [7] are still true if the approximating domains of Ω are not necessarily subdomains of Ω and whether the proofs in [1] and [7] can be extended to those situations. If the results and methods in [1] and [7] can be extended to those situations, then they can be applied to domains, such as the Koch snowflake domain, which can be approximated by a familiar decreasing sequence of polygons from outside. The method in [1, 7] can be extended to situations when the approximating domains are not necessarily subdomains of Ω, but not in a straight forward manner. The purpose of this paper is to work out such an extension and to apply it to the Koch snowflake domain when it is approximated from outside by a decreasing sequence of polygons.

2000 *Mathematics Subject Classification.* 35P05, 35P15.
Key words and phrases. Stability; approximations; Neumann eigenvalues and eigenfunctions.

Submitted November 5, 2010. Published August 7, 2011.

To state our results we first fix notation. Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a bounded Sobolev extension domain. Let $\epsilon_0 > 0$ be sufficiently small, depending on Ω. For each $\epsilon \in (0, \epsilon_0]$, let Ω_ϵ, Ω^ϵ and $\Omega(\epsilon)$ be bounded Sobolev extension domains in $\mathbb{R}^N$ satisfying the following assumptions:

$$\begin{gathered} \Omega_\epsilon \supseteq \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\}, \\ \Omega^\epsilon \subseteq \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Omega) < \epsilon\}, \\ \Omega_\epsilon \subseteq \Omega(\epsilon) \subseteq \Omega^\epsilon. \end{gathered} \tag{1.1}$$

We shall assume that

$$\Omega_{\epsilon_1} \supseteq \Omega_{\epsilon_2} \quad \text{if } 0 < \epsilon_1 \leq \epsilon_2 \leq \epsilon_0 \tag{1.2}$$

and that

$$\Omega^{\epsilon_1} \subseteq \Omega^{\epsilon_2} \quad \text{if } 0 < \epsilon_1 \leq \epsilon_2 \leq \epsilon_0. \tag{1.3}$$

Let $-\Delta_\epsilon$, $-\Delta$, $-\Delta^\epsilon$, $-\Delta(\epsilon)$ be the Neumann Laplacian defined on Ω_ϵ, Ω, Ω^ϵ and $\Omega(\epsilon)$, respectively, and let $P_\epsilon(t,x,y)$, $P(t,x,y)$, $P^\epsilon(t,x,y)$ and $P(\epsilon)(t,x,y)$ be the heat kernel of $e^{-\Delta_\epsilon t}$, $e^{-\Delta t}$, $e^{-\Delta^\epsilon t}$ and $e^{-\Delta(\epsilon)t}$, respectively. We shall assume that there exists a positive continuous function $c : (0,1] \to (0,\infty)$ such that for all $0 < \epsilon \leq \epsilon_0$ and all $0 < t \leq 1$ we have

$$\begin{aligned} P_\epsilon(t,x,y) &\leq c(t) \quad (x, y \in \Omega_\epsilon), \\ P(t,x,y) &\leq c(t) \quad (x, y \in \Omega), \\ P^\epsilon(t,x,y) &\leq c(t) \quad (x, y \in \Omega^\epsilon), \\ P(\epsilon)(t,x,y) &\leq c(t) \quad (x, y \in \Omega(\epsilon)). \end{aligned} \tag{1.4}$$

Since the domains Ω_ϵ, Ω, Ω^ϵ and $\Omega(\epsilon)$ are assumed to be bounded, (1.4) implies that $-\Delta_\epsilon$, $-\Delta$, $-\Delta^\epsilon$ and $-\Delta(\epsilon)$ all have compact resolvents (see [4, Theorem 2.1.5]). We shall write $\{\mu_i\}_{i=1}^\infty$ for the eigenvalues of $-\Delta$, where $\{\mu_i\}_{i=1}^\infty$ is a non-decreasing sequence with $\mu_1 = 0$ and the eigenvalues are listed repeatedly according to multiplicity. Similarly, for $0 < \epsilon \leq \epsilon_0$, we shall write $\{\mu_{i,\epsilon}\}_{i=1}^\infty$, $\{\mu_i^\epsilon\}_{i=1}^\infty$, and $\{\mu_i(\epsilon)\}_{i=1}^\infty$ for the eigenvalues of $-\Delta_\epsilon$, $-\Delta^\epsilon$ and $-\Delta(\epsilon)$, respectively. We shall write $\{\varphi_i\}_{i=1}^\infty$, $\{\varphi_{i,\epsilon}\}_{i=1}^\infty$, $\{\varphi_i^\epsilon\}_{i=1}^\infty$ and $\{\varphi_i(\epsilon)\}_{i=1}^\infty$ for the corresponding eigenfunctions of $-\Delta$, $-\Delta_\epsilon$, $-\Delta^\epsilon$ and $-\Delta(\epsilon)$, respectively. We may, and shall, assume that $\{\varphi_i\}_{i=1}^\infty$, $\{\varphi_{i,\epsilon}\}_{i=1}^\infty$, $\{\varphi_i^\epsilon\}_{i=1}^\infty$ and $\{\varphi_i(\epsilon)\}_{i=1}^\infty$ are complete orthonormal systems in $L^2(\Omega)$, $L^2(\Omega_\epsilon)$, $L^2(\Omega^\epsilon)$ and $L^2(\Omega(\epsilon))$, respectively. We define the sequence $\{k_i\}_{i=1}^\infty$ of positive integers using the multiplicities of the eigenvalues $\{\mu_i\}_{i=1}^\infty$ of $-\Delta$ as follows:

Let $k_1 = 1$ and, for $i = 2, 3, 4, \dots$, we define k_i by:

$$\begin{aligned} 0 = \mu_1 < \mu_2 = \mu_3 = \dots = \mu_{k_2} < \mu_{k_2+1} = \mu_{k_2+2} = \dots = \mu_{k_3} \\ < \mu_{k_3+1} = \mu_{k_3+2} = \dots = \mu_{k_4} < \mu_{k_4+1} = \dots. \end{aligned} \tag{1.5}$$

For all $j = 1, 2, 3, \dots$ and all $\epsilon \in (0, \epsilon_0]$ we write

$$\varphi_j|_{\Omega\cap\Omega(\epsilon)} = \sum_{\ell=1}^\infty a_{j,\ell}(\epsilon)\varphi_\ell(\epsilon) \in L^2(\Omega \cap \Omega(\epsilon)) \subseteq L^2(\Omega(\epsilon)). \tag{1.6}$$

Let $p \geq 1$ be an integer. For $i = k_p + 1, \dots, k_{p+1}$ and $\epsilon \in (0, \epsilon_0]$ let

$$\hat{\psi}_i(\epsilon) = \Big(\sum_{\ell=k_p+1}^{k_{p+1}} a_{i,\ell}(\epsilon)\varphi_\ell(\epsilon) \Big)\Big|_{\Omega\cap\Omega(\epsilon)} \tag{1.7}$$

and let
$$\psi_i(\epsilon) = \|\hat{\psi}_i(\epsilon)\|^{-1}_{L^2(\Omega\cap\Omega(\epsilon))}\hat{\psi}_i(\epsilon). \tag{1.8}$$
We now state our results:

Theorem 1.1. *For all $i = 1, 2, 3, \dots,$ we have*
$$\lim_{\epsilon\downarrow 0}\mu_i(\epsilon) = \mu_i. \tag{1.9}$$

Theorem 1.2. *Let K be a compact subset of Ω. Then we have*
$$\lim_{\epsilon\downarrow 0}\{\sup_{x\in K}|\varphi_j(x) - \psi_j(\epsilon)(x)|\} = 0 \tag{1.10}$$
for $j = 1, 2, 3, \dots$.

To apply Theorems 1.1 and 1.2 to the Koch snowflake domain, we have

Theorem 1.3. *Let $\Omega \subseteq \mathbb{R}^2$ be the Koch snowflake domain. Let $\{\Omega_{\rm out}(n)\}_{n=1}^{\infty}$ be the usual decreasing sequence of polygons approximating Ω from outside, with $\Omega_{\rm out}(1)$ being a regular hexagon. Let $\{\Omega_{\rm in}(n)\}_{n=1}^{\infty}$ be the usual increasing sequence of polygons approximating Ω from inside, with $\Omega_{\rm in}(1)$ being an equilateral triangle. Let $P^{\Omega}(t,x,y)$, $P^{\Omega_{\rm out}(n)}(t,x,y)$ and $P^{\Omega_{\rm in}(n)}(t,x,y)$ be the Neumann heat kernels on Ω, $\Omega_{\rm out}(n)$ and $\Omega_{\rm in}(n)$, respectively. Then there exists $c \geq 1$, independent of n, such that*
$$\begin{gathered} P^{\Omega}(t,x,y) \leq ct^{-1} \quad (x,y\in\Omega),\\ P^{\Omega_{\rm out}(n)}(t,x,y) \leq ct^{-1} \quad (x,y\in\Omega_{\rm out}(n)),\\ P^{\Omega_{\rm in}(n)}(t,x,y) \leq ct^{-1} \quad (x,y\in\Omega_{\rm in}(n)), \end{gathered} \tag{1.11}$$
for all $0 < t \leq 1$ and $n = 1, 2, 3, \dots$.

Remark 1.4. (i) The third inequality in (1.11) was proved in [7, Theorem 1.3].

(ii) Since Ω, $\Omega_{\rm out}(n)$, $\Omega_{\rm in}(n)$, $n = 1, 2, 3, \dots$, are bounded Sobolev extension domains, Theorem 1.3 enables one to apply Theorems 1.1 and 1.2 to the case when Ω is the Koch snowflake domain approximated from outside by the sequence $\{\Omega_{\rm out}(n)\}_{n=1}^{\infty}$ by putting $\{\Omega_\epsilon\} = \{\Omega_{\rm in}(n)\}_{n=1}^{\infty}$ and $\{\Omega^\epsilon\} = \{\Omega(\epsilon)\} = \{\Omega_{\rm out}(n)\}_{n=1}^{\infty}$.

In Section 2 we shall prove some abstract approximation results for families of non-negative self-adjoint operators with domains in Hilbert spaces. In Section 3 we consider the case when these non-negative self-adjoint operators are the Neumann Laplacians defined on domains of $\mathbb{R}^N$. It will be seen that results in Sections 2 and 3 imply Theorems 1.1 and 1.2. Theorem 1.3 will be proved in Section 4.

We refer to the references in [1, 7] for recent papers on numerical studies on the Neumann eigenvalues and eigenfunctions of the Koch snowflake domain and on stability results for Neumann eigenvalues and eigenfunctions. In addition, we mention the excellent recent survey paper [2], and references therein, for stability results for eigenvalues and eigenfunctions of elliptic operators defined on domains with either Dirichlet or Neumann boundary conditions.

2. Quadratic forms and approximations

In this section we prove the abstract theorems we shall need in the proofs of the mains results stated in Section 1. If $\mathcal{U}$ and $\mathcal{V}$ are Hilbert spaces and if $\mathcal{U} \subseteq \mathcal{V}$, then we shall denote the orthogonal projection of $\mathcal{V}$ onto U by $P_{\mathcal{V},\mathcal{U}}$ and write $\mathcal{U}^{\perp}\mathcal{V}$ for the orthogonal compliment of $\mathcal{U}$ in $\mathcal{V}$. We shall also write $I_{\mathcal{U}}$ for the identity map on $\mathcal{U}$. We shall let $\mathcal{H}$ be a fixed Hilbert space. For all sufficiently small $\epsilon > 0$, let $\mathcal{H}_\epsilon$ and $\mathcal{H}^\epsilon$ be Hilbert spaces satisfying the following assumptions:

(A1) If $0 < \epsilon_2 \le \epsilon_1$, then $\mathcal{H}^{\epsilon_2} \subseteq \mathcal{H}^{\epsilon_1}$.

(A2) If $0 < \epsilon_2 \le \epsilon_1$, then $\mathcal{H}_{\epsilon_2} \supseteq \mathcal{H}_{\epsilon_1}$.

(A3) $\cap_{\epsilon>0}\mathcal{H}^\epsilon = \mathcal{H} = \cup_{\epsilon>0}\mathcal{H}_\epsilon$.

(A4) For all $f \in \mathcal{H}$ we have

$$\|f - P_{\mathcal{H},\mathcal{H}_\epsilon} f\|_{\mathcal{H}} \to 0 \quad \text{as } \epsilon \downarrow 0,$$

where $\|\cdot\|_{\mathcal{H}}$ denotes the norm in $\mathcal{H}$.

(A5) If $\epsilon_1 > 0$ and if $f \in \mathcal{H}^{\epsilon_1}$, then

$$\|P_{\mathcal{H}^{\epsilon_1},\mathcal{H}} f - P_{\mathcal{H}^{\epsilon_1},\mathcal{H}^\epsilon} f\|_{\mathcal{H}^\epsilon} \to 0 \quad \text{as } \epsilon \downarrow 0.$$

For all sufficiently small $\epsilon > 0$ let $\mathcal{A}(\epsilon)$ and $\mathcal{B}(\epsilon)$ be Hilbert spaces satisfying the following assumptions:

(A6) $\mathcal{H}_\epsilon \subseteq \mathcal{B}(\epsilon) \subseteq \mathcal{H} \cap \mathcal{A}(\epsilon) \subseteq \mathcal{A}(\epsilon) \subseteq \mathcal{H}^\epsilon$,

(A7) For all $f \in \mathcal{H}$ we have

$$\|f - P_{\mathcal{H},\mathcal{B}(\epsilon)} f\|_{\mathcal{H}} \to 0 \quad \text{as } \epsilon \downarrow 0.$$

We assume that for all sufficiently small $\epsilon > 0$ there exists a closed subspace $\mathcal{C}(\epsilon)$ of $\mathcal{A}(\epsilon)$ satisfying the following assumptions:

(A8) $\mathcal{C}(\epsilon) \subseteq \mathcal{H}^\perp\mathcal{H}^\epsilon$.

Lemma 2.1. *If $\epsilon_1 > 0$, then for all $f \in \mathcal{H}^{\epsilon_1}$, we have*

$$\|P_{\mathcal{H}^{\epsilon_1},\mathcal{C}(\epsilon)} f\|_{\mathcal{H}^{\epsilon_1}} \to 0 \quad \textit{as } \epsilon \downarrow 0.$$

Proof. Let $f \in \mathcal{H}^{\epsilon_1}$. Then

$$\begin{aligned}
&\|P_{\mathcal{H}^\perp\mathcal{H}^\epsilon,\mathcal{C}(\epsilon)}(P_{\mathcal{H}^\epsilon,\mathcal{H}^\perp\mathcal{H}^\epsilon}(P_{\mathcal{H}^{\epsilon_1},\mathcal{H}^\epsilon} f))\|_{\mathcal{H}^{\epsilon_1}} \\
&= \|P_{\mathcal{H}^\perp\mathcal{H}^\epsilon,\mathcal{C}(\epsilon)}[(I_{\mathcal{H}^\epsilon} - P_{\mathcal{H}^\epsilon,\mathcal{H}})(P_{\mathcal{H}^{\epsilon_1},\mathcal{H}^\epsilon} f)]\|_{\mathcal{H}^{\epsilon_1}} \\
&\le \|P_{\mathcal{H}^\perp\mathcal{H}^\epsilon,\mathcal{C}(\epsilon)}\| \|P_{\mathcal{H}^{\epsilon_1},\mathcal{H}^\epsilon} f - P_{\mathcal{H}^{\epsilon_1},\mathcal{H}} f\|_{\mathcal{H}^{\epsilon_1}} \\
&\le \|P_{\mathcal{H}^{\epsilon_1},\mathcal{H}^\epsilon} f - P_{\mathcal{H}^{\epsilon_1},\mathcal{H}} f\|_{\mathcal{H}^{\epsilon_1}} \to 0 \quad \text{as } \epsilon \downarrow 0.
\end{aligned}$$

□

We assume that for all sufficiently small $\epsilon > 0$ there exists a closed subspace $\mathcal{D}(\epsilon)$ of $\mathcal{A}(\epsilon)$ satisfying the following assumptions:

(A9) $\mathcal{A}(\epsilon) = \mathcal{B}(\epsilon) \oplus \mathcal{C}(\epsilon) \oplus \mathcal{D}(\epsilon)$, where $\oplus$ denotes orthogonal direct sum.

(A10) If $\epsilon_1 > 0$, then, for all $f \in \mathcal{H}^{\epsilon_1}$,

$$\|P_{\mathcal{H}^{\epsilon_1},\mathcal{D}(\epsilon)} f\|_{\mathcal{H}^{\epsilon_1}} \to 0 \quad \text{as } \epsilon \downarrow 0.$$

For all sufficiently small $\epsilon > 0$ let Q^ϵ and Q_ϵ be non-negative closed quadratic forms with domains $\mathrm{Dom}(Q^\epsilon) \subseteq \mathcal{H}^\epsilon$ and $\mathrm{Dom}(Q_\epsilon) \subseteq \mathcal{H}_\epsilon$, respectively. Let Q be a non-negative closed quadratic form with domain $\mathrm{Dom}(Q) \subseteq \mathcal{H}$. We assume that Q, Q^ϵ and Q_ϵ satisfy the following assumptions:

(A11) For all sufficiently small $\epsilon > 0$, we have
- (i) $\mathrm{Dom}(Q^\epsilon)$ is dense in $\mathcal{H}^\epsilon$,
- (ii) $\mathrm{Dom}(Q_\epsilon)$ is dense in $\mathcal{H}_\epsilon$,
- (iii) $\mathrm{Dom}(Q)$ is dense in $\mathcal{H}$.

(A12) For $0 < \epsilon_2 \le \epsilon_1$ we have
- (i) $P_{\mathcal{H}^{\epsilon_1},\mathcal{H}^{\epsilon_2}}(\mathrm{Dom}(Q^{\epsilon_1})) = \mathrm{Dom}(Q^{\epsilon_2})$,

(ii) $P_{\mathcal{H}^{\epsilon_1},\mathcal{H}}(\mathrm{Dom}(Q^{\epsilon_1})) = \mathrm{Dom}(Q)$.

(A13) If $\epsilon_1 > 0$, then, for all sufficiently small $\epsilon > 0$, we have
$$P_{\mathcal{H}^{\epsilon_1},\mathcal{H}_\epsilon}(\mathrm{Dom}(Q^{\epsilon_1})) = \mathrm{Dom}(Q_\epsilon).$$

(A14) For $0 < \epsilon_2 \le \epsilon_1$ we have
(i) $P_{\mathcal{H}_{\epsilon_2},\mathcal{H}_{\epsilon_1}}(\mathrm{Dom}(Q_{\epsilon_2})) = \mathrm{Dom}(Q_{\epsilon_1})$,
(ii) $P_{\mathcal{H},\mathcal{H}_{\epsilon_1}}(\mathrm{Dom}(Q)) = \mathrm{Dom}(Q_{\epsilon_1})$.

Definition 2.2. Let $\epsilon_0 > 0$ be fixed. For $0 < \epsilon \le \epsilon_0$ let $\hat{Q}^\epsilon$ be the quadratic form with domain
$$\mathrm{Dom}(\hat{Q}^\epsilon) = \mathrm{Dom}(Q^\epsilon) \oplus (\mathcal{H}^\epsilon)^\perp \mathcal{H}^{\epsilon_0}$$
and, if $f, g \in \mathrm{Dom}(Q^\epsilon)$ and $h, i \in (\mathcal{H}^\epsilon)^\perp \mathcal{H}^{\epsilon_0}$, we define $\hat{Q}^\epsilon(f \oplus h, g \oplus i)$ by
$$\hat{Q}^\epsilon(f + h, g + i) = Q^\epsilon(f, g) = Q^\epsilon(P_{\mathcal{H}^{\epsilon_0},\mathcal{H}^\epsilon}(f + h), P_{\mathcal{H}^{\epsilon_0},\mathcal{H}^\epsilon}(g + i)).$$

Similarly we write $\hat{Q}$ and $\hat{Q}_\epsilon$ for the quadratic forms with domains
$$\mathrm{Dom}(\hat{Q}) = \mathrm{Dom}(Q) \oplus \mathcal{H}^\perp \mathcal{H}^{\epsilon_0},$$
$$\mathrm{Dom}(\hat{Q}_\epsilon) = \mathrm{Dom}(Q) \oplus (\mathcal{H}_\epsilon)^\perp \mathcal{H}^{\epsilon_0},$$
respectively, and, for all $f, g \in \mathcal{H}^{\epsilon_0}$, we define $\hat{Q}(f, g)$ and $\hat{Q}_\epsilon(f, g)$ by
$$\hat{Q}(f, g) = Q(P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\, f, P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\, g),$$
$$\hat{Q}_\epsilon(f, g) = Q_\epsilon(P_{\mathcal{H}^{\epsilon_0},\mathcal{H}_\epsilon} f, P_{\mathcal{H}^{\epsilon_0},\mathcal{H}_\epsilon} g),$$
respectively. We assume that these quadratic forms satisfy the following assumptions:

(A15) If $0 < \epsilon_2 \le \epsilon_1 \le \epsilon_0$, then, for all $f \in \mathcal{H}^{\epsilon_1}$, we have
(i) $Q^{\epsilon_1}(f, f) \ge Q^{\epsilon_2}(P_{\mathcal{H}^{\epsilon_1},\mathcal{H}^{\epsilon_2}} f, P_{\mathcal{H}^{\epsilon_1},\mathcal{H}^{\epsilon_2}} f)$,
(ii) $Q^{\epsilon_1}(f, f) \ge Q(P_{\mathcal{H}^{\epsilon_1},\mathcal{H}}\, f, P_{\mathcal{H}^{\epsilon_1},\mathcal{H}}\, f)$.

(A16) For all $\epsilon_1, \epsilon_2 \in (0, \epsilon_0]$ and all $f \in \mathcal{H}^{\epsilon_1}$, we have
$$Q^{\epsilon_1}(f, f) \ge Q_{\epsilon_2}(P_{\mathcal{H}^{\epsilon_1},\mathcal{H}_{\epsilon_2}} f, P_{\mathcal{H}^{\epsilon_1},\mathcal{H}_{\epsilon_2}} f).$$

(A17) If $0 < \epsilon_2 \le \epsilon_1 \le \epsilon_0$, then, for all $f \in \mathcal{H}_{\epsilon_2}$, we have
$$Q_{\epsilon_2}(f, f) \ge Q_{\epsilon_1}(P_{\mathcal{H}_{\epsilon_2},\mathcal{H}_{\epsilon_1}} f, P_{\mathcal{H}_{\epsilon_2},\mathcal{H}_{\epsilon_1}} f).$$

(A18) For all $0 < \epsilon \le \epsilon_0$ and all $f \in \mathcal{H}$, we have
$$Q(f, f) \ge Q_\epsilon(P_{\mathcal{H},\mathcal{H}_\epsilon} f, P_{\mathcal{H},\mathcal{H}_\epsilon} f).$$

(A19) For all $f \in \mathcal{H}$ we have
$$Q(f, f) = \lim_{\epsilon \downarrow 0} Q_\epsilon(P_{\mathcal{H},\mathcal{H}_\epsilon} f, P_{\mathcal{H},\mathcal{H}_\epsilon} f).$$

(A20) For all $f \in \mathrm{Dom}(Q^{\epsilon_0})$ we have
$$Q(P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\, f, P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\, f) = \lim_{\epsilon \downarrow 0} Q^\epsilon(P_{\mathcal{H}^{\epsilon_0},\mathcal{H}^\epsilon} f, P_{\mathcal{H}^{\epsilon_0},\mathcal{H}^\epsilon} f).$$

Definition 2.3. For $0 < \epsilon \le \epsilon_0$ let $H_\epsilon \ge 0$ be the self-adjoint operator associated to Q_ϵ with domain $D(H_\epsilon) \subseteq \mathcal{H}_\epsilon$. Similarly, let $H^\epsilon \ge 0$ and $H \ge 0$ be the self-adjoint operators associated to Q^ϵ and Q, respectively, with domains $D(H^\epsilon) \subseteq \mathcal{H}^\epsilon$ and $D(H) \subseteq \mathcal{H}$.

Assumptions (A11)–(A18) imply that we have an increasing family of non-negative quadratic forms:

$$\cdots \le \hat{Q}_{\epsilon_1} \le \cdots \le \hat{Q}_{\epsilon_2} \le \cdots \le \hat{Q} \le \cdots \le \hat{Q}^{\epsilon_3} \le \cdots \le \hat{Q}^{\epsilon_4} \le \dots \tag{2.1}$$

where

$$0 < \epsilon_2 \le \epsilon_1 \le \epsilon_0 \quad \text{and} \quad 0 < \epsilon_3 \le \epsilon_4 \le \epsilon_0. \tag{2.2}$$

So by [3, Theorem 4.17] we have, for all $\lambda > 0$,

$$\begin{aligned}
&\dots \ge (\lambda + H_{\epsilon_1})^{-1} \oplus \lambda^{-1} \ge \cdots \ge (\lambda + H_{\epsilon_2})^{-1} \oplus \lambda^{-1} \ge \dots \\
&\dots \ge (\lambda + H)^{-1} \oplus \lambda^{-1} \ge \cdots \ge (\lambda + H^{\epsilon_3})^{-1} \oplus \lambda^{-1} \ge \dots \\
&\dots \ge (\lambda + H^{\epsilon_4})^{-1} \oplus \lambda^{-1} \ge \dots
\end{aligned}$$

if $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ satisfy (2.2), where $(\lambda + H_{\epsilon_1})^{-1} \oplus \lambda^{-1}$ is the operator defined on $\mathcal{H}^{\epsilon_0} = \mathcal{H}_{\epsilon_1} \oplus (\mathcal{H}_{\epsilon_1})^{\perp}\mathcal{H}^{\epsilon_0}$ by

$$[(\lambda + H_{\epsilon_1})^{-1} \oplus \lambda^{-1}](f + g) = (\lambda + H_{\epsilon_1})^{-1} f + \lambda^{-1} g$$

for all $f \in \mathcal{H}_{\epsilon_1}$ and all $g \in (\mathcal{H}_{\epsilon_1})^{\perp}\mathcal{H}^{\epsilon_0}$. Similarly the operators $(\lambda + H)^{-1} \oplus \lambda^{-1}$ and $(\lambda + H^{\epsilon})^{-1} \oplus \lambda^{-1}$ are defined on $\mathcal{H}^{\epsilon_0} = \mathcal{H} \oplus \mathcal{H}^{\perp}\mathcal{H}^{\epsilon_0}$ and $\mathcal{H}^{\epsilon_0} = \mathcal{H}^{\epsilon} \oplus (\mathcal{H}^{\epsilon})^{\perp}\mathcal{H}^{\epsilon_0}$, respectively.

For $0 < \epsilon \le \epsilon_0$ let $Q(\epsilon)$ be a closed non-negative quadratic form with domain $\mathrm{Dom}(Q(\epsilon)) \subseteq \mathcal{A}(\epsilon)$ satisfying the following assumptions:

(A21) $\mathrm{Dom}(Q(\epsilon))$ is dense in $\mathcal{A}(\epsilon)$.

(A22) For $0 < \epsilon \le \epsilon_0$ we have
- (i) $P_{\mathcal{H}^{\epsilon}, \mathcal{A}(\epsilon)}(\mathrm{Dom}(Q^{\epsilon})) = \mathrm{Dom}(Q(\epsilon))$,
- (ii) $P_{\mathcal{A}(\epsilon), \mathcal{H}_{\epsilon}}(\mathrm{Dom}(Q(\epsilon))) = \mathrm{Dom}(Q_{\epsilon})$.

(A23) If $0 < \epsilon \le \epsilon_0$, then, for all $f \in \mathcal{H}^{\epsilon}$, we have

$$Q^{\epsilon}(f, f) \ge Q(\epsilon)(P_{\mathcal{H}^{\epsilon}, \mathcal{A}(\epsilon)} f, P_{\mathcal{H}^{\epsilon}, \mathcal{A}(\epsilon)} f),$$

and, for all $g \in \mathcal{A}(\epsilon)$, we have

$$Q(\epsilon)(g, g) \ge Q_{\epsilon}(P_{\mathcal{A}(\epsilon), \mathcal{H}_{\epsilon}} g, P_{\mathcal{A}(\epsilon), \mathcal{H}_{\epsilon}} g).$$

Definition 2.4. For $0 < \epsilon \le \epsilon_0$ we define the quadratic form $\hat{Q}(\epsilon)$, with domain

$$\mathrm{Dom}(\hat{Q}(\epsilon)) = \mathrm{Dom}(Q(\epsilon)) \oplus \mathcal{A}(\epsilon)^{\perp}\mathcal{H}^{\epsilon_0} \subseteq \mathcal{H}^{\epsilon_0}$$

by

$$\hat{Q}(\epsilon)(f, g) = Q(\epsilon)(P_{\mathcal{H}^{\epsilon_0}, \mathcal{A}(\epsilon)} f, P_{\mathcal{H}^{\epsilon_0}, \mathcal{A}(\epsilon)} g)$$

for all $f, g \in \mathcal{H}^{\epsilon_0}$. We let $H(\epsilon) \ge 0$ be the self-adjoint operator associated to $Q(\epsilon)$ with domain $D(H(\epsilon)) \subseteq \mathcal{A}(\epsilon)$.

Assumption (A23) implies that if $0 < \epsilon \le \epsilon_0$, then

$$\hat{Q}_{\epsilon} \le \hat{Q}(\epsilon) \le \hat{Q}^{\epsilon} \tag{2.3}$$

and hence, by [3, Theorem 4.17],

$$(\lambda + H_{\epsilon})^{-1} \oplus \lambda^{-1} \ge (\lambda + H(\epsilon))^{-1} \oplus \lambda^{-1} \ge (\lambda + H^{\epsilon})^{-1} \oplus \lambda^{-1} \tag{2.4}$$

for all $\lambda > 0$, where $(\lambda + H(\epsilon))^{-1} \oplus \lambda^{-1}$ is the operator defined on $\mathcal{H}^{\epsilon_0} = \mathcal{A}(\epsilon) \oplus \mathcal{A}(\epsilon)^{\perp}\mathcal{H}^{\epsilon_0}$ by

$$((\lambda + H(\epsilon))^{-1} \oplus \lambda^{-1})(f + g) = (\lambda + H(\epsilon))^{-1} f + \lambda^{-1} g$$

for all $f \in \mathcal{A}(\epsilon)$ and $g \in \mathcal{A}(\epsilon)^{\perp}\mathcal{H}^{\epsilon_0}$.

Proposition 2.5 ([3, Theorem 4.32]). *Let $K_n \geq 0$ be an increasing sequence of non-negative self-adjoint operators with domains in a Hilbert space $\mathcal{U}$. Put*
$$\mathcal{E} = \cap_n D(K_n^{1/2})$$
and let $\hat{\mathcal{U}}$ be the closure of $\mathcal{E}$. Then there exists a self-adjoint operator $K \geq 0$ with domain $D(K) \subseteq \hat{\mathcal{U}}$ such that its associated quadratic form domain equal $\mathcal{E}$ and that
$$\langle K^{1/2} f, K^{1/2} f\rangle = \lim_{n\to\infty} \langle K_n^{1/2} f, K_n^{1/2} f\rangle \quad (f \in \mathcal{E}).$$
Moreover
$$\lim_{n\to\infty} \{ \sup_{0\leq t\leq a} \|e^{-K_n t} f - e^{-Kt} f\|\} = 0$$
for all $a \geq 0$ and $f \in \hat{\mathcal{U}}$. Hence for all $\lambda > 0$ we have
$$\|(\lambda + K_n)^{-1} f - (\lambda + K)^{-1} f\| \to 0 \quad \text{as } n \to \infty$$
for all $f \in \hat{\mathcal{U}}$.

Definition 2.6. For $0 < \epsilon \leq \epsilon_0$ we let $\hat{H}_\epsilon$ and $\hat{H}^\epsilon$ be the operators with domains $D(\hat{H}_\epsilon) = D(H_\epsilon) \oplus (\mathcal{H}_\epsilon)^\perp \mathcal{H}^{\epsilon_0}$ and $D(\hat{H}^\epsilon) = D(H^\epsilon) \oplus (\mathcal{H}^\epsilon)^\perp \mathcal{H}^{\epsilon_0}$, respectively, defined by
$$\hat{H}_\epsilon(f+g) = H_\epsilon f = H_\epsilon P_{\mathcal{H}^{\epsilon_0},\mathcal{H}_\epsilon}(f+g)$$
for all $f + g \in D(H_\epsilon) \oplus (\mathcal{H}_\epsilon)^\perp \mathcal{H}^{\epsilon_0}$, and
$$\hat{H}^\epsilon(f+g) = H^\epsilon f = H^\epsilon P_{\mathcal{H}^{\epsilon_0},\mathcal{H}^\epsilon}(f+g)$$
for all $f + g \in D(H^\epsilon) \oplus (\mathcal{H}^\epsilon)^\perp \mathcal{H}^{\epsilon_0}$. Similarly we write $\hat{H}$ to denote the operator with domain $D(\hat{H}) = D(H) \oplus \mathcal{H}^\perp \mathcal{H}^{\epsilon_0}$ defined by
$$\hat{H}(f+g) = \hat{H} f = \hat{H} P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}(f+g)$$
for all $f + g \in D(H) \oplus \mathcal{H}^\perp \mathcal{H}^{\epsilon_0}$.

We also write $\hat{\hat{H}}_\epsilon$ for the operator with domain $D(\hat{\hat{H}}_\epsilon) = D(H_\epsilon) \oplus (\mathcal{H}_\epsilon)^\perp \mathcal{H}$ defined by
$$\hat{\hat{H}}_\epsilon(f+g) = H_\epsilon f = H_\epsilon P_{\mathcal{H},\mathcal{H}_\epsilon}(f+g)$$
for all $f + g \in D(H_\epsilon) \oplus (\mathcal{H}_\epsilon)^\perp \mathcal{H}$.

Lemma 2.7. *We have*

(i) $\lim_{\epsilon\downarrow 0}\{\sup_{0\leq t\leq a} \|e^{-\hat{\hat{H}}_\epsilon t} f - e^{-Ht} f\|_{\mathcal{H}}\} = 0$ *for all $f \in \mathcal{H}$ and $a \geq 0$. Also*
$$\lim_{\epsilon\downarrow 0} \|(\lambda + \hat{\hat{H}}_\epsilon)^{-1} f - (\lambda + H)^{-1} f\|_{\mathcal{H}} = 0$$
for all $f \in \mathcal{H}$ and $a \geq 0$.

(ii) $\lim_{\epsilon\downarrow 0}\{\sup_{0\leq t\leq a} \|e^{-\hat{H}_\epsilon t} f - e^{-\hat{H} t} f\|_{\mathcal{H}^{\epsilon_0}}\} = 0$ *for all $f \in \mathcal{H}^{\epsilon_0}$ and $a \geq 0$. Also*
$$\lim_{\epsilon\downarrow 0} \|(\lambda + \hat{H}_\epsilon)^{-1} f - (\lambda + \hat{H})^{-1} f\|_{\mathcal{H}^{\epsilon_0}} = 0$$
for all $f \in \mathcal{H}^{\epsilon_0}$ and $\lambda > 0$.

Proof. To prove (i) we apply Proposition 2.5 with $\mathcal{U} = \mathcal{H}$, $K_n = \hat{\hat{\mathcal{H}}}_\varepsilon$ and then use Assumptions (A17), (A18) and (A19). Similarly, to prove (ii) we apply Proposition 2.5 with $\mathcal{U} = \mathcal{H}^{\epsilon_0}$ and $K_n = \hat{\mathcal{H}}_\varepsilon$, and then use Assumptions (A17), (A18) and (A19). □

Definition 2.8. Let $\mathcal{U}$ be a Hilbert space and let $Q \geq 0$ be a closed quadratic form with domain $\mathrm{Dom}(Q) \subseteq \mathcal{U}$. (Note that $\mathrm{Dom}(Q)$ is not necessarily dense in $\mathcal{U}$.) Let $H \geq 0$ be the self-adjoint operator associated to Q with domain $D(H) \subseteq \overline{\mathrm{Dom}(Q)}$. If $\phi : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function, then we define the bounded operator $\phi(H)$ on $\mathcal{U} = \overline{\mathrm{Dom}(Q)} \oplus ((\overline{\mathrm{Dom}(Q)})^{\perp}\mathcal{U})$ by

$$\phi(Q)(f+g) = \phi(H)f = \phi(H)(P_{\mathcal{U},\overline{\mathrm{Dom}(Q)}}(f+g)) \tag{2.5}$$

for all $f \in \overline{\mathrm{Dom}(Q)}$ and $g \in (\overline{\mathrm{Dom}(Q)})^{\perp}\mathcal{U}$.

Similarly, on $\mathcal{U} = \overline{\mathrm{Dom}(Q)} \oplus ((\overline{\mathrm{Dom}(Q)})^{\perp}\mathcal{U})$, we define the bounded operator $[\phi(Q)]_M$ by

$$\begin{aligned}[\phi(Q)]_M(f+g) &= \phi(H)f + g \\ &= \phi(H)P_{\mathcal{U},\overline{\mathrm{Dom}(Q)}}(f+g) + P_{\mathcal{U},(\overline{\mathrm{Dom}(Q)})^{\perp}\mathcal{U}}(f+g)\end{aligned} \tag{2.6}$$

for all $f \in \overline{\mathrm{Dom}(Q)}$ and $g \in (\overline{\mathrm{Dom}(Q)})^{\perp}\mathcal{U}$.

In both (2.5) and (2.6), $\phi(H)$ is the bounded operator on $\overline{\mathrm{Dom}(Q)}$ defined using the spectral theorem.

Definition 2.9. Let $\mathcal{U}$ be a Hilbert space and for $n = 1, 2, 3, \dots$ let $Q_n \geq 0$ be a closed quadratic form with domain $\mathrm{Dom}(Q_n) \subseteq \mathcal{U}$. ($\mathrm{Dom}(Q_n)$ is not necessarily dense in $\mathcal{U}$.) Let $Q \geq 0$ be a closed quadratic form with domain in $\mathcal{U}$. ($\mathrm{Dom}(Q)$ is not necessarily dense in $\mathcal{U}$.) We say that Q_n converges to Q in the strong resolvent sense (srs) if for some $\lambda > 0$ we have

$$\lim_{n\to\infty} (\lambda + Q_n)^{-1} f = (\lambda + Q)^{-1} f \quad (f \in \mathcal{U}).$$

Lemma 2.10. *Let $\mathcal{U}$, Q_n and Q be as in Definitions 2.8 and 2.9. Let P_n be the orthogonal projection of $\mathcal{U}$ onto $\overline{\mathrm{Dom}(Q_n)}$. Suppose that for all $f \in \mathcal{U}$ we have*

$$\|P_n f - f\| \to 0 \quad \text{as } n \to \infty.$$

Suppose also that $\overline{\mathrm{Dom}(Q)} = \mathcal{U}$. Then $Q_n \xrightarrow{\text{srs}} Q$ as $n \to \infty$ is equivalent to

$$[(\lambda + Q_n)^{-1}]_M f \to [(\lambda + Q)^{-1}]_M f \quad \text{as } n \to \infty$$

for some $\lambda > 0$ and for all $f \in \mathcal{U}$.

The proof of this lemma is obvious.

Proposition 2.11 ([4, Theorem 1.2.3])**.** *Let $K_n \geq 0$, $n = 1, 2, 3, \dots,$ and $K \geq 0$ be self-adjoint operators with domains in a Hilbert space $\mathcal{U}$. Suppose that*

$$K_1 \geq K_2 \geq \dots \geq K_n \geq K_{n+1} \geq \dots \geq K$$

and that their associated quadratic forms satisfy

$$\langle K^{1/2} f, K^{1/2} f\rangle = \lim_{n\to\infty} \langle K_n^{1/2} f, K_n^{1/2} f\rangle$$

for all f in a form core of K. Then K_n converges to K in the strong resolvent sense.

Definition 2.12. We let $\mathcal{C}$ be the subspace of $\mathrm{Dom}(\hat{Q})$ defined by

$$\mathcal{C} = \cup_{0<\epsilon\leq\epsilon_0} \mathrm{Dom}(Q^{\epsilon}) \oplus (\mathcal{H}^{\epsilon})^{\perp}\mathcal{H}^{\epsilon_0}.$$

(Note that $\mathcal{C}$ is a subspace of $\mathrm{Dom}(\hat{Q})$ by Assumption (A15).)

Lemma 2.13. *$\mathcal{C}$ is a form core of $\hat{Q}$.*

Proof. We first recall that, by Assumption (A12),

$$\mathrm{Dom}(Q) = P_{\mathcal{H}^\epsilon,\mathcal{H}}(\mathrm{Dom}(Q^\epsilon)) \quad (0 < \epsilon \le \epsilon_0).$$

Let $f = g + h \in \mathrm{Dom}(\hat{Q})$, where $g \in \mathrm{Dom}(Q)$ and $h \in \mathcal{H}^\perp\mathcal{H}^{\epsilon_0}$. Let $\alpha \in \mathrm{Dom}(Q^{\epsilon_0})$ such that

$$P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\alpha = g.$$

For $0 < \epsilon \le \epsilon_0$ let

$$\begin{aligned} g_\epsilon &= P_{\mathcal{H}^{\epsilon_0},\mathcal{H}^\epsilon}\alpha = P_{\mathcal{H}^\epsilon,\mathcal{H}}\, g_\epsilon + P_{\mathcal{H}^\epsilon,\mathcal{H}^\perp\mathcal{H}^\epsilon} g_\epsilon \\ &= P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\alpha + P_{\mathcal{H}^\epsilon,\mathcal{H}^\perp\mathcal{H}^\epsilon} g_\epsilon \\ &= g + P_{\mathcal{H}^\epsilon,\mathcal{H}^\perp\mathcal{H}^\epsilon} g_\epsilon \end{aligned}$$

and let

$$h_\epsilon = P_{\mathcal{H}^\perp\mathcal{H}^{\epsilon_0},(\mathcal{H}^\epsilon)^\perp\mathcal{H}^{\epsilon_0}} h$$

and let

$$\begin{aligned} f_\epsilon &= g_\epsilon + h_\epsilon \\ &= P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\alpha + P_{\mathcal{H}^\epsilon,\mathcal{H}^\perp\mathcal{H}^\epsilon} g_\epsilon + h_\epsilon \\ &= g + P_{\mathcal{H}^\epsilon,\mathcal{H}^\perp\mathcal{H}^\epsilon} g_\epsilon + h_\epsilon. \end{aligned} \tag{2.7}$$

Then, by (A12), $f_\epsilon \in \mathrm{Dom}(Q^\epsilon) \oplus (\mathcal{H}^\epsilon)^\perp\mathcal{H}^{\epsilon_0}$. Since

$$\begin{aligned} h &= P_{\mathcal{H}^\perp\mathcal{H}^{\epsilon_0},(\mathcal{H}^\epsilon)^\perp\mathcal{H}^{\epsilon_0}} h + (I_{\mathcal{H}^\perp\mathcal{H}^{\epsilon_0}} - P_{\mathcal{H}^\perp\mathcal{H}^{\epsilon_0},\,(\mathcal{H}^\epsilon)^\perp\mathcal{H}^{\epsilon_0}})h \\ &= h_\epsilon + (I_{\mathcal{H}^\perp\mathcal{H}^{\epsilon_0}} - P_{\mathcal{H}^\perp\mathcal{H}^{\epsilon_0},(\mathcal{H}^\epsilon)^\perp\mathcal{H}^{\epsilon_0}})h, \end{aligned}$$

we have

$$\begin{aligned} f - f_\epsilon &= g + h - (g + P_{\mathcal{H}^\epsilon,\mathcal{H}^\perp\mathcal{H}^\epsilon} g_\epsilon + h_\epsilon) \\ &= g + h_\epsilon + (I_{\mathcal{H}^\perp\mathcal{H}^{\epsilon_0}} - P_{\mathcal{H}^\perp\mathcal{H}^{\epsilon_0},(\mathcal{H}^\epsilon)^\perp\mathcal{H}^{\epsilon_0}})h - (g + P_{\mathcal{H}^\epsilon,\mathcal{H}^\perp\mathcal{H}^\epsilon} g_\epsilon + h_\epsilon) \\ &= (I_{\mathcal{H}^\perp\mathcal{H}^{\epsilon_0}} - P_{\mathcal{H}^\perp\mathcal{H}^{\epsilon_0},(\mathcal{H}^\epsilon)^\perp\mathcal{H}^{\epsilon_0}})h - P_{\mathcal{H}^\epsilon,\mathcal{H}^\perp\mathcal{H}^\epsilon} g_\epsilon \\ &= (I_{\mathcal{H}^\perp\mathcal{H}^{\epsilon_0}} - P_{\mathcal{H}^\perp\mathcal{H}^{\epsilon_0},(\mathcal{H}^\epsilon)^\perp\mathcal{H}^{\epsilon_0}})h - P_{\mathcal{H}^{\epsilon_0},\mathcal{H}^\perp\mathcal{H}^\epsilon}\alpha \\ &= (I_{\mathcal{H}^\epsilon} - P_{\mathcal{H}^\epsilon,\mathcal{H}})P_{\mathcal{H}^{\epsilon_0},\mathcal{H}^\epsilon} h - (I_{\mathcal{H}^\epsilon} - P_{\mathcal{H}^\epsilon,\mathcal{H}})P_{\mathcal{H}^{\epsilon_0},\mathcal{H}^\epsilon}\alpha \\ &\to 0 \quad \text{as } \epsilon \downarrow 0 \quad (\text{by (A5)}). \end{aligned} \tag{2.8}$$

Also, by (2.8),

$$\begin{aligned} \hat{Q}(f - f_\epsilon, f - f_\epsilon) &= \hat{Q}(P_{\mathcal{H}^\epsilon,\,\mathcal{H}^\perp\mathcal{H}^\epsilon} P_{\mathcal{H}^{\epsilon_0},\mathcal{H}^\epsilon}(h - \alpha), P_{\mathcal{H}^\epsilon,\mathcal{H}^\perp\mathcal{H}^\epsilon} P_{\mathcal{H}^{\epsilon_0},\mathcal{H}^\epsilon}(h - \alpha)) \\ &= 0 \end{aligned}$$

since $P_{\mathcal{H}^\epsilon,\mathcal{H}^\perp\mathcal{H}^\epsilon} P_{\mathcal{H}^{\epsilon_0},\mathcal{H}^\epsilon}(h - \alpha) \in \mathcal{H}^\perp\mathcal{H}^{\epsilon_0}$. Also, by Assumption (A12), it is not difficult to show that $\mathcal{C}$ is closed under addition and scalar multiplication. Hence $\mathcal{C}$ is a form core of $\hat{Q}$. □

Theorem 2.14. *We have $\hat{H}^\epsilon \xrightarrow{\text{srs}} \hat{H}$ as $\epsilon \downarrow 0$.*

Proof. Let $\delta \in (0, \epsilon_0]$ and let $f \in \mathrm{Dom}(Q^\delta) \oplus (\mathcal{H}^\delta)^\perp\mathcal{H}^{\epsilon_0}$. Then, for $0 < \epsilon < \delta$, we have, by (A5),

$$P_{\mathcal{H}^{\epsilon_0},\mathcal{H}^\epsilon} f = P_{\mathcal{H}^{\epsilon_0},\mathcal{H}^\epsilon} f - P_{\mathcal{H}^{\epsilon_0},\mathcal{H}} f + P_{\mathcal{H}^{\epsilon_0},\mathcal{H}} f \to P_{\mathcal{H}^{\epsilon_0},\mathcal{H}} f \quad \text{as } \epsilon \downarrow 0.$$

Hence, for $0 < \epsilon < \delta$, we have, by (A20),

$$\begin{aligned} \hat{Q}^\epsilon(f, f) &= Q^\epsilon(P_{\mathcal{H}^{\epsilon_0},\mathcal{H}^\epsilon} f, P_{\mathcal{H}^{\epsilon_0},\mathcal{H}^\epsilon} f) \\ &\to Q(P_{\mathcal{H}^{\epsilon_0},\mathcal{H}} f, P_{\mathcal{H}^{\epsilon_0},\mathcal{H}} f) = \hat{Q}(f, f) \quad \text{as } \epsilon \downarrow 0. \end{aligned}$$

Thus for all $f \in \mathcal{C}$ we have

$$\hat{Q}(f,f) = \lim_{\epsilon \downarrow 0} \hat{Q}^{\epsilon}(f,f). \tag{2.9}$$

The theorem now follows from Proposition 2.11 together with (2.1), (2.9) and Lemma 2.13. □

Definition 2.15. For $0 < \epsilon \le \epsilon_0$ we let $\hat{H}(\epsilon)$ be the operator with domain $D(\hat{H}(\epsilon)) = D(H(\epsilon)) \oplus \mathcal{A}(\epsilon)^{\perp}\mathcal{H}^{\epsilon_0}$ defined by

$$\hat{H}(\epsilon)(f+g) = H(\epsilon)f = H(\epsilon)P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}(f+g)$$

for all $f \in D(H(\epsilon))$ and $g \in \mathcal{A}(\epsilon)^{\perp}\mathcal{H}^{\epsilon_0}$. Thus $\hat{H}(\epsilon) \ge 0$ is the self-adjoint operator associated to the quadratic form $\hat{Q}(\epsilon)$ defined in Definition 2.4. By (2.4) we have

$$(\lambda + \hat{H}_{\epsilon})^{-1} \ge (\lambda + \hat{H}(\epsilon))^{-1} \ge (\lambda + \hat{H}^{\epsilon})^{-1}$$

for all $\lambda > 0$ and $0 < \epsilon \le \epsilon_0$; i.e.,

$$\langle (\lambda + \hat{H}_{\epsilon})^{-1} f, f\rangle \ge \langle (\lambda + \hat{H}(\epsilon))^{-1} f, f\rangle \ge \langle (\lambda + \hat{H}^{\epsilon})^{-1} f, f\rangle \tag{2.10}$$

for all $f \in \mathcal{H}^{\epsilon_0}$, $\lambda > 0$ and $0 < \epsilon \le \epsilon_0$.

Lemma 2.16. *We have*

$$\langle (\lambda + \hat{H})^{-1} f, f\rangle = \lim_{\epsilon \downarrow 0} \langle (\lambda + \hat{H}(\epsilon))^{-1} f, f\rangle$$

for all $f \in \mathcal{H}^{\epsilon_0}$ and $\lambda > 0$.

Proof. This lemma follows from the second inequality of Lemma 2.7(ii), Theorem 2.14 and (2.10). □

Theorem 2.17. *For all $\lambda > 0$ we have*

$$(\lambda + \hat{H})^{-1} f = \lim_{\epsilon \downarrow 0} (\lambda + \hat{H}(\epsilon))^{-1} f$$

for all $f \in \mathcal{H}^{\epsilon_0}$. Hence for all $a > 0$ and $f \in \mathcal{H}^{\epsilon_0}$ we have

$$\lim_{\epsilon \downarrow 0} \{ \sup_{0 \le t \le a} \| e^{-\hat{H}(\epsilon)t} f - e^{-\hat{H}t} f \|_{\mathcal{H}^{\epsilon_0}} \} = 0.$$

Proof. By Lemma 2.7(ii), we have, for all $\lambda > 0$,

$$(\lambda + \hat{H})^{-1} f = \lim_{\epsilon \downarrow 0} (\lambda + \hat{H}_{\epsilon})^{-1} f \quad (f \in \mathcal{H}^{\epsilon_0}).$$

This is equivalent to having

$$\lim_{\epsilon \downarrow 0} \{ \sup_{0 \le t \le a} \| e^{-\hat{H}_{\epsilon}t} f - e^{-\hat{H}t} f \|_{\mathcal{H}^{\epsilon_0}} \} = 0 \quad (f \in \mathcal{H}^{\epsilon_0}) \tag{2.11}$$

for all $a > 0$ (see, for example [3, Theorem 3.17]). Similarly, Theorem 2.14 is equivalent to

$$\lim_{\epsilon \downarrow 0} \{ \sup_{0 \le t \le a} \| e^{-\hat{H}^{\epsilon}t} f - e^{-\hat{H}t} f \|_{\mathcal{H}^{\epsilon_0}} \} = 0 \quad (f \in \mathcal{H}^{\epsilon_0}) \tag{2.12}$$

for all $a > 0$. Since, for $\lambda > 0$, we have

$$(\lambda + \hat{H}_{\epsilon})^{-1/2} f = \int_0^{\infty} \frac{1}{\sqrt{\pi t}} e^{-\lambda t} e^{-\hat{H}_{\epsilon}t} f \, dt \quad (f \in \mathcal{H}^{\epsilon_0}),$$

$$(\lambda + \hat{H}^{\epsilon})^{-1/2} f = \int_0^{\infty} \frac{1}{\sqrt{\pi t}} e^{-\lambda t} e^{-\hat{H}^{\epsilon}t} f \, dt \quad (f \in \mathcal{H}^{\epsilon_0}),$$

we have, from (2.11) and (2.12),

$$\lim_{\epsilon\downarrow 0}(\lambda+\hat{H}_\epsilon)^{-1/2}f=\lim_{\epsilon\downarrow 0}(\lambda+\hat{H}^\epsilon)^{-1/2}f=(\lambda+\hat{H})^{-1/2}f \tag{2.13}$$

for all $\lambda>0$ and $f\in\mathcal{H}^{\epsilon_0}$. Since, for all $\lambda>0$,

$$(\lambda+\hat{H}_\epsilon)^{-1}\geq(\lambda+\hat{H}(\epsilon))^{-1}\geq(\lambda+\hat{H}^\epsilon)^{-1},$$

we have

$$(\lambda+\hat{H}_\epsilon)^{-1/2}\geq(\lambda+\hat{H}(\epsilon))^{-1/2}\geq(\lambda+\hat{H}^\epsilon)^{-1/2}$$

for all $\lambda>0$ (see, for example, [3, Lemma 4.19]); i.e., for all $0<\epsilon\leq\epsilon_0$, $\lambda>0$ and $f\in\mathcal{H}^{\epsilon_0}$, we have

$$\langle(\lambda+\hat{H}_\epsilon)^{-1/2}f,f\rangle\geq\langle(\lambda+\hat{H}(\epsilon))^{-1/2}f,f\rangle\geq\langle(\lambda+\hat{H}^\epsilon)^{-1/2}f,f\rangle. \tag{2.14}$$

Hence, from (2.13) and (2.14), we have

$$\langle(\lambda+\hat{H})^{-1/2}f,f\rangle=\lim_{\epsilon\downarrow 0}\langle(\lambda+\hat{H}(\epsilon))^{-1/2}f,f\rangle \tag{2.15}$$

for all $\lambda>0$ and $f\in\mathcal{H}^{\epsilon_0}$. The polarization identity (see, for example, [3, p.103]) and (2.15) imply that

$$\langle(\lambda+\hat{H})^{-1/2}f,g\rangle=\lim_{\epsilon\downarrow 0}\langle(\lambda+H(\epsilon))^{-1/2}f,g\rangle \tag{2.16}$$

for all $\lambda>0$ and $f,g\in\mathcal{H}^{\epsilon_0}$. We now need the following result.

Proposition 2.18 (See [3, Problem 4.11]). *Let $\mathcal{U}$ be a Hilbert space and let $f,f_n\in\mathcal{U}$ for $n=1,2,3,\dots$. Suppose that*

$$\langle f,g\rangle=\lim_{n\to\infty}\langle f_n,g\rangle\quad(g\in\mathcal{U}).$$

Then

$$\lim_{n\to\infty}\|f_n-f\|=0\quad\text{if and only if}\quad\lim_{n\to\infty}\|f_n\|=\|f\|.$$

By Lemma 2.16, we have

$$\lim_{\epsilon\downarrow 0}\|(\lambda+\hat{H}(\epsilon))^{-1/2}f\|_{\mathcal{H}^{\epsilon_0}}=\|(\lambda+\hat{H})^{-1/2}f\|_{\mathcal{H}^{\epsilon_0}} \tag{2.17}$$

for all $\lambda>0$ and $f\in\mathcal{H}^{\epsilon_0}$. Proposition 2.18 together with (2.16) and (2.17) imply that

$$(\lambda+\hat{H})^{-1/2}f=\lim_{\epsilon\downarrow 0}(\lambda+\hat{H}(\epsilon))^{-1/2}f \tag{2.18}$$

for all $\lambda>0$ and $f\in\mathcal{H}^{\epsilon_0}$. Hence

$$\begin{aligned}&(\lambda+\hat{H}(\epsilon))^{-1}f-(\lambda+\hat{H})^{-1}f\\&=(\lambda+\hat{H}(\epsilon))^{-1/2}[(\lambda+\hat{H}(\epsilon))^{-1/2}f-(\lambda+\hat{H})^{-1/2}f]\\&\quad+(\lambda+\hat{H}(\epsilon))^{-1/2}(\lambda+\hat{H})^{-1/2}f-(\lambda+\hat{H})^{-1/2}(\lambda+\hat{H})^{-1/2}f\to 0\quad\text{as }\epsilon\downarrow 0\end{aligned}$$

for all $\lambda>0$ and $f\in\mathcal{H}^{\epsilon_0}$. The strong convergence of $e^{-\hat{H}(\epsilon)t}$ to $e^{-\hat{H}t}$ now follows from [3, Theorem 3.17]. □

We next impose more assumptions on the operators H, H^ϵ, H_ϵ and $H(\epsilon)$, $0<\epsilon\leq\epsilon_0$:

- (A24) H, H^ϵ, H_ϵ, $H(\epsilon)$, $0<\epsilon\leq\epsilon_0$, have compact resolvents in the Hilbert spaces $\mathcal{H}$, $\mathcal{H}^\epsilon$, $\mathcal{H}_\epsilon$ and $\mathcal{A}(\epsilon)$, respectively.
- (A25) $0\in Sp(H)$, $0\in Sp(H^\epsilon)$, $0\in Sp(H_\epsilon)$ and $0\in Sp(H(\epsilon))$, $0<\epsilon\leq\epsilon_0$.

Definition 2.19. We shall write $\{\mu_i\}_{i=1}^\infty$ for the eigenvalues of H, where $\{\mu_i\}_{i=1}^\infty$ is a non-decreasing sequence and the eigenvalues are listed repeatedly according to multiplicity. Similarly, for $0<\epsilon\le\epsilon_0$, we shall write $\{\mu_i^\epsilon\}_{i=1}^\infty$, $\{\mu_{i,\epsilon}\}_{i=1}^\infty$, and $\{\mu_i(\epsilon)\}_{i=1}^\infty$ for the eigenvalues of H^ϵ, H_ϵ and $H(\epsilon)$, respectively. Thus, by (A25), we have

$$0=\mu_1=\mu_1^\epsilon=\mu_{1,\epsilon}=\mu_1(\epsilon)\quad(0<\epsilon\le\epsilon_0).$$

We shall also write $\{\varphi_i\}_{i=1}^\infty$, $\{\varphi_i^\epsilon\}_{i=1}^\infty$, $\{\varphi_{i,\epsilon}\}_{i=1}^\infty$ and $\{\varphi_i(\epsilon)\}_{i=1}^\infty$ for the corresponding normalized eigenvectors of H, H^ϵ, H_ϵ and $H(\epsilon)$, respectively. We shall also assume that $\{\varphi_i\}_{i=1}^\infty$, $\{\varphi_i^\epsilon\}_{i=1}^\infty$, $\{\varphi_{i,\epsilon}\}_{i=1}^\infty$ and $\{\varphi_i(\epsilon)\}_{i=1}^\infty$ are complete orthonormal systems in their respective Hilbert spaces $\mathcal{H}$, $\mathcal{H}^\epsilon$, $\mathcal{H}_\epsilon$ and $\mathcal{H}(\epsilon)$.

(A26) μ_1, μ_1^ϵ, $\mu_{1,\epsilon}$, $\mu_1(\epsilon)$, $0<\epsilon\le\epsilon_0$, all have multiplicity 1.

(A27) For $0<\epsilon\le\epsilon_0$, we assume that $P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0}$, $P_{\mathcal{H}^{\epsilon_0},\mathcal{H}^\epsilon}\varphi_1^{\epsilon_0}$, $P_{\mathcal{H}^{\epsilon_0},\mathcal{H}_\epsilon}\varphi_1^{\epsilon_0}$ and $P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0}$ are eigenvectors of H, H^ϵ, H_ϵ and $H(\epsilon)$, respectively, associated to the eigenvalue $0=\mu_1=\mu_1^\epsilon=\mu_{1,\epsilon}=\mu_1(\epsilon)$. In fact we assume that φ_1, φ_1^ϵ, $\varphi_{1,\epsilon}$ and $\varphi_1(\epsilon)$ are chosen so that

$$\begin{gathered}
\varphi_1=\|P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0}\|_{\mathcal{H}}^{-1}P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0},\\
\varphi_1^\epsilon=\|P_{\mathcal{H}^{\epsilon_0},\mathcal{H}^\epsilon}\varphi_1^{\epsilon_0}\|_{\mathcal{H}^\epsilon}^{-1}P_{\mathcal{H}^{\epsilon_0},\mathcal{H}^\epsilon}\varphi_1^{\epsilon_0},\\
\varphi_{1,\epsilon}=\|P_{\mathcal{H}^{\epsilon_0},\mathcal{H}_\epsilon}\varphi_1^{\epsilon_0}\|_{\mathcal{H}_\epsilon}^{-1}P_{\mathcal{H}^{\epsilon_0},\mathcal{H}_\epsilon}\varphi_1^{\epsilon_0},\\
\varphi_1(\epsilon)=\|P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0}\|_{\mathcal{A}(\epsilon)}^{-1}P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0}.
\end{gathered}$$

(A28) For all $0<t\le1$ and $n=1,2,3,\dots$, we assume that

$$\begin{gathered}
\lim_{\epsilon\downarrow0}\|P_{\mathcal{H},\mathcal{B}(\epsilon)}e^{-Ht}\varphi_n-e^{-H(\epsilon)t}P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_n\|_{\mathcal{A}(\epsilon)}=0,\\
\lim_{\epsilon\downarrow0}\|e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_n(\epsilon)-P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}e^{-H(\epsilon)t}\varphi_n(\epsilon)\|_{\mathcal{H}}=0,\\
\lim_{\epsilon\downarrow0}\|P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_n(\epsilon)\|_{\mathcal{H}}=1.
\end{gathered}$$

Theorem 2.20. *We have* $\lim_{\epsilon\downarrow0}\mu_2(\epsilon)=\mu_2$.

Proof. For $0<\epsilon\le\epsilon_0$ let

$$\beta_1(\epsilon)=\langle P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2,\varphi_1(\epsilon)\rangle_{\mathcal{A}(\epsilon)}.\tag{2.19}$$

Then

$$\begin{aligned}
&e^{-\mu_2(\epsilon)t}\\
&\ge\|P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2-\beta_1(\epsilon)\varphi_1(\epsilon)\|_{\mathcal{A}(\epsilon)}^{-2}\\
&\quad\times\langle e^{-H(\epsilon)t}(P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2-\beta_1(\epsilon)\varphi_1(\epsilon)),(P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2-\beta_1(\epsilon)\varphi_1(\epsilon))\rangle_{\mathcal{A}(\epsilon)}\\
&=\|P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2-\beta_1(\epsilon)\varphi_1(\epsilon)\|_{\mathcal{A}(\epsilon)}^{-2}\{\langle e^{-H(\epsilon)t}P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2,P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2\rangle_{\mathcal{A}(\epsilon)}\\
&\quad-2\beta_1(\epsilon)\langle e^{-H(\epsilon)t}P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2,\varphi_1(\epsilon)\rangle_{\mathcal{A}(\epsilon)}+\beta_1(\epsilon)^2\}\\
&=\|P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2-\beta_1(\epsilon)\varphi_1(\epsilon)\|_{\mathcal{A}(\epsilon)}^{-2}\{\langle e^{-H(\epsilon)t}P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2,P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2\rangle_{\mathcal{A}(\epsilon)}\\
&\quad-\beta_1(\epsilon)^2\}.
\end{aligned}\tag{2.20}$$

Consider the following term in (2.20):

$$\begin{aligned}
&\langle e^{-H(\epsilon)t}P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2,P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2\rangle_{\mathcal{A}(\epsilon)}\\
&=\langle e^{-H(\epsilon)t}P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2-P_{\mathcal{H},\mathcal{B}(\epsilon)}e^{-Ht}\varphi_2,P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2\rangle_{\mathcal{A}(\epsilon)}
\end{aligned}$$

$$
\begin{aligned}
&+ \langle P_{\mathcal{H},\mathcal{B}(\epsilon)} e^{-Ht}\varphi_2, P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2\rangle_{\mathcal{A}(\epsilon)} \\
&= \langle e^{-H(\epsilon)t} P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2 - P_{\mathcal{H},\,\mathcal{B}(\epsilon)} e^{-Ht}\varphi_2, P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2\rangle_{\mathcal{A},(\epsilon)} \\
&\quad + e^{-\mu_2 t}\langle P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2, P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2\rangle_{\mathcal{A}(\epsilon)} \\
&= \langle e^{-H(\epsilon)t} P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2 - P_{\mathcal{H},\mathcal{B}(\epsilon)} e^{-Ht}\varphi_2, P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2\rangle_{\mathcal{A}(\epsilon)} \\
&\quad + e^{-\mu_2 t}\langle P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2, P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2\rangle_{\mathcal{H}} \\
&= \langle e^{-H(\epsilon)t} P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2 - P_{\mathcal{H},\mathcal{B}(\epsilon)} e^{-Ht}\varphi_2, P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2\rangle_{\mathcal{A}(\epsilon)} \\
&\quad + e^{-\mu_2 t}\langle (P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2 - \varphi_2) + \varphi_2, (P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2 - \varphi_2) + \varphi_2\rangle_{\mathcal{H}} \\
&= \langle e^{-H(\epsilon)t} P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2 - P_{\mathcal{H},\mathcal{B}(\epsilon)} e^{-Ht}\varphi_2, P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2\rangle_{\mathcal{A}(\epsilon)} \\
&\quad + e^{-\mu_2 t}\{\langle P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2 - \varphi_2, P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2 - \varphi_2\rangle_{\mathcal{H}} \\
&\quad + 2\langle P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2 - \varphi_2, \varphi_2\rangle_{\mathcal{H}} + 1\} \\
&= \langle e^{-H(\epsilon)t} P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2 - P_{\mathcal{H},\mathcal{B}(\epsilon)} e^{-Ht}\varphi_2, P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2\rangle_{\mathcal{A}(\epsilon)} \\
&\quad + e^{-\mu_2 t}\{\langle P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2 - \varphi_2, P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2 - \varphi_2\rangle_{\mathcal{H}} \\
&\quad + 2\langle P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2 - \varphi_2, \varphi_2\rangle_{\mathcal{H}}\} + e^{-\mu_2 t}.
\end{aligned}
$$

Hence, by (A7) and (A28), we have

$$
\lim_{\epsilon\downarrow 0}\langle e^{-H(\epsilon)t} P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2, P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2\rangle_{\mathcal{A}(\epsilon)} = e^{-\mu_2 t}. \tag{2.21}
$$

Next we consider the term $\beta_1(\epsilon)$ defined in (2.19). We note that, by (A27),

$$
\begin{aligned}
P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_1(\epsilon) &= \|P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0}\|_{\mathcal{A}(\epsilon)}^{-1} P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)} P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0} \\
&= \|P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0}\|_{\mathcal{A}(\epsilon)}^{-1} P_{\mathcal{H}^{\epsilon_0},\mathcal{B}(\epsilon)}\varphi_1^{\epsilon_0} \\
&= \|P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0}\|_{\mathcal{A}(\epsilon)}^{-1} P_{\mathcal{H},\mathcal{B}(\epsilon)} P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0} \\
&= \|P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0}\|_{\mathcal{A}(\epsilon)}^{-1} \|P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0}\|_{\mathcal{H}} P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_1.
\end{aligned} \tag{2.22}
$$

Consider the term $\|P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0}\|_{\mathcal{A}(\epsilon)}$ in (2.22). We have, by (A9),

$$
\begin{aligned}
P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0} &= P_{\mathcal{H}^{\epsilon_0},\mathcal{B}(\epsilon)}\varphi_1^{\epsilon_0} + P_{\mathcal{H}^{\epsilon_0},\mathcal{C}(\epsilon)}\varphi_1^{\epsilon_0} + P_{\mathcal{H}^{\epsilon_0},\mathcal{D}(\epsilon)}\varphi_1^{\epsilon_0} \\
&= P_{\mathcal{H},\mathcal{B}(\epsilon)} P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0} + P_{\mathcal{H}^{\epsilon_0},\mathcal{C}(\epsilon)}\varphi_1^{\epsilon_0} + P_{\mathcal{H}^{\epsilon_0},\mathcal{D}(\epsilon)}\varphi_1^{\epsilon_0} \\
&= \|P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0}\|_{\mathcal{H}} P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_1 + P_{\mathcal{H}^{\epsilon_0},\mathcal{C}(\epsilon)}\varphi_1^{\epsilon_0} + P_{\mathcal{H}^{\epsilon_0},\mathcal{D}(\epsilon)}\varphi_1^{\epsilon_0}.
\end{aligned}
$$

Thus, by (A7), Lemma 2.1 and (A10),

$$
\begin{aligned}
\|P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0}\|_{\mathcal{A}(\epsilon)}^2 &= \|P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0}\|_{\mathcal{H}}^2 \|P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_1\|_{\mathcal{B}(\epsilon)}^2 \\
&\quad + \|P_{\mathcal{H}^{\epsilon_0},\mathcal{C}(\epsilon)}\varphi_1^{\epsilon_0}\|_{\mathcal{C}(\epsilon)}^2 + \|P_{\mathcal{H}^{\epsilon_0},\mathcal{D}(\epsilon)}\varphi_1^{\epsilon_0}\|_{\mathcal{D}(\epsilon)}^2 \\
&= \|P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0}\|_{\mathcal{H}}^2 \|(P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_1 - \varphi_1) + \varphi_1\|_{\mathcal{H}}^2 \\
&\quad + \|P_{\mathcal{H}^{\epsilon_0},\mathcal{C}(\epsilon)}\varphi_1^{\epsilon_0}\|_{\mathcal{C}(\epsilon)}^2 + \|P_{\mathcal{H}^{\epsilon_0},\mathcal{D}(\epsilon)}\varphi_1^{\epsilon_0}\|_{\mathcal{D}(\epsilon)}^2 \\
&\to \|P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0}\|_{\mathcal{H}}^2 \quad \text{as } \epsilon\downarrow 0.
\end{aligned} \tag{2.23}
$$

Thus, by (A9),

$$
\begin{aligned}
\varphi_1(\epsilon) &= P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_1(\epsilon) + P_{\mathcal{A}(\epsilon),\mathcal{C}(\epsilon)}\varphi_1(\epsilon) + P_{\mathcal{A}(\epsilon),\mathcal{D}(\epsilon)}\varphi_1(\epsilon) \\
&= \|P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0}\|_{\mathcal{A}(\epsilon)}^{-1} \|P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0}\|_{\mathcal{H}} P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_1 \\
&\quad + P_{\mathcal{A}(\epsilon),\mathcal{C}(\epsilon)}\varphi_1(\epsilon) + P_{\mathcal{A}(\epsilon),\mathcal{D}(\epsilon)}\varphi_1(\epsilon).
\end{aligned} \tag{2.24}
$$

Since the second and third terms in the last line of (2.24) are in $\mathcal{C}(\epsilon)$ and $\mathcal{D}(\epsilon)$, respectively, they are orthogonal to $P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_1(\epsilon)$ by (A9). Hence, by (2.24), (2.23) and (A7),

$$\begin{aligned}
\beta_1(\epsilon) &= \langle P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2, \varphi_1(\epsilon)\rangle_{\mathcal{A}(\epsilon)} \\
&= \langle P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2, \|P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0}\|_{\mathcal{A}(\epsilon)}^{-1}\|P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0}\|_{\mathcal{H}}P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_1\rangle_{\mathcal{H}} \\
&= \langle (P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2 - \varphi_2) + \varphi_2, \|P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0}\|_{\mathcal{A}(\epsilon)}^{-1}\|P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0}\|_{\mathcal{H}} \\
&\quad \times (P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_1 - \varphi_1) + \|P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0}\|_{\mathcal{A}(\epsilon)}^{-1}\|P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0}\|_{\mathcal{H}}\varphi_1\rangle_{\mathcal{H}} \\
&\to \langle \varphi_2, \varphi_1\rangle_{\mathcal{H}} = 0 \quad \text{as } \epsilon \downarrow 0.
\end{aligned} \tag{2.25}$$

Therefore we can deal with the term

$$\|P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2 - \beta_1(\epsilon)\varphi_1(\epsilon)\|_{\mathcal{A}(\epsilon)}^{-2}$$

of (2.20) as follows:

$$\begin{aligned}
&\|P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2 - \beta_1(\epsilon)\varphi_1(\epsilon)\|_{\mathcal{A}(\epsilon)}^{-2} \\
&= \|P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2 - \beta_1(\epsilon)P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_1(\epsilon) - \beta_1(\epsilon)P_{\mathcal{A}(\epsilon),\mathcal{C}(\epsilon)}\varphi_1(\epsilon) \\
&\quad - \beta_1(\epsilon)P_{\mathcal{A}(\epsilon),\mathcal{D}(\epsilon)}\varphi_1(\epsilon)\|_{\mathcal{A}(\epsilon)}^{-2} \\
&= \|P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2 - \beta_1(\epsilon)P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_1(\epsilon)\|_{\mathcal{B}(\epsilon)}^2 \\
&\quad + \beta_1(\epsilon)^2\|P_{\mathcal{A}(\epsilon),\,\mathcal{C}(\epsilon)}\varphi_1(\epsilon)\|_{\mathcal{C}(\epsilon)}^2 + \beta_1(\epsilon)^2\|P_{\mathcal{A}(\epsilon),\,\mathcal{D}(\epsilon)}\varphi_1(\epsilon)\|_{\mathcal{D}(\epsilon)}^2 \\
&= \|(P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_2 - \varphi_2) + \varphi_2 - \beta_1(\epsilon)P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_1(\epsilon)\|_{\mathcal{H}}^2 \\
&\quad + \beta_1(\epsilon)^2\|P_{\mathcal{A}(\epsilon),\mathcal{C}(\epsilon)}\varphi_1(\epsilon)\|_{\mathcal{C}(\epsilon)}^2 + \beta_1(\epsilon)^2\|P_{\mathcal{A}(\epsilon),\mathcal{D}(\epsilon)}\varphi_1(\epsilon)\|_{\mathcal{D}(\epsilon)}^2 \\
&\to \|\varphi_2\|_{\mathcal{H}}^2 = 1 \quad \text{as } \epsilon \downarrow 0,
\end{aligned} \tag{2.26}$$

by (A7) and (2.25). Therefore, by (2.20), (2.21), (2.25) and (2.26), we have, for all $\delta > 0$, there exists $\epsilon_1 \in (0, \epsilon_0]$ such that

$$\mu_2(\epsilon)t \le \mu_2 t + \delta \tag{2.27}$$

for all $0 < \epsilon \le \epsilon_1$. We next prove the reverse inequality of (2.27). For all $0 < \epsilon \le \epsilon_0$ let

$$\gamma_1(\epsilon) = \langle P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon), \varphi_1\rangle_{\mathcal{H}}. \tag{2.28}$$

Then

$$\begin{aligned}
e^{-\mu_2 t} &\ge \|P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon) - \gamma_1(\epsilon)\varphi_1\|_{\mathcal{H}}^{-2} \\
&\quad \times \langle e^{-Ht}(P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon) - \gamma_1(\epsilon)\varphi_1), P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon) - \gamma_1(\epsilon)\varphi_1\rangle_{\mathcal{H}} \\
&= \|P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon) - \gamma_1(\epsilon)\varphi_1\|_{\mathcal{H}}^{-2} \\
&\quad \times \{\langle e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon), P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon)\rangle_{\mathcal{H}} \\
&\quad - 2\gamma_1(\epsilon)\langle e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon), \varphi_1\rangle_{\mathcal{H}} + \gamma_1(\epsilon)^2\} \\
&= \|P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon) - \gamma_1(\epsilon)\varphi_1\|_{\mathcal{H}}^{-2} \\
&\quad \times \{\langle e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon), P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon)\rangle_{\mathcal{H}} - \gamma_1(\epsilon)^2\}.
\end{aligned} \tag{2.29}$$

Consider the term

$$\begin{aligned}
&\langle e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon), P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon)\rangle_{\mathcal{H}} \\
&= \langle e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon) - P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}e^{-H(\epsilon)t}\varphi_2(\epsilon), P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon)\rangle_{\mathcal{H}} \\
&\quad + \langle P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}e^{-H(\epsilon)t}\varphi_2(\epsilon), P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon)\rangle_{\mathcal{H}} \\
&= \langle e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon) - P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}e^{-H(\epsilon)t}\varphi_2(\epsilon), P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon)\rangle_{\mathcal{H}} \\
&\quad + e^{-\mu_2(\epsilon)t}\langle P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon), P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon)\rangle_{\mathcal{H}}.
\end{aligned} \tag{2.30}$$

Therefore, by (A28), (2.27) and (2.30), for all $\delta > 0$ there exists $\epsilon_1 \in (0, \epsilon_0]$ such that

$$\langle e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon), P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon)\rangle_{\mathcal{H}} \geq e^{-\mu_2(\epsilon)t-\delta} \tag{2.31}$$

for all $\epsilon \in (0, \epsilon_1]$.

We next consider the term $\gamma_1(\epsilon)$ defined in (2.28):

$$\gamma_1(\epsilon) = \langle P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon), \varphi_1\rangle_{\mathcal{H}} = \langle P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon), P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_1 + P_{\mathcal{H},\mathcal{B}(\epsilon)^{\perp\mathcal{H}}}\varphi_1\rangle_{\mathcal{H}}. \tag{2.32}$$

But, by (A27),

$$\begin{aligned}
P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_1 &= \|P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0}\|_{\mathcal{H}}^{-1}P_{\mathcal{H},\mathcal{B}(\epsilon)}P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0} \\
&= \|P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0}\|_{\mathcal{H}}^{-1}P_{\mathcal{H}^{\epsilon_0},\mathcal{B}(\epsilon)}\varphi_1^{\epsilon_0} \\
&= \|P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0}\|_{\mathcal{H}}^{-1}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0} \\
&= \|P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0}\|_{\mathcal{H}}^{-1}\|P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0}\|_{\mathcal{A}(\epsilon)}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_1(\epsilon).
\end{aligned} \tag{2.33}$$

Hence, from (2.32) and (2.33),

$$\begin{aligned}
\gamma_1(\epsilon) &= \langle P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon), \|P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0}\|_{\mathcal{H}}^{-1}\|P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0}\|_{\mathcal{A}(\epsilon)} \\
&\quad \times P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_1(\epsilon) + P_{\mathcal{H},\mathcal{B}(\epsilon)^{\perp\mathcal{H}}}\varphi_1\rangle_{\mathcal{H}} \\
&= \langle P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon), \|P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0}\|_{\mathcal{H}}^{-1}\|P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0}\|_{\mathcal{A}(\epsilon)} \\
&\quad \times P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_1(\epsilon)\rangle_{\mathcal{A}(\epsilon)} + \langle P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon), P_{\mathcal{H},\mathcal{B}(\epsilon)^{\perp\mathcal{H}}}\varphi_1\rangle_{\mathcal{H}} \\
&= \langle P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon), \|P_{\mathcal{H}^{\epsilon_0},\mathcal{H}}\varphi_1^{\epsilon_0}\|_{\mathcal{H}}^{-1}\|P_{\mathcal{H}^{\epsilon_0},\mathcal{A}(\epsilon)}\varphi_1^{\epsilon_0}\|_{\mathcal{A}(\epsilon)} \\
&\quad \times P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_1(\epsilon)\rangle_{\mathcal{A}(\epsilon)}.
\end{aligned} \tag{2.34}$$

We show that the last line of (2.34) tends to 0 as $\epsilon \downarrow 0$: We have

$$\begin{aligned}
0 &= \langle \varphi_2(\epsilon), \varphi_1(\epsilon)\rangle_{\mathcal{A}(\epsilon)} \\
&= \langle P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon), P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_1(\epsilon)\rangle_{\mathcal{B}(\epsilon)} \\
&\quad + \langle P_{\mathcal{A}(\epsilon),\mathcal{C}(\epsilon)}\varphi_2(\epsilon), P_{\mathcal{A}(\epsilon),\mathcal{C}(\epsilon)}\varphi_1(\epsilon)\rangle_{\mathcal{C}(\epsilon)} \\
&\quad + \langle P_{\mathcal{A}(\epsilon),\mathcal{D}(\epsilon)}\varphi_2(\epsilon), P_{\mathcal{A}(\epsilon),\mathcal{D}(\epsilon)}\varphi_1(\epsilon)\rangle_{\mathcal{D}(\epsilon)}.
\end{aligned} \tag{2.35}$$

Since, by (A28),

$$\lim_{\epsilon\downarrow 0} \|P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_n(\epsilon)\|_{\mathcal{B}(\epsilon)} = 1, \tag{2.36}$$

we have

$$\lim_{\epsilon\downarrow 0} \|P_{\mathcal{A}(\epsilon),\mathcal{C}(\epsilon)}\varphi_n(\epsilon)\|_{\mathcal{C}(\epsilon)} = \lim_{\epsilon\downarrow 0} \|P_{\mathcal{A}(\epsilon),\mathcal{D}(\epsilon)}\varphi_n(\epsilon)\|_{\mathcal{D}(\epsilon)} = 0. \tag{2.37}$$

From (2.35), (2.36) and (2.37) we obtain

$$\lim_{\epsilon\downarrow 0}\langle P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon), P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_1(\epsilon)\rangle_{\mathcal{B}(\epsilon)} = 0. \tag{2.38}$$

Hence, by (2.34), (2.38) and (2.23), we have

$$\lim_{\epsilon\downarrow 0}\gamma_1(\epsilon)=0. \tag{2.39}$$

Thus, by (2.39) and (A28), we have

$$\begin{aligned}
&\|P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon)-\gamma_1(\epsilon)\varphi_1\|_{\mathcal{H}}^2\\
&=\|P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon)-\gamma_1(\epsilon)P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_1-\gamma_1(\epsilon)P_{\mathcal{H},\mathcal{B}(\epsilon)^{\perp_{\mathcal{H}}}}\varphi_1\|_{\mathcal{H}}^2\\
&=\|P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_2(\epsilon)-\gamma_1(\epsilon)P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_1\|_{\mathcal{B}(\epsilon)}^2+\gamma_1(\epsilon)^2\|P_{\mathcal{H},\mathcal{B}(\epsilon)^{\perp_{\mathcal{H}}}}\varphi_1\|_{\mathcal{H}}^2\\
&\to 1\quad\text{as }\epsilon\downarrow 0.
\end{aligned} \tag{2.40}$$

Combining (2.29), (2.31), (2.39) and (2.40), we see that for all $\delta>0$, there exists $\epsilon_1\in(0,\epsilon_0]$ such that

$$\mu_2 t\le \mu_2(\epsilon)t+\delta \tag{2.41}$$

for all $\epsilon\in(0,\epsilon_1]$. The theorem now follows from (2.27) and (2.41). □

Definition 2.21. We now define the sequence $\{k_i\}_{i=1}^\infty$ of positive integers as follows: Suppose we list the eigenvalues $\{\mu_n\}_{n=1}^\infty$ of H in a way reflecting their multiplicities. Then the positive integers k_i are defined by:

$$\begin{aligned}
0=\mu_1<\mu_2=\mu_3=\dots=\mu_{k_2}<\mu_{k_2+1}=\dots=\mu_{k_3}\\
<\mu_{k_3+1}=\dots=\mu_{k_4}<\mu_{k_4+1}=\dots.
\end{aligned} \tag{2.42}$$

We also define $k_1=1$.

Lemma 2.22. *Let $p\ge 1$ be an integer and let i be an integer satisfying*

$$k_p+1<i\le k_{p+1}.$$

Suppose, for $j=1,2,\dots,i-1$, we have

$$\lim_{\epsilon\downarrow 0}\mu_j(\epsilon)=\mu_j. \tag{2.43}$$

Then

$$\lim_{\epsilon\downarrow 0}\mu_i(\epsilon)=\mu_i=\mu_{k_p+1}=\mu_{k_{p+1}}.$$

Proof. Assume, for a contradiction, that

$$\mu_i(\epsilon)\not\to\mu_i\quad\text{as }\epsilon\downarrow 0. \tag{2.44}$$

Then there exist $\eta>0$ and a strictly decreasing sequence $\{\epsilon_m\}_{m=1}^\infty$ of positive numbers such that $\epsilon_m\downarrow 0$ as $m\to\infty$, and that

$$\mu_i(\epsilon_m)\ge\mu_i+\eta\quad(m=1,2,3,\dots).$$

For $j=1,2,3,\dots$ we regard $P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_j$ as a vector in $\mathcal{A}(\epsilon)$ and write

$$P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_j=\sum_{\ell=1}^\infty a_{j,\ell}(\epsilon)\varphi_\ell(\epsilon). \tag{2.45}$$

Then, for all $0<t\le 1$ and $j=1,2,3,\dots,$

$$\begin{aligned}
&e^{-H(\epsilon)t}P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_j-P_{\mathcal{H},\mathcal{B}(\epsilon)}e^{-Ht}\varphi_j\\
&=\Big(\sum_{\ell=1}^{i-1}+\sum_{\ell=i}^\infty\Big)a_{j,\ell}(\epsilon)(e^{-\mu_\ell(\epsilon)t}-e^{-\mu_j t})\varphi_\ell(\epsilon).
\end{aligned} \tag{2.46}$$

By (A28), (2.43), (2.46) and the orthogonality of $\{\varphi_i(\epsilon)\}_{i=1}^{\infty}$, we have

$$\lim_{\epsilon\downarrow 0}\sum_{\ell=i}^{\infty} a_{j,\ell}(\epsilon)^2(e^{-\mu_\ell(\epsilon)t}-e^{-\mu_j t})^2=0. \tag{2.47}$$

Since, for $\ell=i,i+1,i+2,\dots$ and $j=1,2,\dots,k_{p+1}$ and $m=1,2,3,\dots$, we have $\mu_\ell(\epsilon_m)\ge\mu_j+\eta$. Equation (2.47) implies

$$\lim_{m\to\infty}\sum_{\ell=i}^{\infty} a_{j,\ell}(\epsilon_m)^2=0.$$

Hence, for $j=1,2,\dots,k_{p+1}$,

$$\lim_{m\to\infty}\Big\|\sum_{\ell=i}^{\infty} a_{j,\ell}(\epsilon_m)\varphi_\ell(\epsilon_m)\Big\|_{\mathcal{A}(\epsilon_m)}=0. \tag{2.48}$$

Since, by (A7), $\lim_{\epsilon\downarrow 0}\|P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_j\|_{\mathcal{A}(\epsilon)}=1$, (2.48) implies that, for $j=1,2,\dots,k_{p+1}$,

$$\lim_{m\to\infty}\Big\|\sum_{\ell=1}^{i-1} a_{j,\ell}(\epsilon_m)\varphi_\ell(\epsilon_m)\Big\|_{\mathcal{A}(\epsilon_m)}=1. \tag{2.49}$$

For $j=1,2,\dots,k_{p+1}$ and $m=1,2,3,\dots$ let

$$u_{j,i}(m)=\sum_{\ell=1}^{i-1} a_{j,\ell}(\epsilon_m)\varphi_\ell(\epsilon_m). \tag{2.50}$$

Then, for $\sigma,\tau\in\{1,2,\dots,i\}$ with $\sigma\ne\tau$, we have

$$\begin{aligned}&\langle P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_\sigma, P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_\tau\rangle_{\mathcal{B}(\epsilon)}\\ &=\langle\varphi_\sigma,\varphi_\tau\rangle_{\mathcal{H}}-\langle(I_{\mathcal{H}}-P_{\mathcal{H},\mathcal{B}(\epsilon)})\varphi_\sigma,(I_{\mathcal{H}}-P_{\mathcal{H},\mathcal{B}(\epsilon)})\varphi_\tau\rangle_{\mathcal{B}(\epsilon)^{\perp}\mathcal{H}}\\ &\to 0\quad\text{as } \epsilon\downarrow 0 \text{ (by (A7))}.\end{aligned} \tag{2.51}$$

But for $m=1,2,3,\dots$

$$\begin{aligned}&\langle P_{\mathcal{H},\mathcal{B}(\epsilon_m)}\varphi_\sigma, P_{\mathcal{H},\mathcal{B}(\epsilon_m)}\varphi_\tau\rangle_{\mathcal{B}(\epsilon_m)}\\ &=\langle u_{\sigma,i}(m),u_{\tau,i}(m)\rangle_{\mathcal{A}(\epsilon_m)}\\ &\quad+\Big\langle\sum_{\ell=i}^{\infty} a_{\sigma,\ell}(\epsilon_m)\varphi_\ell(\epsilon_m),\sum_{\ell=i}^{\infty} a_{\tau,\ell}(\epsilon_m)\varphi_\ell(\epsilon_m)\Big\rangle_{\mathcal{A}(\epsilon_m)}.\end{aligned} \tag{2.52}$$

From (2.48), (2.51) and (2.52), we obtain

$$\lim_{m\to\infty}\langle u_{\sigma,i}(m),u_{\tau,i}(m)\rangle_{\mathcal{A}(\epsilon_m)}=0. \tag{2.53}$$

From (2.49), (2.50) and (2.53), we have a set of i vectors $\{u_{1,i}(m),\dots,u_{i,i}(m)\}$ in an $(i-1)$-dimensional inner product space spanned by $\{\varphi_1(\epsilon_m),\dots,\varphi_{i-1}(\epsilon_m)\}$ which, as $m\to\infty$, is almost orthonormal. This gives a contradiction. Thus we must have $\lim_{\epsilon\downarrow 0}\mu_i(\epsilon)=\mu_i$. $\square$

Lemma 2.23. *Let $p\ge 2$ be an integer. Suppose that $\lim_{\epsilon\downarrow 0}\mu_i(\epsilon)=\mu_i$ for all $i=1,2,\dots,k_p$. Then there exists $\eta>0$ such that for all sufficiently small $\epsilon>0$ we have*

$$\mu_{k_p+1}(\epsilon)\ge\mu_{k_p}+\eta.$$

Proof. For $i = 1, 2, 3, \dots$ let

$$P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_i(\epsilon) = \sum_{\ell=1}^{\infty} b_{i,\ell}(\epsilon)\varphi_\ell, \tag{2.54}$$

regarding $P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_i(\epsilon)$ as a vector in $\mathcal{H}$. Suppose the lemma is false. Then there exists a strictly decreasing sequence of positive numbers $\{\epsilon_m\}_{m=1}^{\infty}$ such that $\epsilon_m \downarrow 0$ as $m \to \infty$ and that

$$\mu_{k_p+1}(\epsilon_m) \to \mu_{k_p} \quad \text{as } m \to \infty. \tag{2.55}$$

Then, for all $0 < t \le 1$ and $n = 1, 2, 3, \dots$, we have

$$\begin{aligned} &e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_n(\epsilon) - P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}e^{-H(\epsilon)t}\varphi_n(\epsilon) \\ &= \Big(\sum_{\ell=1}^{k_{p-1}} + \sum_{\ell=k_{p-1}+1}^{k_p} + \sum_{\ell=k_p+1}^{\infty}\Big) b_{n,\ell}(\epsilon)(e^{-\mu_\ell t} - e^{-\mu_n(\epsilon)t})\varphi_\ell \\ &= A + B + C. \end{aligned} \tag{2.56}$$

By (A28) and the orthogonality of $\{\varphi_\ell\}_{\ell=1}^{\infty}$ we have

$$\lim_{\epsilon\downarrow 0} \|A\|_{\mathcal{H}} = \lim_{\epsilon\downarrow 0} \|B\|_{\mathcal{H}} = \lim_{\epsilon\downarrow 0} \|C\|_{\mathcal{H}} = 0. \tag{2.57}$$

For $n = k_{p-1} + 1, \dots, k_p + 1$, (2.55) and (2.57) imply

$$\lim_{m\to\infty} \Big\{ \sum_{\ell=1}^{k_{p-1}} b_{n,\ell}(\epsilon_m)^2 + \sum_{\ell=k_p+1}^{\infty} b_{n,\ell}(\epsilon_m)^2 \Big\} = 0. \tag{2.58}$$

Using (A28) and (2.58) we obtain

$$\lim_{m\to\infty} \Big\| \sum_{\ell=k_{p-1}+1}^{k_p} b_{n,\ell}(\epsilon_m)\varphi_\ell \Big\|_{\mathcal{H}} = 1 \tag{2.59}$$

for all $n = k_{p-1} + 1, \dots, k_p + 1$.

For $\sigma, \tau \in \{k_{p-1} + 1, \dots, k_p + 1\}$ with $\sigma \ne \tau$, we have

$$\begin{aligned} 0 &= \langle \varphi_\sigma(\epsilon), \varphi_\tau(\epsilon)\rangle_{\mathcal{A}(\epsilon)} \\ &= \langle P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_\sigma(\epsilon), P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_\tau(\epsilon)\rangle_{\mathcal{B}(\epsilon)} \\ &\quad + \langle P_{\mathcal{A}(\epsilon),\mathcal{C}(\epsilon)\oplus\mathcal{D}(\epsilon)}\varphi_\sigma(\epsilon), P_{\mathcal{A}(\epsilon),\mathcal{C}(\epsilon)\oplus\mathcal{D}(\epsilon)}\varphi_\tau(\epsilon)\rangle_{\mathcal{C}(\epsilon)\oplus\mathcal{D}(\epsilon)} \end{aligned} \tag{2.60}$$

and since

$$1 = \|\varphi_\sigma(\epsilon)\|^2_{\mathcal{A}(\epsilon)} = \|P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_\sigma(\epsilon)\|^2_{\mathcal{B}(\epsilon)} + \|P_{\mathcal{A}(\epsilon),\mathcal{C}(\epsilon)\oplus\mathcal{D}(\epsilon)}\varphi_\sigma(\epsilon)\|^2_{\mathcal{C}(\epsilon)\oplus\mathcal{D}(\epsilon)} \tag{2.61}$$

and

$$\lim_{\epsilon\downarrow 0} \|P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_\sigma(\epsilon)\|^2_{\mathcal{B}(\epsilon)} = 1, \tag{2.62}$$

we have

$$\lim_{\epsilon\downarrow 0} \langle P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_\sigma(\epsilon), P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_\tau(\epsilon)\rangle_{\mathcal{B}(\epsilon)} = 0. \tag{2.63}$$

Hence, from (2.54), (2.58), (2.59) and (2.63), we have

$$\lim_{m\to\infty} \Big\langle \sum_{\ell=k_{p-1}+1}^{k_p} b_{\sigma,\ell}(\epsilon_m)\varphi_\ell, \sum_{\ell=k_{p-1}+1}^{k_p} b_{\tau,\ell}(\epsilon_m)\varphi_\ell \Big\rangle_{\mathcal{H}} = 0. \tag{2.64}$$

For $n = k_{p-1}+1, \dots, k_p+1$ and $m = 1, 2, 3, \dots$ let

$$u_n(m) = \sum_{\ell=k_{p-1}+1}^{k_p} b_{n,\ell}(\epsilon_m)\varphi_\ell. \tag{2.65}$$

Then we have a set of $k_p + 1 - k_{p-1}$ vectors $\{u_{k_{p-1}+1}(m), \dots, u_{k_p+1}(m)\}$ in a $(k_p - k_{p-1})$-dimensional inner product space spanned by the set $\{\varphi_{k_{p-1}+1}, \dots, \varphi_{k_p}\}$ which, by (2.59) and (2.64), is almost orthonormal. This gives a contradiction. Hence (2.55) cannot be true and the lemma is proved. $\square$

Lemma 2.24. *Let $p \geq 2$ be an integer. Suppose that $\lim_{\epsilon\downarrow 0}\mu_i(\epsilon) = \mu_i$ for $i = 1, 2, \dots, k_p$. Then*

$$\lim_{\epsilon\downarrow 0}\mu_{k_p+1}(\epsilon) = \mu_{k_p+1}.$$

Proof. For $0 < \epsilon \leq \epsilon_0$ let

$$P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_{k_p+1} = \sum_{\ell=1}^{\infty}\beta_\ell(\epsilon)\varphi_\ell(\epsilon) \tag{2.66}$$

regarded as a vector in $\mathcal{A}(\epsilon)$ and let

$$f_{k_p+1}(\epsilon) = P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_{k_p+1} - \sum_{\ell=1}^{k_p}\beta_\ell(\epsilon)\varphi_\ell(\epsilon) \in \mathcal{A}(\epsilon). \tag{2.67}$$

Then

$$\begin{aligned}
&e^{-\mu_{k_p+1}(\epsilon)t}\\
&\geq \|f_{k_p+1}(\epsilon)\|^{-2}_{\mathcal{A}(\epsilon)}\langle e^{-H(\epsilon)t}f_{k_p+1}(\epsilon), f_{k_p+1}(\epsilon)\rangle_{\mathcal{A}(\epsilon)}\\
&= \|f_{k_p+1}(\epsilon)\|^{-2}_{\mathcal{A}(\epsilon)}\Big\langle e^{-H(\epsilon)t}P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_{k_p+1} - \sum_{\ell=1}^{k_p}\beta_\ell(\epsilon)e^{-\mu_\ell(\epsilon)t}\varphi_\ell(\epsilon),\\
&\quad P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_{k_p+1} - \sum_{\ell=1}^{k_p}\beta_\ell(\epsilon)\varphi_\ell(\epsilon)\Big\rangle_{\mathcal{A}(\epsilon)}\\
&= \|f_{k_p+1}(\epsilon)\|^{-2}_{\mathcal{A}(\epsilon)}\Big\{\langle e^{-H(\epsilon)t}P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_{k_p+1}, P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_{k_p+1}\rangle_{\mathcal{A}(\epsilon)}\\
&\quad - 2\Big\langle\sum_{\ell=1}^{k_p}\beta_\ell(\epsilon)e^{-\mu_\ell(\epsilon)t}\varphi_\ell(\epsilon), P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_{k_p+1}\Big\rangle_{\mathcal{A}(\epsilon)} + \sum_{\ell=1}^{k_p}e^{-\mu_\ell(\epsilon)t}\beta_\ell(\epsilon)^2\Big\}\\
&= \|f_{k_p+1}(\epsilon)\|^{-2}_{\mathcal{A}(\epsilon)}\Big\{\langle e^{-H(\epsilon)t}P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_{k_p+1} - P_{\mathcal{H},\mathcal{B}(\epsilon)}e^{-Ht}\varphi_{k_p+1},\\
&\quad P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_{k_p+1}\rangle_{\mathcal{A}(\epsilon)} + \langle P_{\mathcal{H},\mathcal{B}(\epsilon)}e^{-Ht}\varphi_{k_p+1}, P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_{k_p+1}\rangle_{\mathcal{A}(\epsilon)}\\
&\quad - \sum_{\ell=1}^{k_p}e^{-\mu_\ell(\epsilon)t}\beta_\ell(\epsilon)^2\Big\} \qquad (2.68)\\
&= \|f_{k_p+1}(\epsilon)\|^{-2}_{\mathcal{A}(\epsilon)}\Big\{\langle e^{-H(\epsilon)t}P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_{k_p+1} - P_{\mathcal{H},\mathcal{B}(\epsilon)}e^{-Ht}\varphi_{k_p+1},\\
&\quad P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_{k_p+1}\rangle_{\mathcal{A}(\epsilon)} + e^{-\mu_{k_p+1}t}\sum_{\ell=1}^{\infty}\beta_\ell(\epsilon)^2
\end{aligned}$$

$$- \sum_{\ell=1}^{k_p} e^{-\mu_\ell(\epsilon)t}\beta_\ell(\epsilon)^2\Big\}.$$

Now

$$\begin{aligned} &e^{-H(\epsilon)t}P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_{k_p+1} - P_{\mathcal{H},\mathcal{B}(\epsilon)}e^{-Ht}\varphi_{k_p+1} \\ &= \sum_{\ell=1}^{\infty} \beta_\ell(\epsilon)(e^{-\mu_\ell(\epsilon)t} - e^{-\mu_{k_p+1}t})\varphi_\ell(\epsilon). \end{aligned} \tag{2.69}$$

So, by (A28), we have

$$\lim_{\epsilon\downarrow 0} \sum_{\ell=1}^{\infty} \beta_\ell(\epsilon)^2(e^{-\mu_\ell(\epsilon)t} - e^{-\mu_{k_p+1}t})^2 = 0, \tag{2.70}$$

in particular

$$\lim_{\epsilon\downarrow 0} \sum_{\ell=1}^{k_p} \beta_\ell(\epsilon)^2(e^{-\mu_\ell(\epsilon)t} - e^{-\mu_{k_p+1}t})^2 = 0. \tag{2.71}$$

But for $\ell = 1, \dots, k_p$ we have, by assumption, $\lim_{\epsilon\downarrow 0}\mu_\ell(\epsilon) = \mu_\ell$. Hence (2.71) implies

$$\lim_{\epsilon\downarrow 0} \sum_{\ell=1}^{k_p} \beta_\ell(\epsilon)^2 = 0. \tag{2.72}$$

So, by (A7), (2.67) and (2.72), we have

$$\lim_{\epsilon\downarrow 0} \sum_{\ell=k_p+1}^{\infty} \beta_\ell(\epsilon)^2 = \lim_{\epsilon\downarrow 0} \|f_{k_p+1}(\epsilon)\|^2_{\mathcal{A}(\epsilon)} = 1. \tag{2.73}$$

From (2.68), (2.69), (2.70), (2.72), (A7) and (2.66), we have for all $\delta > 0$ there exists $\epsilon_1 \in (0, \epsilon_0]$ such that

$$e^{-\mu_{k_p+1}(\epsilon)t} \ge e^{-\mu_{k_p+1}t} - \delta \tag{2.74}$$

for all $\epsilon \in (0, \epsilon_1]$.

Next we prove the reverse inequality of (2.74). For $i = 1, 2, 3, \dots$ and $\epsilon \in (0, \epsilon_0]$, let

$$P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_i(\epsilon) = \sum_{\ell=1}^{\infty} \gamma_{i,\ell}(\epsilon)\varphi_\ell \in \mathcal{B}(\epsilon) \subseteq \mathcal{H} \tag{2.75}$$

and let

$$g_{k_p+1}(\epsilon) = P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_{k_p+1}(\epsilon) - \sum_{\ell=1}^{k_p} \gamma_{k_p+1,\ell}(\epsilon)\varphi_\ell. \tag{2.76}$$

Then

$$\begin{aligned} &e^{-\mu_{k_p+1}t} \\ &\ge \|g_{k_p+1}(\epsilon)\|^{-2}_{\mathcal{H}}\langle e^{-Ht}g_{k_p+1}(\epsilon), g_{k_p+1}(\epsilon)\rangle_{\mathcal{H}} \\ &= \|g_{k_p+1}(\epsilon)\|^{-2}_{\mathcal{H}}\{\langle e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_{k_p+1}(\epsilon), P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_{k_p+1}(\epsilon)\rangle_{\mathcal{H}} \\ &\quad - 2\Big\langle e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_{k_p+1}(\epsilon), \sum_{\ell=1}^{k_p} \gamma_{k_p+1,\ell}(\epsilon)\varphi_\ell\Big\rangle_{\mathcal{H}} \end{aligned}$$

$$
\begin{aligned}
&+ \Big\langle e^{-Ht}\sum_{\ell=1}^{k_p}\gamma_{k_{p+1},\ell}(\epsilon)\varphi_\ell, \sum_{\ell=1}^{k_p}\gamma_{k_p+1,\ell}(\epsilon)\varphi_\ell\Big\rangle_{\mathcal{H}}\Big\}\\
&= \|g_{k_p+1}(\epsilon)\|_{\mathcal{H}}^{-2}\Big\{\langle e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_{k_p+1}(\epsilon) - P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}e^{-H(\epsilon)t}\varphi_{k_p+1}(\epsilon),\\
&\quad P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_{k_p+1}(\epsilon)\rangle_{\mathcal{H}} + \langle P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}e^{-H(\epsilon)t}\varphi_{k_p+1}(\epsilon),\\
&\quad P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_{k_p+1}(\epsilon)\rangle_{\mathcal{H}} - 2\sum_{\ell=1}^{k_p} e^{-\mu_\ell t}\gamma_{k_p+1,\ell}(\epsilon)^2\\
&\quad + \sum_{\ell=1}^{k_p} e^{-\mu_\ell t}\gamma_{k_p+1,\ell}(\epsilon)^2\Big\}\\
&= \|g_{k_p+1}(\epsilon)\|_{\mathcal{H}}^{-2}\Big\{\langle e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_{k_p+1}(\epsilon) - P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}e^{-H(\epsilon)t}\varphi_{k_p+1}(\epsilon),\\
&\quad P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_{k_p+1}(\epsilon)\rangle_{\mathcal{H}} + e^{-\mu_{k_p+1}(\epsilon)t}\sum_{\ell=1}^{\infty}\gamma_{k_p+1,\ell}(\epsilon)^2\\
&\quad - \sum_{\ell=1}^{k_p} e^{-\mu_\ell t}\gamma_{k_p+1,\ell}(\epsilon)^2\Big\}.
\end{aligned} \tag{2.77}
$$

Now, by (A28),

$$
\begin{aligned}
&\|e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_{k_p+1}(\epsilon) - P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}e^{-H(\epsilon)t}\varphi_{k_p+1}(\epsilon)\|_{\mathcal{H}}^2\\
&= \sum_{\ell=1}^{\infty}\gamma_{k_p+1,\ell}(\epsilon)^2(e^{-\mu_\ell t} - e^{-\mu_{k_p+1}(\epsilon)t})^2 \to 0 \quad \text{as } \epsilon\downarrow 0,
\end{aligned} \tag{2.78}
$$

in particular we have

$$
\lim_{\epsilon\downarrow 0}\sum_{\ell=1}^{k_p}\gamma_{k_p+1,\ell}(\epsilon)^2(e^{-\mu_\ell t} - e^{-\mu_{k_p+1}(\epsilon)t})^2 = 0. \tag{2.79}
$$

But, by Lemma 2.23, there exists $\eta > 0$ such that for all sufficiently small $\epsilon > 0$ we have

$$
\mu_{k_p+1}(\epsilon) \ge \mu_{k_p} + \eta. \tag{2.80}
$$

Thus, from (2.79) and (2.80), we have

$$
\lim_{\epsilon\downarrow 0}\sum_{\ell=1}^{k_p}\gamma_{k_p+1,\ell}(\epsilon)^2 = 0. \tag{2.81}
$$

Hence, by (A28), (2.81) and (2.76), we obtain

$$
\lim_{\epsilon\downarrow 0}\|g_{k_p+1}(\epsilon)\|_{\mathcal{H}} = 1. \tag{2.82}
$$

Therefore, by (2.75), (2.77), (2.81), (2.82) and (A28), given any $\delta > 0$, there exists $\epsilon_1 \in (0, \epsilon_0]$ such that

$$
e^{-\mu_{k_p+1}t} \ge e^{-\mu_{k_p+1}(\epsilon)t} - \delta \tag{2.83}
$$

for all $\epsilon \in (0, \epsilon_1]$. The lemma now follows from (2.74) and (2.83). $\square$

Theorem 2.25. *For all $i = 1, 2, 3, \dots,$ we have $\lim_{\epsilon\downarrow 0}\mu_i(\epsilon) = \mu_i$.*

The above theorem follows from Theorem 2.20, and Lemmas 2.22 and 2.24.

Theorem 2.26. *For all $j = 1, 2, 3, \dots$ and $\epsilon \in (0, \epsilon_0]$ let*

$$P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_j = \sum_{\ell=1}^{\infty} a_{j,\ell}(\epsilon)\varphi_\ell(\epsilon) \in \mathcal{B}(\epsilon) \subseteq \mathcal{A}(\epsilon). \tag{2.84}$$

Let $p \geq 1$ be an integer. For $i = k_p + 1, \dots, k_{p+1}$ and $\epsilon \in (0, \epsilon_0]$ let

$$\hat{\psi}_i(\epsilon) = P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\Big(\sum_{\ell=k_p+1}^{k_{p+1}} a_{i,\ell}(\epsilon)\varphi_\ell(\epsilon)\Big) \tag{2.85}$$

and let

$$\psi_i(\epsilon) = \|\hat{\psi}_i(\epsilon)\|_{\mathcal{B}(\epsilon)}^{-1}\hat{\psi}_i(\epsilon). \tag{2.86}$$

Then for each $i = k_p + 1, \dots, k_{p+1}$ we have

$$\lim_{\epsilon\downarrow 0} \|\varphi_i - \psi_i(\epsilon)\|_{\mathcal{H}} = 0. \tag{2.87}$$

Proof. For $i = k_p + 1, \dots, k_{p+1}$ and $\epsilon \in (0, \epsilon_0]$ we have

$$\begin{aligned} &\|\varphi_i - \psi_i(\epsilon)\|_{\mathcal{H}} \\ &\leq \|\varphi_i - P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_i\|_{\mathcal{H}} + \|P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_i - \hat{\psi}_i(\epsilon)\|_{\mathcal{B}(\epsilon)} + \|\hat{\psi}_i(\epsilon) - \psi_i(\epsilon)\|_{\mathcal{B}(\epsilon)} \\ &\leq \|\varphi_i - P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_i\|_{\mathcal{H}} + \Big\|P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_i - \sum_{\ell=k_p+1}^{k_{p+1}} a_{i,\ell}(\epsilon)\varphi_\ell(\epsilon)\Big\|_{\mathcal{A}(\epsilon)} \\ &\quad + \Big\|\sum_{\ell=k_p+1}^{k_{p+1}} a_{i,\ell}(\epsilon)[\varphi_\ell(\epsilon) - P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_\ell(\epsilon)]\Big\|_{\mathcal{A}(\epsilon)} + \|\hat{\psi}_i(\epsilon) - \psi_i(\epsilon)\|_{\mathcal{B}(\epsilon)}. \end{aligned} \tag{2.88}$$

Consider the term

$$\Big\|P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_i - \sum_{\ell=k_p+1}^{k_{p+1}} a_{i,\ell}(\epsilon)\varphi_\ell(\epsilon)\Big\|_{\mathcal{A}(\epsilon)}$$

in (2.88). We have

$$\begin{aligned} &e^{-H(\epsilon)t}P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_i - P_{\mathcal{H},\mathcal{B}(\epsilon)}e^{-Ht}\varphi_i \\ &= \Big(\sum_{\ell=1}^{k_p} + \sum_{\ell=k_p+1}^{k_{p+1}} + \sum_{\ell=k_{p+1}+1}^{\infty}\Big)a_{i,\ell}(\epsilon)[e^{-\mu_\ell(\epsilon)t} - e^{-\mu_i t}]\varphi_\ell(\epsilon). \end{aligned} \tag{2.89}$$

By (A28) and the orthogonality of $\{\varphi_\ell(\epsilon)\}_{\ell=1}^{\infty}$, each of the three sums in (2.89) approaches 0 as $\epsilon \downarrow 0$. Hence, together with Theorem 2.25, we have, for $i = k_p + 1, \dots, k_{p+1}$,

$$\Big(\sum_{\ell=1}^{k_p} + \sum_{\ell=k_{p+1}+1}^{\infty}\Big)a_{i,\ell}(\epsilon)^2 \to 0 \quad \text{as } \epsilon \downarrow 0. \tag{2.90}$$

Thus, for $i = k_p + 1, \dots, k_{p+1}$,

$$\begin{aligned} &\lim_{\epsilon\downarrow 0}\Big\|P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_i - \sum_{\ell=k_p+1}^{k_{p+1}} a_{i,\ell}(\epsilon)\varphi_\ell(\epsilon)\Big\|_{\mathcal{A}(\epsilon)}^2 \\ &= \lim_{\epsilon\downarrow 0}\Big(\sum_{\ell=1}^{k_p} + \sum_{\ell=k_{p+1}+1}^{\infty}\Big)a_{i,\ell}(\epsilon)^2 = 0. \end{aligned} \tag{2.91}$$

By (A9) and (A28) we have

$$\lim_{\epsilon\downarrow 0}\|\varphi_\ell(\epsilon)-P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_\ell(\epsilon)\|_{\mathcal{A}(\epsilon)}=0$$

for all $\ell=1,2,3,\dots$. Thus

$$\lim_{\epsilon\downarrow 0}\Big\|\sum_{\ell=k_p+1}^{k_{p+1}}a_{i,\ell}(\epsilon)[\varphi_\ell(\epsilon)-P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_\ell(\epsilon)]\Big\|_{\mathcal{A}(\epsilon)}=0. \tag{2.92}$$

By (A7) and (2.91) we have, for $i=k_p+1,\dots,k_{p+1}$,

$$\lim_{\epsilon\downarrow 0}\sum_{\ell=k_p+1}^{k_{p+1}}a_{i,\ell}(\epsilon)^2=1. \tag{2.93}$$

Thus, for $i=k_p+1,\dots,k_{p+1}$, (2.85), (2.92) and (2.93) imply

$$\begin{aligned}
\lim_{\epsilon\downarrow 0}\|\hat\psi_i(\epsilon)\|_{\mathcal{B}(\epsilon)}&=\lim_{\epsilon\downarrow 0}\|\hat\psi_i(\epsilon)\|_{\mathcal{A}(\epsilon)}\\
&=\lim_{\epsilon\downarrow 0}\Big\|\Big(\hat\psi_i(\epsilon)-\sum_{\ell=k_p+1}^{k_{p+1}}a_{i,\ell}(\epsilon)\varphi_\ell(\epsilon)\Big)+\sum_{\ell=k_p+1}^{k_{p+1}}a_{i,\ell}(\epsilon)\varphi_\ell(\epsilon)\Big\|_{\mathcal{A}(\epsilon)}\\
&=\lim_{\epsilon\downarrow 0}\Big\|\sum_{\ell=k_p+1}^{k_{p+1}}a_{i,\ell}(\epsilon)\varphi_\ell(\epsilon)\Big\|_{\mathcal{A}(\epsilon)}\\
&=\lim_{\epsilon\downarrow 0}\Big\{\sum_{\ell=k_p+1}^{k_{p+1}}a_{i,\ell}(\epsilon)^2\Big\}^{1/2}=1.
\end{aligned} \tag{2.94}$$

Therefore,

$$\lim_{\epsilon\downarrow 0}\|\hat\psi_i(\epsilon)-\psi_i(\epsilon)\|_{\mathcal{B}(\epsilon)}=\lim_{\epsilon\downarrow 0}|1-\|\hat\psi_i(\epsilon)\|^{-1}_{\mathcal{B}(\epsilon)}|\|\hat\psi_i(\epsilon)\|_{\mathcal{B}(\epsilon)}=0. \tag{2.95}$$

The theorem now follows from (2.88), (A7), (2.91), (2.92) and (2.95). □

3. Application to Neumann Laplacians on domains in $\mathbb{R}^N$

The purpose of this section is to show that the assumptions (A1)–(A28) in Section 2 are all satisfied when applying the abstract theory in Section 2 to the situation studied in this section. For our application, it will be easy to check that (A1)–(A27) are satisfied. So we shall show that (A28) holds for our application. Throughout this section we let $\Omega\subseteq\mathbb{R}^N$ be a bounded Sobolev extension domain. Fix a sufficiently small $\epsilon_0>0$. For each $\epsilon\in(0,\epsilon_o]$ let Ω_ϵ, Ω^ϵ and $\Omega(\epsilon)$ be bounded Sobolev extension domains in $\mathbb{R}^N$ satisfying

$$\begin{gathered}
\Omega_\epsilon\supseteq\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)>\epsilon\},\\
\Omega^\epsilon\subseteq\{x\in\mathbb{R}^N:\operatorname{dist}(x,\Omega)<\epsilon\},\\
\Omega_\epsilon\subseteq\Omega(\epsilon)\subseteq\Omega^\epsilon.
\end{gathered} \tag{3.1}$$

We shall assume that $\{\Omega_\epsilon\}_{0<\epsilon\le\epsilon_0}$ is a decreasing family of domains in the sense that

$$\Omega_{\epsilon_1}\supseteq\Omega_{\epsilon_2}\quad\text{if } 0<\epsilon_1\le\epsilon_2. \tag{3.2}$$

Similarly we shall assume that $\{\Omega^\epsilon\}_{0<\epsilon\le\epsilon_0}$ is an increasing family of domains in the sense that

$$\Omega^{\epsilon_1} \subseteq \Omega^{\epsilon_2} \quad \text{if } 0 < \epsilon_1 \le \epsilon_2. \tag{3.3}$$

We shall apply the abstract theory in Section 2 by putting:

$$\begin{gathered}\mathcal{H}^\epsilon = L^2(\Omega^\epsilon), \quad \mathcal{H}_\epsilon = L^2(\Omega_\epsilon), \quad \mathcal{H}(\epsilon) = L^2(\Omega(\epsilon)) = \mathcal{A}(\epsilon),\\ \mathcal{B}(\epsilon) = L^2(\Omega\cap\Omega(\epsilon)), \quad \mathcal{C}(\epsilon) = L^2(\Omega(\epsilon)\backslash\Omega), \quad \mathcal{D}(\epsilon) = \{0\}.\end{gathered} \tag{3.4}$$

Let $-\Delta_\epsilon$, $-\Delta$, $-\Delta^\epsilon$, $-\Delta(\epsilon)$ be the Neumann Laplacian defined on Ω_ϵ, Ω, Ω^ϵ and $\Omega(\epsilon)$, respectively. When applying the abstract theory in Section 2 we shall put

$$H_\epsilon = -\Delta_\epsilon, \quad H = -\Delta, \quad H^\epsilon = -\Delta^\epsilon, \quad H(\epsilon) = -\Delta(\epsilon). \tag{3.5}$$

We shall write $P_\epsilon(t,x,y)$, $P(t,x,y)$, $P^\epsilon(t,x,y)$ and $P(\epsilon)(t,x,y)$ for the heat kernel of $e^{\Delta_\epsilon t}$, $e^{\Delta t}$, $e^{\Delta^\epsilon t}$ and $e^{\Delta(\epsilon)t}$, respectively. We shall assume that there exists a positive continuous functions $c:(0,1]\to(0,\infty)$ such that

$$\begin{gathered}P_\epsilon(t,x,y) \le c(t) \quad (x,y\in\Omega_\epsilon),\\ P(t,x,y) \le c(t) \quad (x,y\in\Omega),\\ P^\epsilon(t,x,y) \le c(t) \quad (x,y\in\Omega^\epsilon)\\ P(\epsilon)(t,x,y) \le c(t) \quad (x,y\in\Omega(\epsilon))\end{gathered} \tag{3.6}$$

for all $0<\epsilon\le\epsilon_0$ and all $0<t\le 1$.

We shall need the parabolic Harnack inequality:

Proposition 3.1 ([6, Lemma 4.10]). *Let Σ be a domain in $\mathbb{R}^d$, let u be a solution of the parabolic equation:*

$$\frac{\partial u}{\partial t} - \omega^{-1}\sum_{i,j=1}^d \{\frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j})\} = 0$$

in $\Sigma\times(\tau_1,\tau_2)$, where ω and $\{a_{ij}\}$ satisfy

$$\begin{gathered}0<\lambda^{-1}\le\omega(x)\le\lambda<\infty \quad (x\in\Sigma),\\ 0<\lambda^{-1}\le\{a_{ij}(x)\}\le\lambda<\infty \quad (x\in\Sigma),\end{gathered}$$

for some $\lambda\ge 1$. Let Σ' be a subdomain of Σ and suppose that $\operatorname{dist}(\Sigma',\partial\Sigma)>\eta$ *and $t_1-\tau_1\ge\eta^2$. Then*

$$|u(x,t)-u(y,s)| \le A[|x-y|+|t-s|^{1/2}]^\alpha$$

for all $x,y\in\Sigma'$ and $t,s\in[t_1,\tau_2)$, where $\alpha\in(0,1]$ depends only on d and λ, and

$$A = (\frac{4}{\eta})^\alpha\theta$$

where θ is the oscillation of u in $\Sigma\times(\tau_1,\tau_2)$.

Theorem 3.2. *We have* $\lim_{\epsilon\downarrow 0} P(\epsilon)(t,x,y) = P(t,x,y)$ *for all $t\in(0,1]$ and $x,y\in\Omega$.*

Proof. Suppose, for a contradiction, that for some $t_0\in(0,1]$ and some $x_0,y_0\in\Sigma$ we have

$$P(\epsilon)(t_o,x_0,y_0) \not\to P(t_o,x_o,y_o) \quad \text{as } \epsilon\downarrow 0. \tag{3.7}$$

Then there exist $c_1 \geq 1$ and a decreasing sequence $\{\epsilon_n\}_{n=1}^{\infty}$ of positive numbers such that $\epsilon_n \downarrow 0$ as $n \to \infty$ and that

$$c_1^{-1} \leq |P(\epsilon_n)(t_0, x_0, y_0) - P(t_0, x_0, y_0)| \quad (n = 1, 2, 3, \dots). \tag{3.8}$$

Applying Proposition 3.1 with

$$\begin{gathered} \Sigma = B\big(x_0, \frac{5}{8}\operatorname{dist}(x_0, \partial\Omega)\big), \quad \Sigma' = B\big(x_0, \frac{1}{8}\operatorname{dist}(x_0, \partial\Omega)\big), \\ u(t,x) = P(t, x, y_0), \quad \lambda = 1, \quad \tau_1 = \frac{1}{4}t_0, \quad \tau_2 = 1, \quad t_1 = \frac{1}{2}t_0, \\ \eta = \min\big\{\frac{3}{8}\operatorname{dist}(x_0, \partial\Omega), \frac{1}{2}t_0^{1/2}\big\}, \end{gathered} \tag{3.9}$$

we obtain, for all $s, t \in (t_1, \tau_2) = (t_0/2, 1)$ and all $x \in B(x_0, \operatorname{dist}(x_0, \partial\Omega)/8)$,

$$|P(t, x, y_0) - P(s, x_0, y_0)| \leq A[|x - x_0| + |t - s|^{1/2}]^{\alpha} \tag{3.10}$$

where $\alpha \in (0, 1]$ depends only on N and

$$A = \big(\frac{4}{\eta}\big)^{\alpha}\theta \tag{3.11}$$

where

$$\theta = \sup_{\frac{1}{4}t_0 \leq t \leq 1} c(t). \tag{3.12}$$

(Hence A depends only on N, $\operatorname{dist}(x_0, \partial\Omega)$ and t_0.) We may assume that, for all $n = 1, 2, 3, \dots$, we have

$$0 < \epsilon_n < \min\{\frac{3}{8}\operatorname{dist}(x_0, \partial\Omega), \frac{3}{8}\operatorname{dist}(y_0, \partial\Omega)\}.$$

By a similar argument we deduce that

$$|P(\epsilon_n)(t, x, y_0) - P(\epsilon_n)(s, x_0, y_0)| \leq A[|x - x_0| + |t - s|^{1/2}]^{\alpha} \tag{3.13}$$

for all $s, t \in (t_1, \tau_2) = (t_0/2, 1)$, all $x \in B(x_0, \operatorname{dist}(x_0, \partial\Omega)/8)$ and all $n = 1, 2, 3, \dots$, and where α and A in (3.13) have the same values as those in (3.10). Let

$$R = \min\big\{(4Ac_1)^{-\frac{1}{\alpha}}, \frac{1}{8}\operatorname{dist}(x_0, \partial\Omega)\big\}. \tag{3.14}$$

Then, for all $x \in B(x_0, R)$, $t \in (t_0/2, 1)$ and $n = 1, 2, 3, \dots$, we have

$$|P(t, x, y_0) - P(t, x_0, y_0)| \leq (4c_1)^{-1}, \tag{3.15}$$

$$|P(\epsilon_n)(t, x, y_0) - P(\epsilon_n)(t, x_0, y_0)| \leq (4c_1)^{-1}. \tag{3.16}$$

For $x \in B(x_0, R)$ and $n = 1, 2, 3, \dots$ we have

$$\begin{aligned} &|P(\epsilon_n)(t_0, x_0, y_0) - P(t_0, x_0, y_0)| \\ &\leq |P(\epsilon_n)(t_0, x_0, y_0) - P(\epsilon_n)(t_0, x, y_0)| + |P(\epsilon_n)(t_0, x, y_0) - P(t_0, x, y_0)| \\ &\quad + |P(t_0, x, y_0) - P(t_0, x_0, y_0)|. \end{aligned} \tag{3.17}$$

So, by (3.8), (3.15), (3.16) and (3.17), we obtain

$$\frac{1}{2}c_1^{-1} \leq |P(\epsilon_n)(t_0, x, y_0) - P(t_0, x, y_0)| \tag{3.18}$$

for all $x \in B(x_0, R)$ and $n = 1, 2, 3, \dots$. Integrating (3.18) over $B(x_0, R)$ we obtain

$$\begin{aligned} &\frac{1}{2} c_1^{-1} |B(x_0, R)| \\ &\leq \Big| \int_\Omega P(t_0, x, y_0) 1_{B(x_0,R)}(x)\, dx - \int_{\Omega(\epsilon_n)} P(\epsilon_n)(t_0, x, y_0) 1_{B(x_0,R)}(x)\, dx \Big|. \end{aligned} \tag{3.19}$$

Put

$$u(t, y) = \int_\Omega P(t, x, y) 1_{B(x_0,R)}(x)\, dx$$

and, for $n = 1, 2, 3, \dots$, put

$$u_n(t, y) = \int_{\Omega(\epsilon_n)} P(\epsilon_n)(t, x, y) 1_{B(x_0,R)}(x)\, dx.$$

Then $u(t, y)$ and $u_n(t, y)$ satisfy the parabolic equations

$$\begin{gathered} \frac{\partial u}{\partial t} = \Delta u \quad \text{in } (0, 1) \times \Omega, \\ \frac{\partial u_n}{\partial t} = \Delta u_n \quad \text{in } (0, 1) \times \Omega(\epsilon_n), \end{gathered}$$

respectively. So we can apply the parabolic Harnack inequality (Proposition 3.1) to $u(t, y)$ and $u_n(t, y)$ and, as in (3.10) and (3.13), obtain

$$|u(t, y) - u(s, y_0)| \leq \tilde{A}[|y - y_0| + |t - s|^{1/2}|^{\tilde{\alpha}} \tag{3.20}$$

for all $y \in B(y_0, \operatorname{dist}(y_0, \partial\Omega)/8)$ and $t, s \in (t_0/2, 1)$, and

$$|u_n(t, y) - u_n(s, y_0)| \leq \tilde{A}[|y - y_0| + |t - s|^{1/2}]^{\tilde{\alpha}} \tag{3.21}$$

for all $y \in B(y_0, \operatorname{dist}(y_0, \partial\Omega)/8)$ and $t, s \in (t_0/2, 1)$ where $\tilde{\alpha} \in (0, 1]$ depends only on N and

$$\tilde{A} = \big(\frac{4}{\tilde{\eta}}\big)^{\tilde{\alpha}} \tilde{\theta} \leq \big(\frac{4}{\tilde{\eta}}\big)^{\tilde{\alpha}}$$

where

$$\tilde{\eta} = \min\{\frac{3}{8} \operatorname{dist}(y_0, \partial\Omega), \frac{1}{2} t_0^{1/2}\}$$

and

$$\begin{aligned} \tilde{\theta} &= \sup\big\{\frac{1}{4} t_0 \leq t \leq 1, |y - y_0| \leq \frac{5}{8} \operatorname{dist}(y_0, \partial\Omega) : u(t, y)\big\} \\ &\leq \sup\big\{\frac{1}{4} t_0 \leq t \leq 1,\ |y - y_0| \leq \frac{5}{8} \operatorname{dist}(y_0, \partial\Omega) : \int_\Omega P(t, x, y)\, dx\big\} \leq 1. \end{aligned}$$

(Hence $\tilde{A}$ depends only on N, t_0 and $\operatorname{dist}(y_0, \partial\Omega)$.) Let

$$\tilde{R} = \min\big\{\big[\frac{1}{8} |B(x_0, R)| c_1^{-1} \tilde{A}^{-1}\big]^{\frac{1}{\tilde{\alpha}}}, \frac{1}{8} \operatorname{dist}(y_0, \partial\Omega)\big\}.$$

Then, by (3.20) and (3.21),

$$|u(t_0, y) - u(t_0, y_0)| \leq \frac{1}{8} |B(x_0, R)| c_1^{-1}, \tag{3.22}$$

$$|u_n(t_0, y) - u_n(t_0, y_0)| \leq \frac{1}{8} |B(x_0, R)| c_1^{-1} \tag{3.23}$$

for all $y \in B(y_0, \tilde{R})$. Thus, for all $y \in B(y_0, \tilde{R})$, we have

$$\begin{aligned} &|u_n(t_0, y_0) - u(t_0, y_0)| \\ &\le |u_n(t_0, y_0) - u_n(t_0, y)| + |u_n(t_0, y) - u(t_0, y)| + |u(t_0, y) - u(t_0, y_0)| \\ &\le \frac{1}{4}|B(x_0, R)|c_1^{-1} + |u_n(t_0, y) - u(t_0, y)|. \end{aligned} \tag{3.24}$$

So, by (3.19) and (3.24), we have

$$\frac{1}{4}|B(x_0, R)|c_1^{-1} \le |u_n(t_0, y) - u(t_0, y)| \tag{3.25}$$

for all $y \in B(y_0, \tilde{R})$. But

$$u_n(t_0, y) = \int_{\Omega(\epsilon_n)} P(\epsilon_n)(t_0, x, y) 1_{B(x_0,R)}(x)\, dx = (e^{-H(\epsilon_n)t_0}\, 1_{B(x_0,R)})(y)$$

and

$$u(t_0, y) = \int_{\Omega} P(t_0, x, y) 1_{B(x_0,R)}(x)\, dx = (e^{-Ht_0}\, 1_{B(x_0,R)})(y).$$

Thus (3.25) implies

$$\begin{aligned} &\int_{B(y_0,\tilde{R})} |(e^{-H(\epsilon_n)t_0} 1_{B(x_0,R)})(y) - (e^{-Ht_0} 1_{B(x_0,R)})(y)|^2\, dy \\ &\ge \frac{1}{16} c_1^{-2} |B(x_0, R)|^2 |B(y_0, \tilde{R})|, \end{aligned}$$

hence, for all $n = 1, 2, 3, \dots$,

$$\begin{aligned} &\|e^{-H(\epsilon_n)t_0} 1_{B(x_0,R)} - e^{-Ht_0} 1_{B(x_0,R)}\|^2_{L^2(B(y_0,\tilde{R}))} \\ &\ge \frac{1}{16} c_1^{-2} |B(x_0, R)|^2 |B(y_0, \tilde{R})|. \end{aligned} \tag{3.26}$$

Let $f \in L^2(\Omega^{\epsilon_0})$ be the function defined by

$$f(y) = \begin{cases} 1 & \text{if } |y - x_0| < R \\ 0 & \text{if } y \in \Omega^{\epsilon_0} \text{ and } |y - x_0| \ge R. \end{cases}$$

By Theorem 2.17 we have, for all $n = 1, 2, 3, \dots$,

$$\lim_{\epsilon \downarrow 0} \|e^{-\hat{H}(\epsilon)t_0} f - e^{-\hat{H}t_0} f\|_{L^2(B(y_0,\tilde{R}))} \le \lim_{\epsilon \downarrow 0} \|e^{-\hat{H}(\epsilon)t_0} f - e^{-\hat{H}t_0} f\|_{L^2(\Omega^{\epsilon_0})} = 0,$$

thus

$$\lim_{\epsilon \downarrow 0} \int_{B(y_0,\tilde{R})} |(e^{-\tilde{H}(\epsilon)t_0} f)(y) - (e^{-\hat{H}t_0} f)(y)|^2\, dy = 0$$

and hence

$$\lim_{\epsilon \downarrow 0} \int_{B(y_0,\tilde{R})} |(e^{-H(\epsilon)t_0} 1_{B(x_0,R)})(y) - (e^{-Ht} 1_{B(x_0,R)})(y)|^2\, dy = 0.$$

But this implies

$$\lim_{n \to \infty} \|e^{-H(\epsilon_n)t_0} 1_{B(x_0,R)} - e^{-Ht_0} 1_{B(x_0,R)}\|^2_{L^2(B(y_0,\tilde{R}))} = 0$$

which contradicts (3.26). Therefore assumption (3.7) must be false and the theorem is proved. $\square$

We now show that the first equality of the assumption (A28) is satisfied. We note that for any $f \in \mathcal{H} = L^2(\Omega)$.

$$P_{\mathcal{H},\mathcal{B}(\epsilon)} f = P_{L^2(\Omega),L^2(\Omega\cap\Omega(\epsilon))} f = 1_{\Omega\cap\Omega(\epsilon)} f.$$

Thus for all $\epsilon \in (0,\epsilon_0]$, $x \in \Omega(\epsilon)$ and $n = 1,2,3,\dots$,

$$\begin{aligned}
&[P_{\mathcal{H},\mathcal{B}(\epsilon)} e^{-Ht}\varphi_n - e^{-H(\epsilon)t}P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_n](x)\\
&= 1_{\Omega\cap\Omega(\epsilon)}(x)\int_\Omega P(t,x,y)\varphi_n(y)\,dy - \int_{\Omega(\epsilon)} P(\epsilon)(t,x,y)1_{\Omega\cap\Omega(\epsilon)}(y)\varphi_n(y)\,dy\\
&= 1_{\Omega\cap\Omega(\epsilon)}(x)\Big(\int_{\Omega\setminus\Omega(\epsilon)} + \int_{\Omega\cap\Omega(\epsilon)}\Big) P(t,x,y)\varphi_n(y)\,dy\\
&\quad - \Big(\int_{\Omega(\epsilon)\setminus\Omega} + \int_{\Omega\cap\Omega(\epsilon)}\Big) P(\epsilon)(t,x,y)1_{\Omega\cap\Omega(\epsilon)}(y)\varphi_n(y)\,dy\\
&= 1_{\Omega\cap\Omega(\epsilon)}(x)\int_{\Omega\setminus\Omega(\epsilon)} P(t,x,y)\varphi_n(y)\,dy\\
&+ \int_{\Omega\cap\Omega(\epsilon)} 1_{\Omega\cap\Omega(\epsilon)}(x)P(t,x,y)\varphi_n(y) - P(\epsilon)(t,x,y)1_{\Omega\cap\Omega(\epsilon)}(y)\varphi_n(y)\,dy
\end{aligned} \tag{3.27}$$

since

$$\int_{\Omega(\epsilon)\setminus\Omega} P(\epsilon)(t,x,y)1_{\Omega\cap\Omega(\epsilon)}(y)\varphi_n(y)\,dy = 0.$$

Let B be a ball such that $B \subseteq \Omega_\epsilon$ ($0 < \epsilon \le \epsilon_0$). For each $k = 1,2,3,\dots$ let $\lambda_k(\epsilon)$ and $\lambda_k(B)$ be the k-th eigenvalue of the Dirichlet Laplacian defined on $\Omega(\epsilon)$ and B, respectively. Then, by min-max,

$$\mu_k(\epsilon) \le \lambda_k(\epsilon) \le \lambda_k(B). \tag{3.28}$$

By the assumption (3.6), we have

$$\sum_{k=1}^\infty e^{-\mu_k(\epsilon)t}\varphi_k(\epsilon)(x)^2 \le c(t)$$

for all $0 < t \le 1$, $0 < \epsilon \le \epsilon_0$ and $x \in \Omega(\epsilon)$. Hence

$$|\varphi_k(\epsilon)(x)| \le [c(t)e^{\mu_k(\epsilon)t}]^{1/2} \le c(t)^{1/2}e^{\lambda_k(B)t/2} \tag{3.29}$$

for all $0 < \epsilon \le \epsilon_0$, $x \in \Omega(\epsilon)$, $0 < t \le 1$ and $k = 1,2,3,\dots$. Similarly we have

$$|\varphi_k(x)| \le c(t)^{1/2}e^{\lambda_k(B)t/2} \tag{3.30}$$

for all $0 < t \le 1$, $x \in \Omega$ and $k = 1,2,3,\dots$. Since

$$|\Omega\setminus\Omega(\epsilon)| \le |\Omega\setminus\Omega_\epsilon| \to 0 \quad \text{as } \epsilon \downarrow 0,$$

we have, for all $0 < t \le 1$,

$$|1_{\Omega\cap\Omega(\epsilon)}(x)\int_{\Omega\setminus\Omega(\epsilon)} P(t,x,y)\varphi_n(y)\,dy| \le c(t)|\Omega\setminus\Omega(\epsilon)|c(1)^{1/2}e^{\lambda_n(B)/2} \to 0 \quad \text{as } \epsilon \downarrow 0\,. \tag{3.31}$$

Next we consider the term

$$\int_{\Omega\cap\Omega(\epsilon)} [1_{\Omega\cap\Omega(\epsilon)}(x)P(t,x,y) - P(\epsilon)(t,x,y)]\varphi_n(y)\,dy$$

in (3.27). Let $0 < \epsilon_1 < \epsilon_0$. For $0 < \epsilon \le \epsilon_1$, $x \in \Omega \cap \Omega(\epsilon)$ and $n = 1, 2, 3, \dots$, by (3.30), we have

$$\begin{aligned}
&\Big| \int_{\Omega\cap\Omega(\epsilon)} [1_{\Omega\cap\Omega(\epsilon)}(x) P(t,x,y) - P(\epsilon)(t,x,y)] \varphi_n(y)\, dy \Big| \\
&\le \int_{\Omega\cap\Omega(\epsilon)} |P(t,x,y) - P(\epsilon)(t,x,y)| |\varphi_n(y)|\, dy \\
&\le c(1)^{1/2} e^{\lambda_n(B)/2} \int_{\Omega\cap\Omega(\epsilon)} |P(t,x,y) - P(\epsilon)(t,x,y)|\, dy \\
&\le c(1)^{1/2} e^{\lambda_n(B)/2} \Big\{ \int_{(\Omega\cap\Omega(\epsilon))\backslash\Omega_{\epsilon_1}} |P(t,x,y) - P(\epsilon)(t,x,y)|\, dy \\
&\quad + \int_{\Omega_{\epsilon_1}} |P(t,x,y) - P(\epsilon)(t,x,y)|\, dy \Big\} \\
&\le c(1)^{1/2} e^{\lambda_n(B)/2} \{ |\Omega^{\epsilon_1} \backslash \Omega_{\epsilon_1}| 2c(t) + \int_{\Omega_{\epsilon_1}} |P(t,x,y) - P(\epsilon)(t,x,y)|\, dy \Big\}.
\end{aligned} \tag{3.32}$$

For $x \in \Omega(\epsilon) \backslash \Omega$, $0 < t \le 1$ and $n = 1, 2, 3, \dots$, we have

$$\begin{aligned}
&\Big| \int_{\Omega\cap\Omega(\epsilon)} [1_{\Omega\cap\Omega(\epsilon)}(x) P(t,x,y) - P(\epsilon)(t,x,y)] \varphi_n(y)\, dy \Big| \\
&= \int_{\Omega\cap\Omega(\epsilon)} P(\epsilon)(t,x,y) \varphi_n(y)\, dy \\
&\le |\Omega \cap \Omega(\epsilon)| c(t) c(1)^{1/2} e^{\lambda_n(B)/2}.
\end{aligned} \tag{3.33}$$

For every $0 < t \le 1$, $0 < \epsilon \le \epsilon_0$ and $n = 1, 2, 3, \dots$,

$$\begin{aligned}
&\int_{\Omega(\epsilon)} |(P_{\mathcal{H},\mathcal{B}(\epsilon)} e^{-Ht} \varphi_n - e^{-H(\epsilon)t} P_{\mathcal{H},\mathcal{B}(\epsilon)} \varphi_n)(x)|^2\, dx \\
&= \int_{\Omega(\epsilon)\backslash\Omega} |(P_{\mathcal{H},\mathcal{B}(\epsilon)} e^{-Ht} \varphi_n - e^{-H(\epsilon)t} P_{\mathcal{H},\mathcal{B}(\epsilon)} \varphi_n)(x)|^2\, dx \\
&\quad + \int_{\Omega(\epsilon)\cap\Omega} |(P_{\mathcal{H},\mathcal{B}(\epsilon)} e^{-Ht} \varphi_n - e^{-H(\epsilon)t} P_{\mathcal{H},\mathcal{B}(\epsilon)} \varphi_n)(x)|^2\, dx.
\end{aligned} \tag{3.34}$$

For the first term on the right side of (3.34) we have, by (3.27), (3.30) and (3.33),

$$\begin{aligned}
&\int_{\Omega(\epsilon)\backslash\Omega} |(P_{\mathcal{H},\mathcal{B}(\epsilon)} e^{-Ht} \varphi_n - e^{-H(\epsilon)t} P_{\mathcal{H},\mathcal{B}(\epsilon)} \varphi_n)(x)|^2\, dx \\
&\le |\Omega^{\epsilon} \backslash \Omega| \{ |\Omega \cap \Omega(\epsilon)| c(t) c(1)^{1/2} e^{\lambda_n(B)/2} \}^2 \to 0 \quad \text{as } \epsilon \downarrow 0.
\end{aligned} \tag{3.35}$$

Next we consider the second term on the right side of (3.34). For $t \in (0, 1]$, $n = 1, 2, 3, \dots$ and $\epsilon \in (0, \epsilon_0]$ let $F_{t,n,\epsilon} : \Omega \to \mathbb{R}$ be defined by

$$F_{t,n,\epsilon}(x) = \begin{cases} 0 & \text{if } x \in \Omega \backslash \Omega(\epsilon) \\ |(P_{\mathcal{H},\mathcal{B}(\epsilon)} e^{-Ht} \varphi_n - e^{-H(\epsilon)t} P_{\mathcal{H},\mathcal{B}(\epsilon)} \varphi_n)(x)|^2 & \text{if } x \in \Omega \cap \Omega(\epsilon). \end{cases} \tag{3.36}$$

If $0 < \epsilon \le \epsilon_1 \le \epsilon_0$ and $x \in \Omega \cap \Omega(\epsilon)$, then

$$\begin{aligned}
F_{t,n,\epsilon}(x) &\le \{ |\Omega \backslash \Omega(\epsilon)| c(t) c(1)^{1/2} e^{\lambda_n(B)/2} + c(1)^{1/2} e^{\lambda_n(B)/2} [|\Omega \backslash \Omega_{\epsilon_1}| 2c(t) \\
&\quad + \int_{\Omega_{\epsilon_1}} |P(t,x,y) - P(\epsilon)(t,x,y)|\, dy] \}^2.
\end{aligned} \tag{3.37}$$

For fixed $t \in (0,1]$, $n = 1,2,3,\dots$ and $x \in \Omega$, given any $\delta > 0$ we can first choose $\epsilon_1 \in (0, \epsilon_0]$ such that

$$c(1)^{1/2} e^{\lambda_n(B)/2} |\Omega \backslash \Omega_{\epsilon_1}| 2c(t) \le \delta/2,$$

then, by Theorem 3.2, we can choose $\epsilon_2 \in (0, \epsilon_1]$ such that

$$c(1)^{1/2} e^{\lambda_n(B)/2} \int_{\Omega_{\epsilon_1}} |P(t,x,y) - P(\epsilon)(t,x,y)|\, dy \le \delta/2$$

for all $\epsilon \in (0, \epsilon_2]$. Thus for all $t \in (0,1]$, $n = 1,2,3, \dots$ and $x \in \Omega$, we have

$$\lim_{\epsilon \downarrow 0} F_{t,n,\epsilon}(x) = 0. \tag{3.38}$$

From (3.37) we see that

$$F_{t,n,\epsilon}(x) \le [c(1)^{1/2} e^{\lambda_n(B)/2} 5c(t)|\Omega|]^2 \tag{3.39}$$

for all $t \in (0,1]$, $n = 1,2,3,\dots$, $\epsilon \in (0, \epsilon_0]$ and $x \in \Omega$. Therefore, by (3.38), (3.39) and the dominated convergence theorem, we have

$$\begin{aligned} &\lim_{\epsilon \downarrow 0} \int_{\Omega \cap \Omega(\epsilon)} |(P_{\mathcal{H},\mathcal{B}(\epsilon)} e^{-Ht} \varphi_n - e^{-H(\epsilon)t} P_{\mathcal{H},\mathcal{B}(\epsilon)} \varphi_n)(x)|^2\, dx \\ &= \lim_{\epsilon \downarrow 0} \int_{\Omega} F_{t,n,\epsilon}(x)\, dx = 0 \end{aligned} \tag{3.40}$$

for all $t \in (0,1]$ and $n = 1,2,3,\dots$. So

$$\begin{aligned} &\|P_{\mathcal{H},\mathcal{B}(\epsilon)} e^{-Ht} \varphi_n - e^{-H(\epsilon)t} P_{\mathcal{H},\mathcal{B}(\epsilon)} \varphi_n\|^2_{\mathcal{A}(\epsilon)} \\ &= \int_{\Omega(\epsilon)} |(P_{\mathcal{H},\mathcal{B}(\epsilon)} e^{-Ht} \varphi_n - e^{-H(\epsilon)t} P_{\mathcal{H},\mathcal{B}(\epsilon)} \varphi_n)(x)|^2\, dx \\ &= \int_{\Omega(\epsilon) \backslash \Omega} |(P_{\mathcal{H},\mathcal{B}(\epsilon)} e^{-Ht} \varphi_n - e^{-H(\epsilon)t} P_{\mathcal{H},\mathcal{B}(\epsilon)} \varphi_n)(x)|^2\, dx \\ &\quad + \int_{\Omega(\epsilon) \cap \Omega} |(P_{\mathcal{H},\mathcal{B}(\epsilon)} e^{-Ht} \varphi_n - e^{-H(\epsilon)t} P_{\mathcal{H},\mathcal{B}(\epsilon)} \varphi_n)(x)|^2\, dx \\ &\le |\Omega(\epsilon) \backslash \Omega| \{ |\Omega \cap \Omega(\epsilon)| c(t) c(1)^{1/2} e^{\lambda_n(B)/2} \}^2 \\ &\quad + \int_{\Omega(\epsilon) \cap \Omega} |(P_{\mathcal{H},\mathcal{B}(\epsilon)} e^{-Ht} \varphi_n - e^{-H(\epsilon)t} P_{\mathcal{H},\mathcal{B}(\epsilon)} \varphi_n)(x)|^2\, dx \\ &\to 0 \quad \text{as } \epsilon \downarrow 0 \end{aligned} \tag{3.41}$$

where we have used (3.27), (3.33) and (3.40). Hence the first equality in (A28) holds in this application.

We next consider the second equality in (A28). For $x \in \Omega$, $t \in (0,1]$ and $n = 1,2,3,\dots$ we have

$$\begin{aligned} &(e^{-Ht} P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)} \varphi_n(\epsilon) - P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)} e^{-H(\epsilon)t} \varphi_n(\epsilon))(x) \\ &= \int_{\Omega \cap \Omega(\epsilon)} P(t,x,y) \varphi_n(\epsilon)(y)\, dy - 1_{\Omega \cap \Omega(\epsilon)}(x) f_{t,n,\epsilon}(x) \end{aligned} \tag{3.42}$$

where

$$f_{t,n,\epsilon}(x) = \begin{cases} 0 & \text{if } x \in \Omega \backslash \Omega(\epsilon) \\ \int_{\Omega(\epsilon)} P(\epsilon)(t,x,y) \varphi_n(\epsilon)(y)\, dy & \text{if } x \in \Omega \cap \Omega(\epsilon). \end{cases} \tag{3.43}$$

So for $x \in \Omega\backslash\Omega(\epsilon)$, $t \in (0,1]$, $n = 1,2,3,\dots$ and $\epsilon \in (0,\epsilon_0]$, by (3.30), we have

$$\begin{aligned} &|(e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_n(\epsilon) - P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}e^{-H(\epsilon)t}\varphi_n(\epsilon))(x)| \\ &= \Big| \int_{\Omega\cap\Omega(\epsilon)} P(t,x,y)\varphi_n(\epsilon)(y)\,dy\Big| \\ &\le |\Omega\cap\Omega(\epsilon)|c(t)c(1)^{1/2}e^{\lambda_n(B)/2}. \end{aligned} \tag{3.44}$$

For $t \in (0,1]$, $n = 1,2,3,\dots$ and $\epsilon \in (0,\epsilon_0]$ we define $G_{t,n,\epsilon} : \Omega \to \mathbb{R}$ by

$$G_{t,n,\epsilon}(x) = \begin{cases} 0 & \text{if } x \in \Omega\backslash\Omega(\epsilon) \\ |(e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_n(\epsilon) - P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}e^{-H(\epsilon)t}\varphi_n(\epsilon))(x)| & \text{if } x \in \Omega\cap\Omega(\epsilon). \end{cases} \tag{3.45}$$

If $0 < \epsilon \le \epsilon_1 \le \epsilon_0$ and $x \in \Omega\cap\Omega(\epsilon)$, then

$$\begin{aligned} G_{t,n,\epsilon}(x) &= \Big| \int_{\Omega\cap\Omega(\epsilon)} [P(t,x,y) - P(\epsilon)(t,x,y)]\varphi_n(\epsilon)(y)\,dy \\ &\quad - \int_{\Omega(\epsilon)\backslash\Omega} P(\epsilon)(t,x,y)\varphi_n(\epsilon)(y)\,dy\Big| \\ &\le \Big| \int_{(\Omega\cap\Omega(\epsilon))\backslash\Omega_{\epsilon_1}} [P(t,x,y) - P(\epsilon)(t,x,y)]\varphi_n(\epsilon)(y)\,dy\Big| \\ &\quad + \Big| \int_{\Omega_{\epsilon_1}} [P(t,x,y) - P(\epsilon)(t,x,y)]\varphi_n(\epsilon)(y)\,dy\Big| \\ &\quad + \Big| \int_{\Omega(\epsilon)\backslash\Omega} P(\epsilon)(t,x,y)\varphi_n(\epsilon)(y)\,dy\Big| \\ &\le c(1)^{1/2}e^{\lambda_n(B)/2}\{|\Omega^{\epsilon_1}\backslash\Omega_{\epsilon_1}|3c(t) \\ &\quad + \int_{\Omega_{\epsilon_1}} |P(t,x,y) - P(\epsilon)(t,x,y)|\,dy\}. \end{aligned} \tag{3.46}$$

Thus, for fixed $t \in (0,1]$, $n = 1,2,3,\dots$ and $x \in \Omega$, given any $\delta > 0$ we can first choose $\epsilon_1 \in (0,\epsilon_0]$ such that

$$c(1)^{1/2}e^{\lambda_n(B)/2}|\Omega^{\epsilon_1}\backslash\Omega_{\epsilon_1}|3c(t) \le \delta/2,$$

then, by Theorem 3.2, we can find $\epsilon_2 \in (0,\epsilon_1]$ such that

$$c(1)^{1/2}e^{\lambda_n(B)/2}\int_{\Omega_{\epsilon_1}} |P(t,x,y) - P(\epsilon)(t,x,y)|\,dy \le \delta/2$$

for all $\epsilon \in (0,\epsilon_2]$. Therefore, for all $t \in (0,1]$, $n = 1,2,3,\dots$ and $x \in \Omega$, we have

$$\lim_{\epsilon\downarrow 0} G_{t,n,\epsilon}(x) = 0. \tag{3.47}$$

Also, from (3.46), we have, for all $t \in (0,1]$, $n = 1,2,3,\dots$, $\epsilon \in (0,\epsilon_0]$ and $x \in \Omega$,

$$G_{t,n,\epsilon}(x) \le c(1)^{1/2}e^{\lambda_n(B)/2}5c(t)|\Omega^{\epsilon_0}|. \tag{3.48}$$

Hence we have

$$\begin{aligned} &\lim_{\epsilon\downarrow 0}\int_{\Omega\cap\Omega(\epsilon)} |(e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_n(\epsilon) - P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}e^{-H(\epsilon)t}\varphi_n(\epsilon))(x)|^2\,dx \\ &= \lim_{\epsilon\downarrow 0}\int_{\Omega} G_{t,n,\epsilon}(x)^2\,dx = 0 \end{aligned} \tag{3.49}$$

for all $t \in (0,1]$ and $n = 1,2,3,\dots$, using (3.45), (3.47), (3.48) and the dominated convergence theorem. So, for all $t \in (0,1]$ and $n = 1,2,3,\dots$, we have

$$
\begin{aligned}
&\|e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_n(\epsilon) - P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}e^{-H(\epsilon)t}\varphi_n(\epsilon)\|_{\mathcal{H}}^2 \\
&= \int_\Omega |(e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_n(\epsilon) - P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}e^{-H(\epsilon)t}\varphi_n(\epsilon))(x)|^2\,dx \\
&= \Big(\int_{\Omega\setminus\Omega(\epsilon)} + \int_{\Omega\cap\Omega(\epsilon)}\Big)|(e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_n(\epsilon) - P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}e^{-H(\epsilon)t}\varphi_n(\epsilon))(x)|^2\,dx \\
&\le |\Omega\setminus\Omega(\epsilon)|(|\Omega\cap\Omega(\epsilon)|c(t)c(1)^{1/2}e^{\lambda_n(B)/2})^2 \\
&\quad + \int_{\Omega\cap\Omega(\epsilon)} |(e^{-Ht}P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_n(\epsilon) - P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}e^{-H(\epsilon)t}\varphi_n(\epsilon))(x)|^2\,dx \\
&\to 0 \quad \text{as } \epsilon \downarrow 0
\end{aligned}
$$

where we have used (3.44) and (3.49). Hence the second equality of (A28) holds in this application.

Finally we consider the third equality in (A28). For $\epsilon \in (0,\epsilon_0]$ and $n = 1,2,3,\dots$ we have

$$\varphi_n(\epsilon) = 1_{\Omega(\epsilon)\setminus\Omega}\varphi_n(\epsilon) + 1_{\Omega(\epsilon)\cap\Omega}\varphi_n(\epsilon).$$

By (3.30) we have

$$\int_{\Omega(\epsilon)\setminus\Omega} |\varphi_n(\epsilon)(x)|^2\,dx \le |\Omega^\epsilon\setminus\Omega|c(1)e^{\lambda_n(B)} \to 0 \quad \text{as } \epsilon \downarrow 0.$$

Since $\|\varphi_n(\epsilon)\|_{\mathcal{A}(\epsilon)}^2 = 1$, we must have

$$\|P_{\mathcal{A}(\epsilon),\mathcal{B}(\epsilon)}\varphi_n(\epsilon)\|_{\mathcal{H}}^2 = \int_{\Omega\cap\Omega(\epsilon)} |\varphi_n(\epsilon)(x)|^2\,dx \to 1 \quad \text{as } \epsilon \downarrow 0,$$

hence the third equality of (A28) holds in this application.

Theorem 3.3. *We use the notation in Section 2. In particular, we shall use the notation in Definition 2.21 and Theorem 2.26. Let K be a compact subset of Ω. Then we have*

$$\lim_{\epsilon\downarrow 0}\big\{\sup_{x\in K}|\varphi_i(x) - \psi_i(\epsilon)(x)|\big\} = 0 \tag{3.50}$$

for all $i = 1,2,3,\dots$.

Proof. We need to consider only $i \ge 2$. Let $p \ge 1$ be an integer and let

$$k_p + 1 \le i \le k_{p+1}.$$

We assume, for a contradiction, that (3.50) is false. Then there exist $\delta > 0$ and a decreasing sequence $\{\epsilon_m\}_{m=1}^\infty$ in $(0,\epsilon_0/2]$, with $\lim_{m\to\infty}\epsilon_m = 0$, and a sequence of points $\{x_m\}_{m=1}^\infty$ in K such that

$$|\varphi_i(x_m) - \psi_i(\epsilon_m)(x_m)| \ge \delta \quad (m = 1,2,3,\dots). \tag{3.51}$$

We can choose $\hat\epsilon \in (0,\min\{1,\epsilon_0\})$ such that

$$D = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > \hat\epsilon\} \supseteq K$$

and that, by Theorem 2.25,

$$|\mu_\ell(\epsilon) - \mu_i| \le 1 \tag{3.52}$$

for all $\ell = k_p+1, \dots, k_{p+1}$ and $\epsilon \in (0, \hat\epsilon]$. Applying the parabolic Harnack inequality, Proposition 3.1, with $\Sigma = \Omega$ or $\Sigma = \Omega(\epsilon)$ $(0 < \epsilon \le \frac{1}{2}\hat\epsilon)$, $\Sigma' = D$, $\omega = 1$, $a_{ij} = \delta_{ij}$, $\tau_1 = 1$, $\tau_2 = 2$, $t_1 = 5/4$, $\eta = \frac{1}{2}\hat\epsilon$ and

$$u(x,t) = e^{-\mu_i t}\varphi_i(x) \quad (1 < t < 2, x \in \Omega) \tag{3.53}$$

or

$$u(x,t) = \Big\| \sum_{\ell=k_p+1}^{k_{p+1}} a_{i,\ell}(\epsilon)\varphi_\ell(\epsilon)\Big\|^{-1}_{L^2(\Omega(\epsilon))} a_{i,q}(\epsilon)e^{-\mu_q(\epsilon)t}\varphi_q(\epsilon)(x) \tag{3.54}$$

for $x \in \Omega(\epsilon)$, $1 < t < 2$, $q = k_p+1, \dots, k_{p+1}$ and $0 < \epsilon \le \frac{1}{2}\hat\epsilon$, where

$$P_{\mathcal{H},\mathcal{B}(\epsilon)}\varphi_i = 1_{\Omega\cap\Omega(\epsilon)}\varphi_i = \sum_{\ell=1}^{\infty} a_{i,\ell}(\epsilon)\varphi_\ell(\epsilon),$$

we see that there exists $\alpha \in (0,1]$, depending only on N, such that

$$|\varphi_i(x) - \varphi_i(y)| \le A|x-y|^\alpha \quad (x, y \in D) \tag{3.55}$$

and, for $\epsilon \in (0, \hat\epsilon/2)$,

$$|\psi_i(\epsilon)(x) - \psi_i(\epsilon)(y)| \le A|x-y|^\alpha \quad (x, y \in D), \tag{3.56}$$

where (using (3.52)),

$$A = 2(8/\hat\epsilon)^\alpha(k_{p+1} - k_p)c(1)^{1/2}e^{(\mu_i+1)/4}\max\{e^{\lambda_q(B)/2} : k_p+1 \le q \le k_{p+1}\}.$$

Note that the oscillation of $u(x,t) = e^{-\mu_i t}\varphi_i(x)$ can be estimated using (3.30) as follows:

$$e^{-\mu_i t}|\varphi_i(x)| \le e^{-\mu_i}c(1)^{1/2}e^{\lambda_i(B)/2}.$$

Similarly the oscillation of

$$u(x,t) = \Big\| \sum_{\ell=k_p+1}^{k_{p+1}} a_{i,\ell}(\epsilon)\varphi_\ell(\epsilon)\Big\|^{-1}_{L^2(\Omega(\epsilon))} a_{i,q}(\epsilon)e^{-\mu_q(\epsilon)t}\varphi_q(\epsilon)(x)$$

can be estimated by (3.29) as follows:

$$\Big\| \sum_{\ell=k_p+1}^{k_{p+1}} a_{i,\ell}(\epsilon)\varphi_\ell(\epsilon)\Big\|^{-1}_{L^2(\Omega(\epsilon))} a_{i,q}(\epsilon)e^{-\mu_q(\epsilon)t}\varphi_q(\epsilon) \le e^{-\mu_q(\epsilon)}c(1)^{1/2}e^{\lambda_q(B)/2}.$$

Let $r = \operatorname{dist}(K, \partial D)$ and

$$\mathcal{R} = \min\{r, \Big(\frac{\delta}{6}\Big)^{1/\alpha}A^{-\frac{1}{\alpha}}\}.$$

Then for $m = 1, 2, 3, \dots$ and all $y \in D$ with $|x_m - y| \le \mathcal{R}$, by (3.55) and (3.56), we have

$$|\varphi_i(x_m) - \varphi_i(y)| \le \delta/6, \tag{3.57}$$

$$|\psi_i(\epsilon_m)(x_m) - \psi_i(\epsilon_m)(y)| \le \delta/6, \tag{3.58}$$

and hence, by (3.51), (3.57) and (3.58),

$$\begin{aligned} |\varphi_i(y) - \psi_i(\epsilon_m)(y)| &\ge |\varphi_i(x_m) - \psi_i(\epsilon_m)(x_m)| - |\varphi_i(x_m) - \varphi_i(y)| \\ &\quad - |\psi_i(\epsilon_m)(y) - \psi_i(\epsilon_m)(x_m)| \\ &\ge 2\delta/3. \end{aligned} \tag{3.59}$$

Thus, for $m = 1, 2, 3, \dots$, we have

$$\int_{B(x_m,R)} |\psi_i(\epsilon_m)(y) - \varphi_i(y)|^2\,dy \geq c_2 R^N \delta^2$$

where $c_2 > 0$ depends only on N. But this contradicts (2.89) of Theorem 2.26. Therefore (3.50) must hold and the theorem is proved. $\square$

4. Application to Koch snowflake

In this section we let $\Omega \subseteq \mathbb{R}^2$ be the Koch snowflake. Let $\{\Omega_{\rm in}(n)\}_{n=1}^{\infty}$ be the usual sequence of polygons approximating Ω from inside, with $\Omega_{\rm in}(1)$ being an equilateral triangle. Let $\{\Omega_{\rm out}(n)\}_{n=1}^{\infty}$ be the usual sequence of polygons approximating Ω from outside, with $\Omega_{\rm out}(1)$ being a regular hexagon.

We first recall the definition of (ϵ, δ)-domains (see [5]):

Definition 4.1. Let D be a domain in $\mathbb{R}^d$ and let $\epsilon > 0$ and $0 < \delta \leq \infty$. We say that D is an (ϵ, δ)-domain if for any two distinct points p_1, $p_2 \in D$ with $|p_1 - p_2| \leq \delta$, there exists a rectifiable path $\Gamma \subseteq D$ joining p_1 to p_2 satisfying the following conditions:

(i) length $(\Gamma) \leq \epsilon^{-1}|p_1 - p_2|$,

(ii) for all $p \in \Gamma$ we have

$$\operatorname{dist}(p, \partial D) \geq \epsilon |p_1 - p_2|^{-1} |p - p_1| |p - p_2|. \tag{4.1}$$

We note that if D is an (ϵ, δ)-domain, then any dilation of D is also on (ϵ, δ)-domain.

We shall need the following result.

Proposition 4.2 ([7, Proposition 3.2])**.** *There exist $\hat{\epsilon}, \hat{\delta} > 0$, independent of n, such that $\Omega \times \mathbb{R}$ and $\Omega_{\rm in}(n) \times \mathbb{R}$, $n = 1, 2, 3, \dots$, are all $(\hat{\epsilon}, \hat{\delta})$-domains in $\mathbb{R}^3$.*

Our main result in this section is as follows.

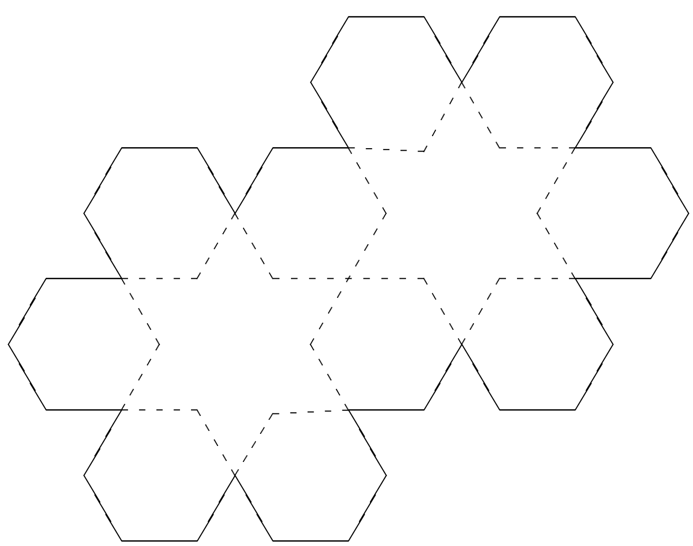

Figure 1. The polygon S in the proof of Theorem 4.3

Theorem 4.3. *There exist $\check{\epsilon}, \check{\delta} \in (0,\infty)$, independent of n, such that $\Omega_{\rm out}(n) \times \mathbb{R}$ is an $(\check{\epsilon}, \check{\delta})$-domain in $\mathbb{R}^3$ for all $n = 1, 2, 3, \dots$.*

Proof. Fix $n \in \mathbb{N}$ and let (x_1, y_1, z_1), $(x_2, y_2, z_2) \in \Omega_{\rm out}(n) \times \mathbb{R}$. By Proposition 4.2 we see that it suffices to consider only the following two cases:
Case 1 Both (x_1, y_1) and (x_2, y_2) are in $\Omega_{\rm out}(n) \backslash \Omega_{\rm in}(n+1)$,
Case 2 $(x_1, y_1) \in \Omega_{\rm out}(n) \backslash \Omega_{\rm in}(n+1)$ but $(x_2, y_2) \in \Omega_{\rm in}(n+1)$.

Let S be the polygon in Figure 1. Then, since $S \times \mathbb{R}$ is a Lipschitz domain, there exist $\tilde{\epsilon}, \tilde{\delta} > 0$ such that $S \times \mathbb{R}$ is an $(\tilde{\epsilon}, \tilde{\delta})$-domain in $\mathbb{R}^3$. Therefore, since any dilation, translation, or rotation of $S \times \mathbb{R}$ is also an $(\tilde{\epsilon}, \tilde{\delta})$-domain, we shall assume that (x_1, y_1) and (x_2, y_2) are not both inside a domain $R \subseteq \Omega_{\rm out}(n)$ that is obtained by a finite sequence of dilations, translations, and rotations of S and that some of the edges of R are also edges of $\Omega_{\rm out}(n)$. This assumption implies that

$$|(x_1, y_1) - (x_2, y_2)| \geq 3L_{\rm in}(n+1) \tag{4.2}$$

where $L_{\rm in}(n+1)$ denotes the length of each side of the polygon $\Omega_{\rm in}(n+1)$. For every edge of $\Omega_{\rm out}(n)$ there corresponds two edges of $\Omega_{\rm in}(n+1)$ as shown in Figure 2.

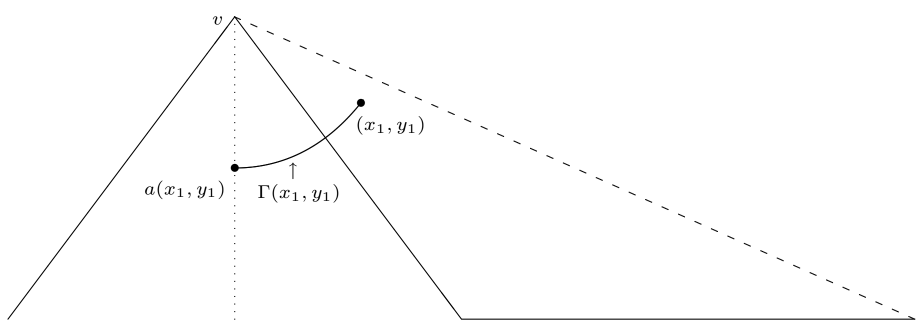

FIGURE 2. Edges: $--$ of $\Omega_{\rm out}(n)$ and — of $\Omega_{\rm in}(n+1)$

Referring to Figure 2, suppose $(x_1, y_1) \in \Omega_{\rm out}(n) \backslash \Omega_{\rm in}(n+1)$ and suppose (x_1, y_1) is within a distance of $\frac{1}{2}\cos(\frac{\pi}{6}) L_{\rm in}(n+1) = \frac{\sqrt{3}}{4} L_{\rm in}(n+1)$ from an acute vertex v of $\Omega_{\rm in}(n+1)$. Then we let $a(x_1, y_1)$ be the point on the angle bisector of $\Omega_{\rm in}(n+1)$ at v that is of the same distance from v as (x_1, y_1) is from v. The arc of the circle, centered at v and with radius $|v - (x_1, y_1)|$, starting at $a(x_1, y_1)$ and ending at (x_1, y_1) will be denoted by $\Gamma(x_1, y_1)$, see Figure 2.

Referring to Figure 3, suppose $(x_1, y_1) \in \Omega_{\rm out}(n) \backslash \Omega_{\rm in}(n+1)$ and suppose (x_1, y_1) is not within a distance of $\frac{1}{2}\cos(\pi/6) L_{\rm in}(n+1) = \frac{\sqrt{3}}{4} L_{\rm in}(n+1)$ from any acute vertex of $\Omega_{\rm in}(n+1)$. Then we let $a(x_1, y_1)$ be the center of the triangle with vertices u, v and w. The straight line segment joining $a(x_1, y_1)$ to (x_1, y_1) will be denoted by $\Gamma(x_1, y_1)$, see Figure 3. Note that in either case we have

$$\text{length}(\Gamma(x_1, y_1)) \leq 3L_{\rm in}(n+1). \tag{4.3}$$

We shall assume that n is sufficiently large so that

$$L_{\rm in}(n+1) \leq \hat{\delta}/9. \tag{4.4}$$

Then if

$$|(x_1, y_1, z_1) - (x_2, y_2, z_2)| \leq \hat{\delta}/3, \tag{4.5}$$

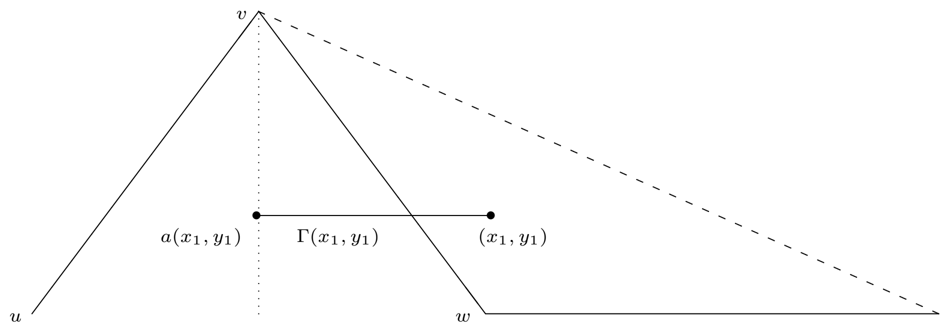

FIGURE 3. Edges: $--$ of $\Omega_{\rm out}(n)$ and — of $\Omega_{\rm in}(n+1)$

then, by (4.3),

$$\begin{aligned}
&|(a(x_1,y_1),z_1)-(a(x_2,y_2),z_2)| \\
&\le |(a(x_1,y_1),z_1)-(x_1,y_1,z_1)|+|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \\
&\quad +|(x_2,y_2,z_2)-(a(x_2,y_2),z_2)| \\
&\le \operatorname{length}(\Gamma(x_1,y_1))+\hat\delta/3+\operatorname{length}(\Gamma(x_2,y_2))\le\hat\delta.
\end{aligned} \tag{4.6}$$

We first present the proof for Case 1. We shall divide the proof for this case into a number of subcases:

Case 1(i). Here we assume that both (x_1, y_1) and (x_2, y_2) are in $\Omega_{\rm out}(n)\backslash\Omega_{\rm in}(n+1)$, that (x_1, y_1) and (x_2, y_2) are not both inside a region $R \subseteq \Omega_{\rm out}(n)$ which is obtained from the polygon S in Figure 1 by a finite sequence of dilations and isometries and that some of the edges of R are also edges of $\Omega_{\rm out}(n)$, that both (x_1, y_1) and (x_2, y_2) are within a distance of $\frac{1}{2}\cos(\pi/6)L_{\rm in}(n+1) = \frac{\sqrt{3}}{4}L_{\rm in}(n+1)$ from some acute vertices v_1 and v_2, respectively, of $\Omega_{\rm in}(n+1)$, and that (4.4) and (4.5) hold.

Since $a(x_1, y_1)$, $a(x_2, y_2) \in \Omega_{\rm in}(n+1)$ and (4.5) holds, Proposition 4.2 and (4.6) imply that there exists a rectifiable path $\Gamma \subseteq \Omega_{\rm in}(n+1)\times\mathbb{R}$ joining $(a(x_1, y_1), z_1)$ to $(a(x_2, y_2), z_2)$ and satisfying:

(A) $\operatorname{length}(\Gamma) \le \hat\epsilon^{-1}|(a(x_1,y_1),z_1)-(a(x_2,y_2),z_2)|$

(B) for all $p \in \Gamma$ we have

$$\begin{aligned}
\operatorname{dist}(p,\partial\Omega_{\rm in}(n+1)\times\mathbb{R}) &\ge \hat\epsilon\,|(a(x_1,y_1),z_1)-(a(x_2,y_2),z_2)|^{-1} \\
&\quad\times|p-(a(x_1,y_1),z_1)||p-(a(x_2,y_2),z_2)|.
\end{aligned} \tag{4.7}$$

Now, by (4.2), for $i = 1, 2$, we have

$$\operatorname{length}(\Gamma(x_i,y_i)) \le \frac{\sqrt{3}}{4}L_{\rm in}(n+1)\frac{\pi}{3} \le \frac{\sqrt{3}\pi}{36}|(x_1,y_1,z_1)-(x_2,y_2,z_2)|. \tag{4.8}$$

Also we have, by (A) and (4.8),

$$\begin{aligned}
\operatorname{length}(\Gamma) &\le \hat\epsilon^{-1}|(a(x_1,y_1),z_1)-(a(x_2,y_2),z_2)| \\
&\le \hat\epsilon^{-1}\big\{|(a(x_1,y_1),z_1)-(x_1,y_1,z_1)|+|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \\
&\quad +|(x_2,y_2,z_2)-(a(x_2,y_2),z_2)|\big\} \\
&\le \hat\epsilon^{-1}\big(\frac{\sqrt{3}\pi}{18}+1\big)|(x_1,y_1,z_1)-(x_2,y_2,z_2)|.
\end{aligned} \tag{4.9}$$

In the reverse direction we have, by (4.8),

$$
\begin{aligned}
&|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \\
&\le |(x_1,y_1,z_1)-(a(x_1,y_1),z_1)| + |(a(x_1,y_1),z_1)-(a(x_2,y_2),z_2)| \\
&\quad + |(a(x_2,y_2),z_2)-(x_2,y_2,z_2)| \\
&\le \frac{\sqrt{3}\pi}{18}|(x_1,y_1,z_1)-(x_2,y_2,z_2)| + |(a(x_1,y_1),z_1)-(a(x_2,y_2),z_2)|
\end{aligned}
\tag{4.10}
$$

and hence

$$
|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \le \big(1-\frac{\sqrt{3}\pi}{18}\big)^{-1}|(a(x_1,y_1),z_1)-(a(x_2,y_2),z_2)|. \tag{4.11}
$$

Let $p\in\Gamma$ and suppose that

$$
|p-(a(x_i,y_i),z_i)| \ge \frac{1}{2}\operatorname{dist}(a(x_i,y_i),\partial\Omega_{\rm in}(n+1)) \tag{4.12}
$$

for $i=1,2$. Then, referring to Figure 2,

$$
\begin{aligned}
|p-(a(x_i,y_i),z_i)| &\ge \frac{1}{2}\sin\big(\frac{\pi}{6}\big)|a(x_i,y_i)-v_i| \\
&= \frac{3}{4\pi}|a(x_i,y_i)-v_i|\frac{\pi}{3} \\
&\ge \frac{3}{4\pi}|(a(x_i,y_i),z_i)-(x_i,y_i,z_i)|
\end{aligned}
\tag{4.13}
$$

for $i=1,2,$. Hence

$$
\begin{aligned}
|p-(x_i,y_i,z_i)| &\le |p-(a(x_i,y_i),z_i)| + |(a(x_i,y_i),z_i)-(x_i,y_i,z_i)| \\
&\le \big(1+\frac{4\pi}{3}\big)|p-(a(x_i,y_i),z_i)|
\end{aligned}
\tag{4.14}
$$

for $i=1,2$. Combining (4.7), (4.9) and (4.14) we get

$$
\begin{aligned}
&\operatorname{dist}(p,\partial\Omega_{\rm out}(n)\times\mathbb{R}) \\
&\ge \operatorname{dist}(p,\partial\Omega_{\rm in}(n+1)\times\mathbb{R}) \\
&\ge \hat\epsilon\,|(a(x_1,y_1),z_1)-(a(x_2,y_2),z_2)|^{-1}|p-(a(x_1,y_1),z_1)| \\
&\quad\times|p-(a(x_2,y_2),z_2)| \\
&\ge \hat\epsilon\big(\frac{\sqrt{3}\pi}{18}+1\big)^{-1}\big(1+\frac{4\pi}{3}\big)^{-2}|(x_1,y_1,z_1)-(x_2,y_2,z_2)|^{-1} \\
&\quad\times|p-(x_1,y_1,z_1)||p-(x_2,y_2,z_2)|.
\end{aligned}
\tag{4.15}
$$

Next let $p\in\Gamma$ and suppose that

$$
|p-(a(x_1,y_1),z_1)| < \frac{1}{2}\operatorname{dist}(a(x_1,y_1),\partial\Omega_{\rm in}(n+1)). \tag{4.16}
$$

Referring to Figure 2, we have, by (4.16),

$$
\begin{aligned}
|p-(x_1,y_1,z_1)| &\le |p-(a(x_1,y_1),z_1)| + |(a(x_1,y_1),z_1)-(x_1,y_1,z_1)| \\
&\le \big(\frac{1}{4}+\frac{\pi}{3}\big)|a(x_1,y_1)-v_1|.
\end{aligned}
\tag{4.17}
$$

So, by (4.16) and (4.17),

$$\begin{aligned} \operatorname{dist}(p, \partial\Omega_{\rm out}(n)\times\mathbb{R}) &\ge \operatorname{dist}(p, \partial\Omega_{\rm in}(n+1)\times\mathbb{R}) \\ &\ge \frac{1}{2}\operatorname{dist}(a(x_1,y_1), \partial\Omega_{\rm in}(n+1)) \\ &= \frac{1}{4}|a(x_1,y_1)-v_1| \\ &\ge \frac{1}{4}\big(\frac{1}{4}+\frac{\pi}{3}\big)^{-1}|p-(x_1,y_1,z_1)|. \end{aligned} \tag{4.18}$$

Also, by (4.16), (4.9), (4.8), (4.2),

$$\begin{aligned} &|p-(x_2,y_2,z_2)| \\ &\le |p-(a(x_1,y_1),z_1)| + |(a(x_1,y_1),z_1)-(a(x_2,y_2),z_2)| \\ &\quad + |(a(x_2,y_2),z_2)-(x_2,y_2,z_2)| \\ &\le \frac{1}{4}|a(x_1,y_1)-v_1| + \big(\frac{\sqrt{3}\pi}{18}+1\big)|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \\ &\quad + \frac{\sqrt{3}\pi}{36}|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \\ &\le \frac{\sqrt{3}}{16}L_{\rm in}(n+1) + \big(\frac{\sqrt{3}\pi}{12}+1\big)|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \\ &\le \frac{\sqrt{3}}{48}|(x_1,y_1)-(x_2,y_2)| + \big(\frac{\sqrt{3}\pi}{12}+1\big)|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \\ &\le \big(\frac{\sqrt{3}}{48}+\frac{\sqrt{3}\pi}{12}+1\big)|(x_1,y_1,z_1)-(x_2,y_2,z_2)|. \end{aligned} \tag{4.19}$$

Combining (4.18) and (4.19) we obtain

$$\begin{aligned} &\operatorname{dist}(p, \partial\Omega_{\rm out}(n)\times\mathbb{R}) \\ &\ge \frac{1}{4}\big(\frac{1}{4}+\frac{\pi}{3}\big)^{-1}\big(\frac{\sqrt{3}}{48}+\frac{\sqrt{3}\pi}{12}+1\big)^{-1}|(x_1,y_1,z_1)-(x_2,y_2,z_2)|^{-1} \\ &\quad \times |p-(x_1,y_1,z_1)||p-(x_2,y_2,z_2)|. \end{aligned} \tag{4.20}$$

Now let $p=(x,y,z_1)\in\Gamma(x_1,y_1)\times\{z_1\}$ and let

$$k_1 = \inf\{b^{-1}\sin b : 0<b<\frac{\pi}{3}\} > 0. \tag{4.21}$$

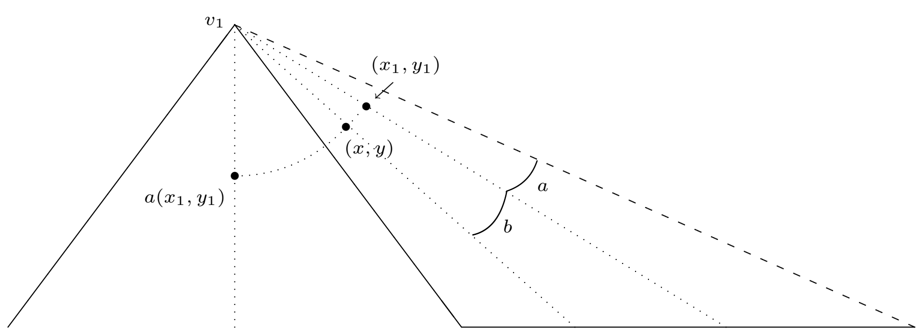

FIGURE 4. Edges: $--$ of $\Omega_{\rm out}(n)$ and — of $\Omega_{\rm in}(n+1)$

Referring to Figure 4 we have

$$\begin{aligned} \operatorname{dist}(p, \partial\Omega_{\rm out}(n)\times\mathbb{R}) &= \operatorname{dist}((x,y), \partial\Omega_{\rm out}(n)) \\ &= |(x_1,y_1)-v_1|\sin(a+b) \\ &= |(x_1,y_1)-v_1| b(b^{-1}\sin(a+b)) \\ &\geq |(x_1,y_1)-v_1| b(b^{-1}\sin b) \\ &\geq |(x_1,y_1)-v_1| bk_1 \\ &\geq k_1|(x_1,y_1)-(x,y)| \\ &= k_1|(x_1,y_1,z_1)-(x,y,z_1)|. \end{aligned} \tag{4.22}$$

Also we have, for $p=(x,y,z_1)\in\Gamma(x_1,y_1)\times\{z_1\}$,

$$\begin{aligned} |p-(x_2,y_2,z_2)| &\leq |p-(x_1,y_1,z_1)| + |(x_1,y_1,z_1)-(x_2,y_2,z_2)| \\ &\leq \frac{\pi\sqrt{3}}{12}L_{\rm in}(n+1) + |(x_1,y_1,z_1)-(x_2,y_2,z_2)| \\ &\leq \big(\frac{\pi\sqrt{3}}{36}+1\big)|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \end{aligned} \tag{4.23}$$

where we have used (4.2). Thus, combining (4.22) and (4.23), we have, for all $p=(x,y,z_1)\in\Gamma(x_1,y_1)\times\{z_1\}$,

$$\begin{aligned} \operatorname{dist}(p, \partial\Omega_{\rm out}(n)\times\mathbb{R}) \geq \Big(\frac{\pi\sqrt{3}}{36}+1\Big)^{-1} & k_1|(x_1,y_1,z_1)-(x_2,y_2,z_1)|^{-1} \\ &\times |p-(x_1,y_1,z_1)||p-(x_2,y_2,z_2)|. \end{aligned} \tag{4.24}$$

Similarly, we have, for all $p=(x,y,z_2)\in\Gamma(x_2,y_2)\times\{z_2\}$,

$$\begin{aligned} \operatorname{dist}(p, \partial\Omega_{\rm out}(n)\times\mathbb{R}) \geq \Big(\frac{\pi\sqrt{3}}{36}+1\Big)^{-1} & k_1|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \\ &\times |p-(x_1,y_1,z_1)||p-(x_2,y_2,z_2)|. \end{aligned} \tag{4.25}$$

From (4.5), (4.6), (4.8), (4.9), (4.15), (4.20), (4.24) and (4.25), we see that if (x_1,y_1,z_1) and (x_2,y_2,z_2) satisfy the assumptions of Case 1(i) and if

$$|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \leq \hat{\delta}/3, \tag{4.26}$$

then there exists a path

$$\tilde{\Gamma} = (\Gamma(x_1,y_1)\times\{z_1\}) + \Gamma + (\Gamma(x_2,y_2)\times\{z_2\}) \tag{4.27}$$

joining (x_1,y_1,z_1) to (x_2,y_2,z_2) satisfying

$$\operatorname{length}(\tilde{\Gamma}) \leq \big[\frac{\sqrt{3}\pi}{18} + \hat{\epsilon}\big(\frac{\sqrt{3}\pi}{18}+1\big)\big]|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \tag{4.28}$$

and for all $p\in\tilde{\Gamma}$ we have

$$\begin{aligned} &\operatorname{dist}(p, \partial\Omega_{\rm out}(n)\times\mathbb{R}) \\ &\geq \epsilon_1|(x_1,y_1,z_1)-(x_2,y_2,z_2)|^{-1}|p-(x_1,y_1,z_1)||p-(x_2,y_2,z_2)| \end{aligned} \tag{4.29}$$

where

$$\begin{aligned} \epsilon_1 = \min\Big\{ & \hat{\epsilon}\Big(\frac{\sqrt{3}\pi}{18}+1\Big)^{-1}\Big(1+\frac{4\pi}{3}\Big)^{-2}, \\ & \frac{1}{4}\Big(\frac{1}{4}+\frac{\pi}{3}\Big)^{-1}\Big(\frac{\sqrt{3}}{48}+\frac{\sqrt{3}\pi}{12}+1\Big)^{-1}, k_1\Big(\frac{\sqrt{3}\pi}{36}+1\Big)^{-1}\Big\}. \end{aligned} \tag{4.30}$$

Case 1(ii). Here we assume that both (x_1, y_1) and (x_2, y_2) are in $\Omega_{\rm out}(n)\backslash\Omega_{\rm in}(n+1)$, that (x_1, y_1) and (x_2, y_2) are not both inside a region $R \subseteq \Omega_{\rm out}(n)$ which is obtained from the polygon S in Figure 1 by a finite sequence of dilations and isometries and that some of the edges of R are also edges of $\Omega_{\rm out}(n)$, that both (x_1, y_1) and (x_2, y_2) are not within a distance of $\frac{1}{2}\cos(\pi/6)L_{\rm in}(n+1) = \frac{\sqrt{3}}{4}L_{\rm in}(n+1)$ from any acute vertex of $\Omega_{\rm in}(n+1)$, and that (4.4) and (4.5) hold.

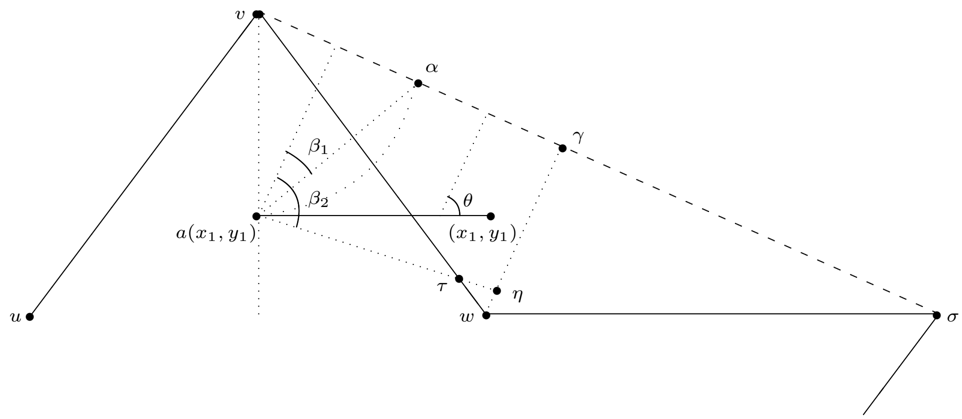

FIGURE 5. Edges: $--$ of $\Omega_{\rm out}(n)$ and — of $\Omega_{\rm in}(n+1)$

In Figure 5, let θ be the angle between the line segment $\overline{(x_1, y_1), a(x_1, y_1)}$ joining (x_1, y_1) to $a(x_1, y_1)$ and the line perpendicular to the line segment $\overline{v, \sigma}$ joining v to σ. Let α be the point on $\overline{v, \sigma}$ such that the length of $\overline{v, \alpha}$ is $\frac{1}{2}\cos(\pi/6)L_{\rm in}(n+1)$. Let γ be the midpoint of $\overline{v, \sigma}$ and let η be a point on $\overline{\gamma, w}$ such that

$$\text{length}(\overline{\gamma, \eta}) = \frac{3}{4}\,\text{length}(\overline{\gamma, w}). \tag{4.31}$$

Let β_1 and β_2 be the angles between the line perpendicular to $\overline{v, \sigma}$ and the line segments $\overline{a(x_1, y_1), \alpha}$ and $\overline{a(x_1, y_1), \eta}$, respectively. Let τ be the point of intersection of the line segments $\overline{v, w}$ and $\overline{a(x_1, y_1), \eta}$. If

$$\beta_1 \le \theta \le \beta_2, \tag{4.32}$$

then, for all $(x, y) \in \overline{a(x_1, y_1), (x_1, y_1)}$, we have

$$\operatorname{dist}((x, y), \partial\Omega_{\rm out}(n)) \ge (\cos\theta)|(x, y) - (x_1, y_1)| \ge (\cos\beta_1)|(x, y) - (x_1, y_1)|. \tag{4.33}$$

If

$$\beta_2 < \theta < \frac{\pi}{2}; \tag{4.34}$$

i.e., if $(x_1, y_1) \in \Delta(\tau, \eta, w)$, where $\Delta(\tau, \eta, w)$ denotes the triangle with vertices τ, η and w, then, for all $(x, y) \in \overline{a(x_1, y_1), (x_1, y_1)}$, we have

$$\begin{aligned}
\operatorname{dist}((x, y), \partial\Omega_{\rm out}(n)) &\ge \operatorname{dist}(\Delta(a(x_1, y_1), \eta, w), \partial\Omega_{\rm out}(n)) \\
&= |\eta - \gamma| \\
&= 3|\eta - w| \\
&= k_2|a(x_1, y_1) - \eta| \\
&\ge k_2|(x, y) - (x_1, y_1)|
\end{aligned} \tag{4.35}$$

for some $k_2 > 0$ independent of n, (x_1, y_1) and (x_2, y_2).

Combining (4.33) and (4.35) we have, for all $p = (x, y, z_1) \in \Gamma(x_1, y_1) \times \{z_1\}$,

$$\operatorname{dist}(p, \partial\Omega_{\text{out}}(n) \times \mathbb{R}) \geq \min\{\cos\beta_1, k_2\}|p - (x_1, y_1, z_1)|. \tag{4.36}$$

Also, by (4.2), for all $p = (x, y, z_1) \in \Gamma(x_1, y_1) \times \{z_1\}$, we have

$$\begin{aligned} |p - (x_2, y_2, z_2)| &\leq |p - (x_1, y_1, z_1)| + |(x_1, y_1, z_1) - (x_2, y_2, z_2)| \\ &\leq L_{\text{in}}(n+1) + |(x_1, y_1, z_1) - (x_2, y_2, z_2)| \\ &\leq \frac{4}{3}|(x_1, y_1, z_1) - (x_2, y_2, z_2)|. \end{aligned} \tag{4.37}$$

Thus combining (4.36) and (4.37) we have, for all $p = (x, y, z_1) \in \Gamma(x_1, y_1) \times \{z_1\}$,

$$\begin{aligned} \operatorname{dist}(p, \partial\Omega_{\text{out}}(n) \times \mathbb{R}) \geq{}& \frac{3}{4}\min\{\cos\beta_1, k_2\}|(x_1, y_1, z_1) - (x_2, y_2, z_2)|^{-1} \\ &\times |p - (x_1, y_1, z_1)||p - (x_2, y_2, z_2)|. \end{aligned} \tag{4.38}$$

Similarly, for all $p = (x, y, z_2) \in \Gamma(x_2, y_2) \times \{z_2\}$, we have

$$\begin{aligned} \operatorname{dist}(p, \partial\Omega_{\text{out}}(n) \times \mathbb{R}) \geq{}& \frac{3}{4}\min\{\cos\beta_1, k_2\}|(x_1, y_1, z_1) - (x_2, y_2, z_2)|^{-1} \\ &\times |p - (x_1, y_1, z_1)||p - (x_2, y_2, z_2)|. \end{aligned} \tag{4.39}$$

Referring to Figure 5 we have, by (4.2),

$$\begin{aligned} \operatorname{length}(\Gamma(x_1, y_1) \times \{z_1\}) &= \operatorname{length}(\Gamma(x_1, y_1)) \\ &\leq L_{\text{in}}(n+1) \\ &\leq \frac{1}{3}|(x_1, y_1, z_1) - (x_2, y_2, z_2)|. \end{aligned} \tag{4.40}$$

Similarly we have

$$\operatorname{length}(\Gamma(x_2, y_2) \times \{z_2\}) \leq \frac{1}{3}|(x_1, y_1, z_1) - (x_2, y_2, z_2)|. \tag{4.41}$$

So, from (4.39) and (4.40), we have

$$\begin{aligned} \operatorname{length}(\Gamma) &\leq \hat{\epsilon}^{-1}|(a(x_1, y_1), z_1) - (a(x_2, y_2), z_2)| \\ &\leq \hat{\epsilon}^{-1}\{|(a(x_1, y_1), z_1) - (x_1, y_1, z_1)| + |(x_1, y_1, z_1) - (x_2, y_2, z_2)| \\ &\quad + |(x_2, y_2, z_2) - (a(x_2, y_2), z_2)| \\ &\leq \frac{5}{3}\hat{\epsilon}^{-1}|(x_1, y_1, z_1) - (x_2, y_2, z_2)|. \end{aligned} \tag{4.42}$$

Now let $p \in \Gamma$ and suppose that

$$|p - (a(x_i, y_i), z_i)| \geq \frac{1}{2}\operatorname{dist}(a(x_i, y_i), \partial\Omega_{\text{in}}(n+1)) \tag{4.43}$$

for $i = 1, 2$. Then, referring to Figure 3, we have

$$|p - (a(x_i, y_i), z_i)| \geq \frac{1}{4\sqrt{3}}L_{\text{in}}(n+1) \geq \frac{1}{4\sqrt{3}}|(a(x_i, y_i), z_i) - (x_i, y_i, z_i)| \tag{4.44}$$

for $i = 1, 2$. Hence

$$\begin{aligned} |p - (x_i, y_i, z_i)| &\leq |p - (a(x_i, y_i), z_i)| + |(a(x_i, y_i), z_i) - (x_i, y_i, z_i)| \\ &\leq (1 + 4\sqrt{3})|p - (a(x_i, y_i), z_i)| \end{aligned} \tag{4.45}$$

for $i = 1, 2$. Thus, by (4.42),

$$\begin{aligned} \operatorname{dist}(p, \partial\Omega_{\rm out}(n) \times \mathbb{R}) &\geq \operatorname{dist}(p, \partial\Omega_{\rm in}(n+1) \times \mathbb{R}) \\ &\geq \hat{\epsilon}|(a(x_1, y_1), z_1) - (a(x_2, y_2), z_2)|^{-1} \\ &\quad \times |p - (a(x_1, y_1), z_1)||p - (a(x_2, y_2), z_2)| \\ &\geq \frac{3}{5}(1 + 4\sqrt{3})^{-2}\hat{\epsilon}|(x_1, y_1, z_1) - (x_2, y_2, z_2)|^{-1} \\ &\quad \times |p - (x_1, y_1, z_1)||p - (x_2, y_2, z_2)|. \end{aligned} \tag{4.46}$$

Next let $p \in \Gamma$ and suppose that

$$|p - (a(x_1, y_1), z_1)| < \frac{1}{2} \operatorname{dist}(a(x_1, y_1), \partial\Omega_{\rm in}(n+1)).$$

Then, referring to Figure 3,

$$\begin{aligned} \operatorname{dist}(p, \partial\Omega_{\rm out}(n) \times \mathbb{R}) &\geq \operatorname{dist}(p, \partial\Omega_{\rm in}(n+1) \times \mathbb{R}) \\ &\geq \frac{1}{2} \operatorname{dist}((a(x_1, y_1), z_1), \partial\Omega_{\rm in}(n+1) \times \mathbb{R}) \\ &= \frac{1}{4\sqrt{3}} L_{\rm in}(n+1) \end{aligned} \tag{4.47}$$

and

$$\begin{aligned} |p - (x_1, y_1, z_1)| &\leq |p - (a(x_1, y_1), z_1)| + |(a(x_1, y_1), z_1) - (x_1, y_1, z_1)| \\ &\leq \frac{1}{2} \operatorname{dist}(a(x_1, y_1), \partial\Omega_{\rm in}(n+1)) + L_{\rm in}(n+1) \\ &= \big(\frac{1}{4\sqrt{3}} + 1\big) L_{\rm in}(n+1), \end{aligned} \tag{4.48}$$

hence, from (4.47) and (4.48),

$$\operatorname{dist}(p, \partial\Omega_{\rm out}(n) \times \mathbb{R}) \geq \frac{1}{4\sqrt{3}}\big(\frac{1}{4\sqrt{3}} + 1\big)^{-1}|p - (x_1, y_1, z_1)|. \tag{4.49}$$

Also, by (4.48) and (4.2), we have

$$\begin{aligned} |p - (x_2, y_2, z_2)| &\leq |p - (x_1, y_1, z_1)| + |(x_1, y_1, z_1) - (x_2, y_2, z_2)| \\ &\leq \big[\big(\frac{1}{4\sqrt{3}} + 1\big)3^{-1} + 1\big]|(x_1, y_1, z_1) - (x_2, y_2, z_2)|. \end{aligned} \tag{4.50}$$

Combining (4.49) and (4.50) we obtain

$$\begin{aligned} &\operatorname{dist}(p, \partial\Omega_{\rm out}(n) \times \mathbb{R}) \\ &\geq \frac{1}{4\sqrt{3}}\Big(\frac{1}{4\sqrt{3}} + 1\Big)^{-1}\Big[\big(\frac{1}{4\sqrt{3}} + 1\big)3^{-1} + 1\Big]^{-1}|(x_1, y_1, z_1) - (x_2, y_2, z_2)|^{-1} \\ &\quad \times |p - (x_1, y_1, z_1)||p - (x_2, y_2, z_2)|. \end{aligned} \tag{4.51}$$

By symmetry, if $p \in \Gamma$ and if

$$|p - (a(x_2, y_2), z_2)| < \frac{1}{2} \operatorname{dist}(a(x_2, y_2), \partial\Omega_{\rm in}(n+1)),$$

then we have

$$\begin{aligned}&\operatorname{dist}(p, \partial\Omega_{\rm out}(n) \times \mathbb{R})\\ &\geq \frac{1}{4\sqrt{3}}\left(\frac{1}{4\sqrt{3}}+1\right)^{-1}\left[\left(\frac{1}{4\sqrt{3}}+1\right)3^{-1}+1\right]^{-1}|(x_1,y_1,z_1)-(x_2,y_2,z_2)|^{-1}\\ &\quad\times |p-(x_1,y_1,z_1)||p-(x_2,y_2,z_2)|.\end{aligned} \tag{4.52}$$

Thus from (4.5), (4.6), (4.38), (4.39), (4.40), (4.41), (4.42), (4.46), (4.51) and (4.52), we see that if (x_1, y_1, z_1) and (x_2, y_2, z_2) satisfy the assumptions of Case 1(ii) and if

$$|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \leq \hat{\delta}/3,$$

then there exists a path

$$\tilde{\Gamma} = (\Gamma(x_1,y_1)\times\{z_1\}) + \Gamma + (\Gamma(x_2,y_2)\times\{z_2\}) \tag{4.53}$$

joining (x_1, y_1, z_1) to (x_2, y_2, z_2) satisfying

$$\operatorname{length}(\tilde{\Gamma}) \leq \big(\frac{2}{3}+\frac{5}{3}\hat{\epsilon}^{-1}\big)|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \tag{4.54}$$

and for all $p \in \tilde{\Gamma}$ we have

$$\begin{aligned}&\operatorname{dist}(p, \partial\Omega_{\rm out}(n)\times\mathbb{R})\\ &\geq \epsilon_2|(x_1,y_1,z_1)-(x_2,y_2,z_2)|^{-1}|p-(x_1,y_1,z_1)||p-(x_2,y_2,z_2)|\end{aligned} \tag{4.55}$$

where

$$\begin{aligned}\epsilon_2 = \min\Big\{&\frac{3}{4}\cos\beta_1, \frac{3}{4}k_2, \frac{3}{5}(1+4\sqrt{3})^{-2}\hat{\epsilon},\\ &\frac{1}{4\sqrt{3}}\Big(\frac{1}{4\sqrt{3}}+1\Big)^{-1}\Big[\big(\frac{1}{4\sqrt{3}}+1\big)3^{-1}+1\Big]^{-1}\Big\}.\end{aligned} \tag{4.56}$$

Case 1(iii). Here we assume that both (x_1, y_1) and (x_2, y_2) are in $\Omega_{\rm out}(n)\backslash\Omega_{\rm in}(n+1)$, that (x_1, y_1) and (x_2, y_2) are not both inside a region $R \subseteq \Omega_{\rm out}(n)$ which is obtained from the polygon S in Figure 1 by a finite sequence of dilations and isometries and that some of the edges of R are also edges of $\Omega_{\rm out}(n)$, that (x_1, y_1) is not within a distance of $\frac{1}{2}\cos(\pi/6)L_{\rm in}(n+1)$ from any acute vertex of $\Omega_{\rm in}(n+1)$, that (x_2, y_2) is within a distance of $\frac{1}{2}\cos(\pi/6)L_{\rm in}(n+1)$ from an acute vertex v_2 of $\Omega_{\rm in}(n+1)$, and that (4.4) and (4.5) hold.

In this case, by (4.2), we have

$$\begin{aligned}\operatorname{length}(\Gamma(x_2,y_2)\times\{z_1\}) &\leq \frac{\pi}{3}|(x_2,y_2,z_2)-v_2|\\ &\leq \frac{\pi\sqrt{3}}{12}L_{\rm in}(n+1)\\ &\leq \frac{\pi\sqrt{3}}{36}|(x_1,y_1,z_1)-(x_2,y_2,z_2)|\end{aligned} \tag{4.57}$$

and, referring to Figure 5,

$$\begin{aligned}\operatorname{length}(\Gamma(x_1,y_1)\times\{z_1\}) &= |a(x_1,y_1)-(x_1,y_1)|\\ &\leq L_{\rm in}(n+1)\\ &\leq \frac{1}{3}|(x_1,y_1,z_1)-(x_2,y_2,z_2)|.\end{aligned} \tag{4.58}$$

Hence

$$\begin{aligned} \operatorname{length}(\Gamma) &\le \hat{\epsilon}^{-1}|(a(x_1,y_1),z_1)-(a(x_2,y_2),z_2)| \\ &\le \hat{\epsilon}^{-1}\{|(a(x_1,y_1),z_1)-(x_1,y_1,z_1)|+|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \\ &\quad +|(x_2,y_2,z_2)-(a(x_2,y_2),z_2)|\} \\ &\le \hat{\epsilon}^{-1}\big(\frac{4}{3}+\frac{\pi\sqrt{3}}{36}\big)|(x_1,y_1,z_1)-(x_2,y_2,z_2)|. \end{aligned} \tag{4.59}$$

From (4.57) and (4.58) we have

$$\begin{aligned} &|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \\ &\le |(x_1,y_1,z_1)-(a(x_1,y_1),z_1)|+|(a(x_1,y_1),z_1)-(a(x_2,y_2),z_2)| \\ &\quad +|(a(x_2,y_2),z_2)-(x_2,y_2,z_2)| \\ &\le \frac{1}{3}|(x_1,y_1,z_1)-(x_2,y_2,z_2)|+|(a(x_1,y_1),z_1)-(a(x_2,y_2),z_2)| \\ &\quad +\frac{\pi\sqrt{3}}{36}|(x_1,y_1,z_1)-(x_2,y_2,z_2)|, \end{aligned}$$

hence

$$|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \le \Big(\frac{2}{3}-\frac{\pi\sqrt{3}}{36}\Big)^{-1}|(a(x_1,y_1),z_1)-(a(x_2,y_2),z_2)|. \tag{4.60}$$

Let $p\in\Gamma$ and suppose that

$$|p-(a(x_i,y_i),z_i)| \ge \frac{1}{2}\operatorname{dist}(a(x_i,y_i),\partial\Omega_{\rm in}(n+1))$$

for $i=1,2$. Then

$$|p-(a(x_1,y_1),z_1)| \ge \frac{1}{4\sqrt{3}}L_{\rm in}(n+1) \ge \frac{1}{4\sqrt{3}}|(a(x_1,y_1),z_1)-(x_1,y_1,z_1)|$$

and so

$$\begin{aligned} |p-(x_1,y_1,z_1)| &\le |p-(a(x_1,y_1),z_1)|+|(a(x_1,y_1),z_1)-(x_1,y_1,z_1)| \\ &\le (1+4\sqrt{3})|p-(a(x_1,y_1),z_1)|. \end{aligned} \tag{4.61}$$

Also

$$\begin{aligned} |p-(a(x_2,y_2),z_2)| &\ge \frac{1}{2}\sin(\pi/6)|a(x_2,y_2)-v_2| \\ &= \frac{3}{4\pi}|a(x_2,y_2)-v_2|\frac{\pi}{3} \\ &\ge \frac{3}{4\pi}|(a(x_2,y_2),z_2)-(x_2,y_2,z_2)|, \end{aligned}$$

and hence

$$\begin{aligned} |p-(x_2,y_2,z_2)| &\le |p-(a(x_2,y_2),z_2)|+|(a(x_2,y_2),z_2)-(x_2,y_2,z_2)| \\ &\le \big(1+\frac{4\pi}{3}\big)|p-(a(x_2,y_2),z_2)|. \end{aligned} \tag{4.62}$$

Thus, by (4.59), (4.61) and (4.62),

$$\begin{aligned}
&\operatorname{dist}(p, \partial\Omega_{\rm out}(n) \times \mathbb{R}) \\
&\ge \operatorname{dist}(p, \partial\Omega_{\rm in}(n+1) \times \mathbb{R}) \\
&\ge \hat\epsilon |(a(x_1,y_1),z_1) - (a(x_2,y_2),z_2)|^{-1} |p - (a(x_1,y_1),z_1)| \\
&\quad \times |p - (a(x_2,y_2),z_2)| \\
&\ge \hat\epsilon \big(\frac{4}{3} + \frac{\pi\sqrt{3}}{36}\big)^{-1} (1+4\sqrt{3})^{-1} \Big(1 + \frac{4\pi}{3}\Big)^{-1} \\
&\quad \times |(x_1,y_1,z_1) - (x_2,y_2,z_2)|^{-1} |p - (x_1,y_1,z_1)||p - (x_2,y_2,z_2)|.
\end{aligned} \tag{4.63}$$

Next let $p \in \Gamma$ and suppose that

$$|p - (a(x_1,y_1),z_1)| < \frac{1}{2} \operatorname{dist}(a(x_1,y_1), \partial\Omega_{\rm in}(n+1)).$$

Then (4.47), (4.48), (4.49) and (4.50) still hold. Hence we have

$$\begin{aligned}
&\operatorname{dist}(p, \partial\Omega_{\rm out}(n) \times \mathbb{R}) \\
&\ge \frac{1}{4\sqrt{3}} \Big(\frac{1}{4\sqrt{3}} + 1\Big)^{-1} \Big[\big(\frac{1}{4\sqrt{3}} + 1\big)3^{-1} + 1\Big]^{-1} \\
&\quad \times |(x_1,y_1,z_1) - (x_2,y_2,z_2)|^{-1} |p - (x_1,y_1,z_1)||p - (x_2,y_2,z_2)|.
\end{aligned} \tag{4.64}$$

Now let $p \in \Gamma$ and suppose that

$$|p - (a(x_2,y_2),z_2)| < \frac{1}{2} \operatorname{dist}(a(x_2,y_2), \partial\Omega_{\rm in}(n+1)).$$

Then

$$\begin{aligned}
|p - (x_2,y_2,z_2)| &\le |p - (a(x_2,y_2),z_2)| + |(a(x_2,y_2),z_2) - (x_2,y_2,z_2)| \\
&\le \big(\frac{1}{4} + \frac{\pi}{3}\big) |a(x_2,y_2) - v_2|.
\end{aligned} \tag{4.65}$$

So

$$\begin{aligned}
\operatorname{dist}(p, \partial\Omega_{\rm out}(n) \times \mathbb{R}) &\ge \operatorname{dist}(p, \partial\Omega_{\rm in}(n+1) \times \mathbb{R}) \\
&\ge \frac{1}{2} \operatorname{dist}(a(x_2,y_2), \partial\Omega_{\rm in}(n+1)) \\
&= \frac{1}{4} |a(x_2,y_2) - v_2| \\
&\ge \frac{1}{4} \Big(\frac{1}{4} + \frac{\pi}{3}\Big)^{-1} |p - (x_2,y_2,z_2)|.
\end{aligned} \tag{4.66}$$

Also we have, by (4.59), (4.58) and (4.2),

$$\begin{aligned}
|p - (x_1,y_1,z_1)| &\le |p - (a(x_2,y_2),z_2)| + |(a(x_2,y_2),z_2) - (a(x_1,y_1),z_1)| \\
&\quad + |(a(x_1,y_1),z_1) - (x_1,y_1,z_1)| \\
&\le \frac{1}{4} |a(x_2,y_2) - v_2| + \big(\frac{4}{3} + \frac{\pi\sqrt{3}}{36}\big) |(x_1,y_1,z_1) - (x_2,y_2,z_2)| \\
&\quad + \frac{1}{3} |(x_1,y_1,z_1) - (x_2,y_2,z_2)| \\
&\le \frac{\sqrt{3}}{16} L_{\rm in}(n+1) + \big(\frac{5}{3} + \frac{\pi\sqrt{3}}{36}\big) |(x_1,y_1,z_1) - (x_2,y_2,z_2)| \\
&\le \big(\frac{\sqrt{3}}{48} + \frac{5}{3} + \frac{\pi\sqrt{3}}{36}\big) |(x_1,y_1,z_1) - (x_2,y_2,z_2)|.
\end{aligned} \tag{4.67}$$

Combining (4.66) and (4.67), we have

$$\begin{aligned} &\operatorname{dist}(p, \partial\Omega_{\rm out}(n) \times \mathbb{R}) \\ &\geq \frac{1}{4}\Big(\frac{1}{4} + \frac{\pi}{3}\Big)^{-1}\Big(\frac{\sqrt{3}}{48} + \frac{5}{3} + \frac{\pi\sqrt{3}}{36}\Big)^{-1}|(x_1, y_1, z_1) - (x_2, y_2, z_2)|^{-1} \\ &\quad \times |p - (x_1, y_1, z_1)||p - (x_2, y_2, z_2)|. \end{aligned} \tag{4.68}$$

Now let $p \in \Gamma(x_2, y_2) \times \{z_2\}$. Let $k_1 > 0$ be the constant defined in (4.21). Then calculations similar to those in (4.22) give

$$\operatorname{dist}(p, \partial\Omega_{\rm out}(n) \times \mathbb{R}) \geq k_1|p - (x_2, y_2, z_2)|. \tag{4.69}$$

Also, by (4.2),

$$\begin{aligned} |p - (x_1, y_1, z_1)| &\leq |p - (x_2, y_2, z_2)| + |(x_2, y_2, z_2) - (x_1, y_1, z_1)| \\ &\leq \frac{\pi\sqrt{3}}{12} L_{\rm in}(n+1) + |(x_1, y_1, z_1) - (x_2, y_2, z_2)| \\ &\leq \big(\frac{\pi\sqrt{3}}{36} + 1\big)|(x_1, y_1, z_1) - (x_2, y_2, z_2)|. \end{aligned} \tag{4.70}$$

Combining (4.69) and (4.70) we obtain

$$\begin{aligned} \operatorname{dist}(p, \partial\Omega_{\rm out}(n) \times \mathbb{R}) &\geq k_1\Big(\frac{\pi\sqrt{3}}{36} + 1\Big)^{-1}|(x_1, y_1, z_1) - (x_2, y_2, z_2)|^{-1} \\ &\quad \times |p - (x_1, y_1, z_1)||p - (x_2, y_2, z_2)|. \end{aligned} \tag{4.71}$$

Let $p \in \Gamma(x_1, y_1) \times \{z_1\}$. Then (4.36) and (4.37), and their proofs, still hold. Thus we have

$$\begin{aligned} \operatorname{dist}(p, \partial\Omega_{\rm out}(n) \times \mathbb{R}) &\geq \frac{3}{4}\min\{\cos\beta_1, k_2\}|(x_1, y_1, z_1) - (x_2, y_2, z_2)|^{-1} \\ &\quad \times |p - (x_1, y_1, z_1)||p - (x_2, y_2, z_2)|, \end{aligned} \tag{4.72}$$

where β_1 and k_2 are constants described in Case 1(ii).

Thus from (4.5), (4.6), (4.57), (4.58), (4.59), (4.63), (4.64), (4.68), (4.71) and (4.78), we see that if (x_1, y_1, z_1) and (x_2, y_2, z_2) satisfy the assumptions of Case 1(iii) and if

$$|(x_1, y_1, z_1) - (x_2, y_2, z_2)| \leq \hat{\delta}/3,$$

then there exists a path

$$\tilde{\Gamma} = (\Gamma(x_1, y_1) \times \{z_1\}) + \Gamma + (\Gamma(x_2, y_2) \times \{z_2\}) \tag{4.73}$$

joining (x_1, y_1, z_1) to (x_2, y_2, z_2) satisfying

$$\operatorname{length}(\tilde{\Gamma}) \leq \big[\frac{1}{3} + \frac{\pi\sqrt{3}}{36} + \hat{\epsilon}^{-1}\big(\frac{4}{3} + \frac{\pi\sqrt{3}}{36}\big)\big]|(x_1, y_1, z_1) - (x_2, y_2, z_2)| \tag{4.74}$$

and for all $p \in \tilde{\Gamma}$ we have

$$\begin{aligned} &\operatorname{dist}(p, \partial\Omega_{\rm out}(n) \times \mathbb{R}) \\ &\geq \epsilon_3|(x_1, y_1, z_1) - (x_2, y_2, z_2)|^{-1}|p - (x_1, y_1, z_1)||p - (x_2, y_2, z_2)| \end{aligned} \tag{4.75}$$

where

$$\begin{aligned}\epsilon_3 = \min\Big\{&\hat{\epsilon}\Big(\frac{4}{3}+\frac{\pi\sqrt{3}}{36}\Big)^{-1}(1+4\sqrt{3})^{-1}\Big(1+\frac{4\pi}{3}\Big)^{-1},\\ &\frac{1}{4\sqrt{3}}\Big(\frac{1}{4\sqrt{3}}+1\Big)^{-1}\Big[\big(\frac{1}{4\sqrt{3}}+1\big)3^{-1}+1\Big]^{-1},\\ &\frac{1}{4}\Big(\frac{1}{4}+\frac{\pi}{3}\Big)^{-1}\Big(\frac{\sqrt{3}}{48}+\frac{5}{3}+\frac{\pi\sqrt{3}}{36}\Big)^{-1}, k_1\left(\frac{\pi\sqrt{3}}{36}+1\right)^{-1},\\ &\frac{3}{4}\cos\beta_1, \frac{3}{4}k_2\Big\}.\end{aligned} \tag{4.76}$$

Case 2(i). Here we assume that $(x_1, y_1) \in \Omega_{\rm out}(n)\backslash\Omega_{\rm in}(n+1)$ and $(x_2, y_2) \in \Omega_{\rm in}(n+1)$, that (x_1, y_1) and (x_2, y_2) are not both inside a region $R \subseteq \Omega_{\rm out}(n)$ which is obtained from the polygon S in Figure 1 by a finite sequence of dilations and isometries and that some of the edges of R are also edges of $\Omega_{\rm out}(n)$, that (x_1, y_1) is within a distance of $\frac{1}{2}\cos(\pi/6)L_{\rm in}(n+1)$ from an acute vertex v_1 of $\Omega_{\rm in}(n+1)$, and that (4.4) and (4.5) hold.

By Proposition 4.2 there exists a path $\Gamma \subseteq \Omega_{\rm in}(n+1)\times\mathbb{R}$ joining $(a(x_1, y_1), z_1)$ to (x_2, y_2, z_2) satisfying

(C) $\operatorname{length}(\Gamma) \le \hat{\epsilon}^{-1}|(a(x_1, y_1), z_1) - (x_2, y_2, z_2)|$,

(D) for all $p \in \Gamma$ we have

$$\begin{aligned}&\operatorname{dist}(p, \partial\Omega_{\rm in}(n+1)\times\mathbb{R})\\ &\ge \hat{\epsilon}|(a(x_1, y_1), z_1) - (x_2, y_2, z_2)|^{-1}|p-(a(x_1, y_1), z_1)||p-(x_2, y_2, z_2)|.\end{aligned} \tag{4.77}$$

By (4.2), we have

$$\begin{aligned}&|(a(x_1, y_1), z_1) - (x_2, y_2, z_2)|\\ &\le |(a(x_1, y_1), z_1) - (x_1, y_1, z_1)| + |(x_1, y_1, z_1) - (x_2, y_2, z_2)|\\ &\le \frac{\pi\sqrt{3}}{12}L_{\rm in}(n+1) + |(x_1, y_1, z_1) - (x_2, y_2, z_2)|\\ &\le \big(\frac{\pi\sqrt{3}}{36}+1\big)|(x_1, y_1, z_1) - (x_2, y_2, z_2)|\end{aligned} \tag{4.78}$$

and

$$\begin{aligned}&|(x_1, y_1, z_1) - (x_2, y_2, z_2)|\\ &\le |(x_1, y_1, z_1) - (a(x_1, y_1), z_1)| + |(a(x_1, y_1), z_1) - (x_2, y_2, z_2)|\\ &\le \frac{\pi\sqrt{3}}{36}|(x_1, y_1, z_1) - (x_2, y_2, z_2)| + |(a(x_1, y_1), z_1) - (x_2, y_2, z_2)|\end{aligned}$$

and hence

$$|(x_1, y_1, z_1) - (x_2, y_2, z_2)| \le \Big(1-\frac{\pi\sqrt{3}}{36}\Big)^{-1}|(a(x_1, y_1), z_1) - (x_2, y_2, z_2)|. \tag{4.79}$$

Thus

$$\begin{aligned}\operatorname{length}(\Gamma) &\le \hat{\epsilon}^{-1}|(a(x_1, y_1), z_1) - (x_2, y_2, z_2)|\\ &\le \hat{\epsilon}^{-1}\big(\frac{\pi\sqrt{3}}{36}+1\big)|(x_1, y_1, z_1) - (x_2, y_2, z_2)|\end{aligned} \tag{4.80}$$

and

$$\begin{aligned} \operatorname{length}(\Gamma(x_1,y_1)\times\{z_1\}) &= \operatorname{length}(\Gamma(x_1,y_1)) \\ &\le \frac{\pi\sqrt{3}}{12} L_{\rm in}(n+1) \\ &\le \frac{\pi\sqrt{3}}{36} |(x_1,y_1,z_1)-(x_2,y_2,z_2)|. \end{aligned} \tag{4.81}$$

Let $p=(x,y,z_1)\in\Gamma(x_1,y_1)\times\{z_1\}$. Let $k_1>0$ be the constant defined by (4.21). Then (4.22) and (4.23) still hold, and hence we have

$$\begin{aligned} \operatorname{dist}(p,\partial\Omega_{\rm out}(n)\times\mathbb{R}) \ge k_1\Big(\frac{\pi\sqrt{3}}{36}+1\Big)^{-1} &|(x_1,y_1,z_1)-(x_2,y_2,z_2)|^{-1} \\ \times |p-(x_1,y_1,z_1)||p-(x_2,y_2,z_2)|&. \end{aligned} \tag{4.82}$$

Let $p\in\Gamma$ and suppose that

$$|p-(a(x_1,y_1),z_1)| \ge \frac{1}{2}\operatorname{dist}(a(x_1,y_1),\partial\Omega_{\rm in}(n+1)).$$

Then

$$\begin{aligned} |p-(a(x_1,y_1),z_1)| &\ge \frac{1}{2}\sin(\pi/6)|a(x_1,y_1)-v_1| \\ &= \frac{3}{4\pi}\left(\frac{\pi}{3}|a(x_1,y_1)-v_1|\right) \\ &\ge \frac{3}{4\pi}|(a(x_1,y_1),z_1)-(x_1,y_1,z_1)|, \end{aligned}$$

hence

$$\begin{aligned} |p-(x_1,y_1,z_1)| &\le |p-(a(x_1,y_1),z_1)| + |(a(x_1,y_1),z_1)-(x_1,y_1,z_1)| \\ &\le \big(1+\frac{4\pi}{3}\big)|p-(a(x_1,y_1),z_1)|. \end{aligned} \tag{4.83}$$

Combining (4.80) and (4.83) we have

$$\begin{aligned} &\operatorname{dist}(p,\partial\Omega_{\rm out}(n)\times\mathbb{R}) \\ &\ge \operatorname{dist}(p,\partial\Omega_{\rm in}(n+1)\times\mathbb{R}) \\ &\ge \hat\epsilon|(a(x_1,y_1),z_1)-(x_2,y_2,z_2)|^{-1}|p-(a(x_1,y_1),z_1)||p-(x_2,y_2,z_2)| \\ &\ge \hat\epsilon\Big(\frac{\pi\sqrt{3}}{36}+1\Big)^{-1}\Big(1+\frac{4\pi}{3}\Big)^{-1}|(x_1,y_1,z_1)-(x_2,y_2,z_2)|^{-1} \\ &\times |p-(x_1,y_1,z_1)||p-(x_2,y_2,z_2)|. \end{aligned} \tag{4.84}$$

Next let $p\in\Gamma$ and suppose that

$$|p-(a(x_1,y_1),z_1)| < \frac{1}{2}\operatorname{dist}(a(x_1,y_1),\partial\Omega_{\rm in}(n+1)).$$

Then

$$\begin{aligned} |p-(x_1,y_1,z_1)| &\le |p-(a(x_1,y_1),z_1)| + |(a(x_1,y_1),z_1)-(x_1,y_1,z_1)| \\ &\le \frac{1}{4}|a(x_1,y_1)-v_1| + \frac{\pi}{3}|a(x_1,y_1)-v_1| \\ &= \big(\frac{1}{4}+\frac{\pi}{3}\big)|a(x_1,y_1)-v_1|, \end{aligned} \tag{4.85}$$

and so

$$\begin{aligned} \operatorname{dist}(p, \partial\Omega_{\rm out}(n)\times\mathbb{R}) &\geq \operatorname{dist}(p, \partial\Omega_{\rm in}(n+1)\times\mathbb{R}) \\ &\geq \frac{1}{2}\operatorname{dist}(a(x_1,y_1), \partial\Omega_{\rm in}(n+1)) \\ &= \frac{1}{4}|a(x_1,y_1)-v_1| \\ &\geq \frac{1}{4}\big(\frac{1}{4}+\frac{\pi}{3}\big)^{-1}|p-(x_1,y_1,z_1)|. \end{aligned} \tag{4.86}$$

Also we have, by (4.80) and (4.2),

$$\begin{aligned} &|p-(x_2,y_2,z_2)| \\ &\leq |p-(a(x_1,y_1),z_1)| + |(a(x_1,y_1),z_1)-(x_2,y_2,z_2)| \\ &\leq \frac{1}{4}|a(x_1,y_1)-v_1| + \big(\frac{\pi\sqrt{3}}{36}+1\big)|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \\ &\leq \frac{\sqrt{3}}{16}L_{\rm in}(n+1) + \big(\frac{\pi\sqrt{3}}{36}+1\big)|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \\ &\leq \big(\frac{\sqrt{3}}{48}+\frac{\pi\sqrt{3}}{36}+1\big)|(x_1,y_1,z_1)-(x_2,y_2,z_2)|. \end{aligned} \tag{4.87}$$

Combining (4.86) and (4.87) we obtain

$$\begin{aligned} &\operatorname{dist}(p, \partial\Omega_{\rm out}(n)\times\mathbb{R}) \\ &\geq \frac{1}{4}\Big(\frac{1}{4}+\frac{\pi}{3}\Big)^{-1}\Big(\frac{\sqrt{3}}{48}+\frac{\pi\sqrt{3}}{36}+1\Big)^{-1}|(x_1,y_1,z_1)-(x_2,y_2,z_2)|^{-1} \\ &\quad\times|p-(x_1,y_1,z_1)||p-(x_2,y_2,z_2)|. \end{aligned} \tag{4.88}$$

Thus from (4.5), (4.6), (4.80), (4.81), (4.82), (4.84), and (4.88), we see that if (x_1,y_1,z_1) and (x_2,y_2,z_2) satisfy the assumptions of Case 2(i) and if

$$|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \leq \hat{\delta}/3,$$

then there exists a path

$$\tilde{\Gamma} = (\Gamma(x_1,y_1)\times\{z_1\}) + \Gamma \tag{4.89}$$

joining (x_1,y_1,z_1) to (x_2,y_2,z_2) satisfying

$$\operatorname{length}(\tilde{\Gamma}) \leq \Big(\frac{\pi\sqrt{3}}{36}+\hat{\epsilon}^{-1}\big(\frac{\pi\sqrt{3}}{36}+1\big)\Big)|(x_1,y_1,z_1)-(x_2,y_2,z_2)| \tag{4.90}$$

and for all $p\in\tilde{\Gamma}$ we have

$$\begin{aligned} &\operatorname{dist}(p, \partial\Omega_{\rm out}(n)\times\mathbb{R}) \\ &\geq \epsilon_4|(x_1,y_1,z_1)-(x_2,y_2,z_2)|^{-1}|p-(x_1,y_1,z_1)||p-(x_2,y_2,z_2)| \end{aligned} \tag{4.91}$$

where

$$\begin{aligned} \epsilon_4 = \min\Big\{&k_1\Big(\frac{\pi\sqrt{3}}{36}+1\Big)^{-1}, \hat{\epsilon}\Big(\frac{\pi\sqrt{3}}{36}+1\Big)^{-1}\Big(1+\frac{4\pi}{3}\Big)^{-1}, \\ &\frac{1}{4}\Big(\frac{1}{4}+\frac{\pi}{3}\Big)^{-1}\Big(\frac{\sqrt{3}}{48}+\frac{\pi\sqrt{3}}{36}+1\Big)^{-1}\Big\}. \end{aligned} \tag{4.92}$$

Case 2(ii). Here we assume that $(x_1,y_1)\in\Omega_{\rm out}(n)\backslash\Omega_{\rm in}(n+1)$ and $(x_2,y_2)\in\Omega_{\rm in}(n+1)$, that (x_1,y_1) and (x_2,y_2) are not both inside a region $R\subseteq\Omega_{\rm out}(n)$ which is obtained from the polygon S in Figure 1 by a finite sequence of dilations and isometries and that some of the edges of R are also edges of $\Omega_{\rm out}(n)$, that (x_1,y_1) is not within a distance of $\frac12\cos(\pi/6)L_{\rm in}(n+1)$ from any acute vertex of $\Omega_{\rm in}(n+1)$, and that (4.4) and (4.5) holds.

Referring to Figure 5, we see that in this case (4.31)-(4.38) and (4.40) still hold, and so, for all $p\in\Gamma(x_1,y_1)\times\{z_1\}$, we have

$$\begin{aligned}\operatorname{dist}(p,\partial\Omega_{\rm out}(n)\times\mathbb{R})\geq\;&\frac34\min\{\cos\beta_1,k_2\}|(x_1,y_1,z_1)-(x_2,y_2,z_2)|^{-1}\\&\times|p-(x_1,y_1,z_1)||p-(x_2,y_2,z_2)|,\end{aligned}\tag{4.93}$$

and

$$\operatorname{length}(\Gamma(x_1,y_1)\times\{z_1\})\leq\frac13|(x_1,y_1,z_1)-(x_2,y_2,z_2)|.\tag{4.94}$$

Hence

$$\begin{aligned}&\operatorname{length}(\Gamma)\\&\leq\hat\epsilon^{-1}|(a(x_1,y_1),z_1)-(x_2,y_2,z_2)|\\&\leq\hat\epsilon^{-1}\{|(a(x_1,y_1),z_1)-(x_1,y_1,z_1)|+|(x_1,y_1,z_1)-(x_2,y_2,z_2)|\}\\&\leq\frac43\hat\epsilon^{-1}|(x_1,y_1,z_1)-(x_2,y_2,z_2)|.\end{aligned}\tag{4.95}$$

Let $p\in\Gamma$ and suppose that

$$|p-(a(x_1,y_1),z_1)|\geq\frac12\operatorname{dist}(a(x_1,y_1),\partial\Omega_{\rm in}(n+1)).$$

Then

$$|p-(a(x_1,y_1),z_1)|\geq\frac{1}{4\sqrt3}L_{\rm in}(n+1)\geq\frac{1}{4\sqrt3}|(a(x_1,y_1),z_1)-(x_1,y_1,z_1)|.\tag{4.96}$$

Thus

$$\begin{aligned}|p-(x_1,y_1,z_1)|&\leq|p-(a(x_1,y_1),z_1)|+|(a(x_1,y_1),z_1)-(x_1,y_1,z_1)|\\&\leq(1+4\sqrt3)|p-(a(x_1,y_1),z_1)|.\end{aligned}\tag{4.97}$$

Also, from (4.2) and Figure 5,

$$\begin{aligned}&|(a(x_1,y_1),z_1)-(x_2,y_2,z_2)|\\&\leq|(a(x_1,y_1),z_1)-(x_1,y_1,z_1)|+|(x_1,y_1,z_1)-(x_2,y_2,z_2)|\\&\leq L_{\rm in}(n+1)+|(x_1,y_1,z_1)-(x_2,y_2,z_2)|\\&\leq\frac43|(x_1,y_1,z_1)-(x_2,y_2,z_2)|.\end{aligned}\tag{4.98}$$

Hence we have, from (4.97) and (4.98),

$$\begin{aligned}\operatorname{dist}(p,\partial\Omega_{\rm out}(n)\times\mathbb{R})&\geq\operatorname{dist}(p,\partial\Omega_{\rm in}(n+1)\times\mathbb{R})\\&\geq\hat\epsilon|(a(x_1,y_1),z_1)-(x_2,y_2,z_2)|^{-1}|p-(a(x_1,y_1),z_1)|\\&\quad\times|p-(x_2,y_2,z_2)|\\&\geq\hat\epsilon\frac34(1+4\sqrt3)^{-1}|(x_1,y_1,z_1)-(x_2,y_2,z_2)|^{-1}\\&\quad\times|p-(x_1,y_1,z_1)||p-(x_2,y_2,z_2)|.\end{aligned}\tag{4.99}$$

Now let $p \in \Gamma$ and suppose that

$$|p - (a(x_1, y_1), z_1)| < \frac{1}{2} \operatorname{dist}(a(x_1, y_1), \partial\Omega_{\rm in}(n+1)).$$

Then

$$\begin{aligned} \operatorname{dist}(p, \partial\Omega_{\rm out}(n) \times \mathbb{R}) &\geq \operatorname{dist}(p, \partial\Omega_{\rm in}(n+1) \times \mathbb{R}) \\ &\geq \frac{1}{2} \operatorname{dist}((a(x_1, y_1), z_1), \partial\Omega_{\rm in}(n+1) \times \mathbb{R}) \\ &= \frac{1}{4\sqrt{3}} L_{\rm in}(n+1) \end{aligned} \tag{4.100}$$

and, referring to Figure 5,

$$\begin{aligned} |p - (x_1, y_1, z_1)| &\leq |p - (a(x_1, y_1), z_1)| + |(a(x_1, y_1), z_1) - (x_1, y_1, z_1)| \\ &\leq \big(\frac{1}{4\sqrt{3}} + 1\big) L_{\rm in}(n+1). \end{aligned} \tag{4.101}$$

Hence, combining (4.100) and (4.101), we get

$$\operatorname{dist}(p, \partial\Omega_{\rm out}(n) \times \mathbb{R}) \geq \frac{1}{4\sqrt{3}} \Big(\frac{1}{4\sqrt{3}} + 1\Big)^{-1} |p - (x_1, y_1, z_1)|. \tag{4.102}$$

Also, by (4.2) and (4.101),

$$\begin{aligned} |p - (x_2, y_2, z_2)| &\leq |p - (x_1, y_1, z_1)| + |(x_1, y_1, z_1) - (x_2, y_2, z_2)| \\ &\leq \big(\frac{1}{4\sqrt{3}} + 1\big) L_{\rm in}(n+1) + |(x_1, y_1, z_1) - (x_2, y_2, z_2)| \\ &\leq \big(\frac{1}{12\sqrt{3}} + \frac{4}{3}\big) |(x_1, y_1, z_1) - (x_2, y_2, z_2)|. \end{aligned} \tag{4.103}$$

Combining (4.102) and (4.103) we obtain

$$\begin{aligned} &\operatorname{dist}(p, \partial\Omega_{\rm out}(n) \times \mathbb{R}) \\ &\geq \frac{1}{4\sqrt{3}} \Big(\frac{1}{4\sqrt{3}} + 1\Big)^{-1} \Big(\frac{1}{12\sqrt{3}} + \frac{4}{3}\Big)^{-1} |(x_1, y_1, z_1) - (x_2, y_2, z_2)|^{-1} \\ &\quad \times |p - (x_1, y_1, z_1)| |p - (x_2, y_2, z_2)|. \end{aligned} \tag{4.104}$$

Thus from (4.5), (4.6), (4.93), (4.94), (4.95), (4.99), and (4.104), we see that if (x_1, y_1, z_1) and (x_2, y_2, z_2) satisfy the assumptions of Case 2(ii) and if

$$|(x_1, y_1, z_1) - (x_2, y_2, z_2)| \leq \hat{\delta}/3,$$

then there exists a path

$$\tilde{\Gamma} = (\Gamma(x_1, y_1) \times \{z_1\}) + \Gamma \tag{4.105}$$

joining (x_1, y_1, z_1) to (x_2, y_2, z_2) satisfying

$$\operatorname{length}(\tilde{\Gamma}) \leq \big(\frac{1}{3} + \frac{4}{3}\hat{\epsilon}^{-1}\big) |(x_1, y_1, z_1) - (x_2, y_2, z_2)| \tag{4.106}$$

and for all $p \in \tilde{\Gamma}$ we have

$$\begin{aligned} &\operatorname{dist}(p, \partial\Omega_{\rm out}(n) \times \mathbb{R}) \\ &\geq \epsilon_5 |(x_1, y_1, z_1) - (x_2, y_2, z_2)|^{-1} |p - (x_1, y_1, z_1)| |p - (x_2, y_2, z_2)| \end{aligned} \tag{4.107}$$

where

$$\begin{aligned}\epsilon_5 = \min\Big\{&\frac{3}{4}\cos\beta_1, \frac{3}{4}k_2, \frac{3}{4}\big(1+4\sqrt{3}\big)^{-1}\hat{\epsilon},\\ &\frac{1}{4\sqrt{3}}\Big(\frac{1}{4\sqrt{3}}+1\Big)^{-1}\Big(\frac{1}{12\sqrt{3}}+\frac{4}{3}\Big)^{-1}\Big\}.\end{aligned} \tag{4.108}$$

To summarize: Cases 1(i),(ii),(iii), 2(i),(ii), exhaust all possibilities of at least one of (x_1, y_1, z_1) or (x_2, y_x, z_2) is in $(\Omega_{\rm out}(n) \times \mathbb{R})\backslash(\Omega_{\rm in}(n+1) \times \mathbb{R})$ with (x_1, y_1) and (x_2, y_2) not both contained in a region $R \subseteq \Omega_{\rm out}(n)$ which is obtained from the polygon S in Figure 1 by a finite sequence of dilations and isometries and that at least one of the edges of R is also an edge of $\Omega_{\rm out}(n)$. Let

$$\delta_6 = \hat{\delta}/3 \tag{4.109}$$

and

$$\begin{aligned}\epsilon_6 = \min\Big\{&\Big[\frac{\sqrt{3}\pi}{18}+\hat{\epsilon}\big(\frac{\sqrt{3}\pi}{18}+1\big)\Big]^{-1}, \Big(\frac{2}{3}+\frac{5}{3}\hat{\epsilon}^{-1}\Big)^{-1},\\ &\Big[\frac{1}{3}+\frac{\pi\sqrt{3}}{36}+\hat{\epsilon}^{-1}\big(\frac{4}{3}+\frac{\pi\sqrt{3}}{36}\big)\Big]^{-1}, \Big[\frac{\pi\sqrt{3}}{36}+\hat{\epsilon}^{-1}\Big(\frac{\pi\sqrt{3}}{36}+1\Big)\Big]^{-1},\\ &\Big(\frac{1}{3}+\frac{4}{3}\hat{\epsilon}^{-1}\Big)^{-1}, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5\Big\}.\end{aligned} \tag{4.110}$$

Then we have proved that in each of these cases, if

$$|(x_1, y_1, z_1) - (x_2, y_2, z_2)| \le \delta_6, \tag{4.111}$$

then there exists a path $\tilde{\Gamma} \subseteq \Omega_{\rm out}(n) \times \mathbb{R}$ joining (x_1, y_1, z_1) to (x_2, y_2, z_2) satisfying

$$\text{length}(\tilde{\Gamma}) \le \epsilon_6^{-1}|(x_1, y_1, z_1) - (x_2, y_2, z_2)| \tag{4.112}$$

and for all $p \in \tilde{\Gamma}$ we have

$$\begin{aligned}&\text{dist}(p, \partial\Omega_{\rm out}(n) \times \mathbb{R})\\ &\ge \epsilon_6|(x_1, y_1, z_1) - (x_2, y_2, z_2)|^{-1}|p - (x_1, y_1, z_1)||p - (x_2, y_2, z_2)|.\end{aligned} \tag{4.113}$$

This together with Proposition 4.2 and the fact $S\times\mathbb{R}$ is a Lipschitz domain, and thus an (ϵ_7, δ_7)-domain in $\mathbb{R}^3$ for some $\epsilon_7, \delta_7 > 0$, complete the proof of Theorem 4.3. $\square$

We finish this section by giving the proof of Theorem 1.3. We shall need the following results:

Proposition 4.4 (see [5, Theorem 1]). *Let $D \in \mathbb{R}^d$ be an (ϵ, δ)-domain. Suppose $k \in \{1, 2, 3, \dots\}$ and $1 \le p \le \infty$. Then there exists a bounded extension operator $\Lambda_{k,p} : W^{k,p}(D) \to W^{k,p}(\mathbb{R}^d)$ such that*

$$(\Lambda_{k,p}f)|_D = f \quad (f \in W^{k,p}(D)).$$

Moreover, the norm $\|\Lambda_{k,p}\|$ depends only on ϵ, δ, k, p and the dimension d.

Proposition 4.5 (see [4, p.47]). *Suppose $D \subseteq \mathbb{R}^d$ is a domain such that for some $p \in [1, d)$ there exists a bounded extension operator $\Lambda_{1,p} : W^{1,p}(D) \to W^{1,p}(\mathbb{R}^d)$ satisfying*

$$(\Lambda_{1,p}f)|_D = f \quad (f \in W^{1,p}(D)).$$

Let q be defined by $\frac{1}{q} = \frac{1}{p} - \frac{1}{d}$. Then there exists $c = c(d) \ge 1$ such that

$$\|f\|_q \le c\|\Lambda_{1,p}\|\{\|\nabla f\|_p^p + \|f\|_p^p\}^{\frac{1}{p}} \quad (f \in W^{1,p}(D)).$$

Proposition 4.6 (see [4, Theorem 2.4.2, Corollaries 2.4.3,and 2.2.8])**.** *Let $D \subseteq \mathbb{R}^d$, $d \geq 3$, be a domain. Suppose there exists $c_1 \geq 1$ such that*

$$\|f\|_{\frac{2d}{d-2}} \leq c_1\{\|\nabla f\|_2^2 + \|f\|_2^2\}^{1/2} \quad (f \in W^{1,2}(D)).$$

Then there exists $c_2 \geq 1$, depending only on c_1 and d, such that

$$P^D(t,x,y) \leq c_2 t^{-d/2} \quad (0 < t \leq 1, x, y \in D),$$

where $P^D(t,x,y)$ denotes the heat kernel associated to the semigroup generated by the Neumann Laplacian on D.

Proof of Theorem 1.3. We follow the arguments in [7]. We also remark that the constants K_i, $i = 1,2,3$, in the argument below will depend only on the values of $\check{\epsilon}$ and $\check{\delta}$ in Theorem 4.3, but not on n.

By Theorem 4.3 and Proposition 4.4 (with $D = \Omega_{\rm out}(n) \times \mathbb{R}$, $k = 1$ and $p = 2$ in Proposition 4.4), there exists a bounded linear extension operator $\Lambda_{1,2} : W^{1,2}(\Omega_{\rm out}(n) \times \mathbb{R}) \to W^{1,2}(\mathbb{R}^3)$ such that

$$(\Lambda_{1,2} f)\,|_{\Omega_{\rm out}(n)\times\mathbb{R}} = f \quad (f \in W^{1,2}(\Omega_{\rm out}(n) \times \mathbb{R})),$$

where the norm $\|\Lambda_{1,2}\|$ depends only on $\check{\epsilon}$ and $\check{\delta}$. So by Proposition 4.5 (with $D = \Omega_{\rm out}(n) \times \mathbb{R}$, $d = 3$, $p = 2$ and $q = 6$), we have

$$\|f\|_6 \leq K_1\{\|\nabla f\|_2^2 + \|f\|_2^2\}^{1/2} \quad (f \in W^{1,2}(\Omega_{\rm out}(n) \times \mathbb{R}))$$

where $K_1 \geq 1$ depends only on $\check{\epsilon}$ and $\check{\delta}$. Hence, by Proposition 4.6, we have

$$P^{\Omega_{\rm out}(n)\times\mathbb{R}}(t,(x_1,y_1,z_1),(x_2,y_2,z_2)) \leq K_2 t^{-3/2} \tag{4.114}$$

for all $0 < t \leq 1$ and all (x_1,y_1,z_1), $(x_2,y_2,z_2) \in \Omega_{\rm out}(n) \times \mathbb{R}$, where $K_2 \geq 1$ depends only on $\check{\epsilon}$ and $\check{\delta}$. Since

$$\begin{aligned} &P^{\Omega_{\rm out}(n)\times\mathbb{R}}(t,(x_1,y_1,z),(x_2,y_2,z)) \\ &= P^{\Omega_{\rm out}(n)}(t,(x_1,y_1),(x_2,y_2))P^{\mathbb{R}}(t,z,z) \\ &= (4\pi t)^{-1/2} P^{\Omega_{\rm out}(n)}(t,(x_1,y_1),(x_2,y_2)) \end{aligned} \tag{4.115}$$

for all $0 < t \leq 1, (x_1,y_1),(x_2,y_2) \in \Omega_{\rm out}(n)$ and $z \in \mathbb{R}$, we have, from (4.114) and (4.115),

$$P^{\Omega_{\rm out}(n)}(t,(x_1,y_1),(x_2,y_2)) \leq K_3 t^{-1} \tag{4.116}$$

for all $0 < t \leq 1$ and $(x_1,y_1),(x_2,y_2) \in \Omega_{\rm out}(n)$, where $K_3 \geq 1$ depends only on $\check{\epsilon}$ and $\check{\delta}$. The proof of Theorem 1.3 is complete by combining (4.116) and [7, Theorem 1.3]. □

Acknowledgement

We thank the Summer Research Fellowship of the University of Missouri for support during this project.

References

[1] Banuelos, R.; Pang, M. M. H.; *Stability and approximations of eigenvalues and eigenfunctions for the Neumann Laplacian, Part 1*, Electron. J. Diff. Eqns., Vol. 2008 (2008), no. 145, 1-13.

[2] Burenkov, V. I.; Lamberti, P. D.; Lanza de Cristoforis, M.; *Spectral stability of non-negative self-adjoint operators* (Russian), Sovrem. Mat. Fundam. Napravl., Vol.15 (2006), 76-111; translation in J. Math. Sci. (N.Y.), Vol. 149 (2008), no.4, 1417-1452.

[3] Davies, E. B.; *One-parameter Semigroups*, Academic Press, London, 1980.

[4] Davies, E. B.; *Heat Kernels and Spectral Theory*, Cambridge University Press, Cambridge, 1989.

[5] Jones, P. W.; *Quasiconformal mappings and extendability of functions in Sobolev spaces*, Acta Math. (1), Vol. 147(1981), 71-88.

[6] Pang, M. M. H.; *L^1 and L^2 properties of a class of singular second order elliptic operators on $\mathbb{R}^N$ with measurable coefficients*, J. Diff. Eqns., Vol. 129 (1996), 1-17.

[7] Pang, M. M. H.; *Stability and approximations of eigenvalues and eigenfunctions for the Neumann Laplacian, Part 2*, J. Math. Anal. and Applications, Vol. 345(2008), no. 1, 485-499.

Michael M. H. Pang

Department of Mathematics, University of Missouri-Columbia, Columbia, MO 65211, USA

E-mail address: pangm@missouri.edu