Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 10, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

VECTOR-VALUED MORREY'S EMBEDDING THEOREM AND HÖLDER CONTINUITY IN PARABOLIC PROBLEMS

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Abstract. If $I \subset \mathbb{R}$ is an open interval and $\Omega \subset \mathbb{R}^N$ an open subset with $\partial\Omega$ Lipschitz continuous, we show that the space $W^{1,p}(I, L^q(\Omega)) \cap L^p(I, W^{1,q}(\Omega))$ is continuously embedded in $C^{0,\frac{1}{p'}-\frac{N}{q}}(\overline{\Omega \times I}) \cap L^\infty(\Omega \times I)$ if $p, q \in (1, \infty)$ and $q > Np'$. When $p = q$, this coincides with Morrey's embedding theorem for $W^{1,p}(\Omega \times I)$. While weaker results have been obtained by various methods, including very technical ones, the proof given here follows that of Morrey's theorem in the scalar case and relies only on widely known properties of the classical Sobolev spaces and of the Bochner integral.

This embedding is useful to formulate nonlinear evolution problems as functional equations, but it has other applications. As an example, we derive apparently new space-time Hölder continuity properties for $u_t = Au + f, u(\cdot, 0) = u_0$ when A generates a holomorphic semigroup on $L^q(\Omega)$.

1. Introduction

Let $I \subset \mathbb{R}$ be an open interval and $\Omega \subset \mathbb{R}^N$ an open subset satisfying the strong local Lipschitz condition [1, p. 83]. For $p, q \in (1, \infty)$, we set

$$\mathcal{V}^{p,q}(I, \Omega) := W^{1,p}(I, L^q(\Omega)) \cap L^p(I, W^{1,q}(\Omega)),$$

a Banach space for the natural norm. Subspaces of $\mathcal{V}^{p,q}(I, \Omega)$ arise naturally in evolution problems and have become especially important due to recent progress in the so-called "L^p maximal regularity" issue and related topics. In such problems, the space of interest is $W^{1,p}(I, L^q(\Omega)) \cap L^p(I, D(A))$, where A is an unbounded linear operator on $L^q(\Omega)$ with domain $D(A)$ usually contained in $W^{1,q}(\Omega)$ or even $W^{k,q}(\Omega)$ with $k \geq 2$. See for instance Denk *et al.* [11] and the references therein, or Arendt and Bu [3] and Arendt and Rabier [4] for the time-periodic case.

Every $u \in \mathcal{V}^{p,q}(I, \Omega)$ can be identified with a measurable real-valued function $u(x, t)$ (further details below). The primary goal of this note is to give a simple proof of the following embedding theorem:

Theorem 1.1. *If $p, q \in (1, \infty)$ and $\frac{1}{p} + \frac{N}{q} < 1$ (i.e., $q > Np'$ with $p' := \frac{p}{p-1}$), then $\mathcal{V}^{p,q}(I, \Omega) \hookrightarrow C^{0,\frac{1}{p'}-\frac{N}{q}}(\overline{\Omega \times I}) \cap L^\infty(\Omega \times I)$. In particular, if I and Ω are bounded, the embedding $\mathcal{V}^{p,q}(I, \Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega \times I})$ is compact for every $0 \leq \alpha < \frac{1}{p'} - \frac{N}{q}$.*

2000 *Mathematics Subject Classification.* 46E35, 46E40, 35K90, 35K55.
Key words and phrases. Morrey's theorem; embedding; vector-valued Sobolev space; mixed norm.
©2011 Texas State University - San Marcos.
Submitted January 11, 2011. Published January 19, 2011.

When $p = q$, then $\mathcal{V}^{p,p}(I,\Omega) = W^{1,p}(\Omega \times I)$, so that Morrey's theorem is recovered. The latter is of notorious importance in elliptic PDEs, especially for the formulation of nonlinear problems as functional equations. In particular, it is often used to prove the definiteness of Nemytskii operators. Theorem 1.1 can be used in the same way in problems of parabolic type, now that the relevance of $\mathcal{V}^{p,q}(I,\Omega)$ and its subspaces is becoming more widely known (see Morris [17] when $p = q$). In time-dependent problems, it is not uncommon to require different integrability properties in space and time, so that the case $p \neq q$ has concrete value. In fact, a completely different application to a regularity question is discussed in Section 4, in which $p = q$ would be far too restrictive.

There are many connections between Theorem 1.1 and the existing literature, starting with the so-called anisotropic Sobolev spaces with mixed norm. First, recall that $L^p(I, L^q(\Omega))$ is isometrically isomorphic to $L^{\mathbf{r}}(\Omega\times I)$ where $\mathbf{r} := (q, \dots, q, p) \in (1,\infty)^{N+1}$ (see Besov *et al.* [6, p. 7] for the definition of $L^{\mathbf{r}}(G)$; the definition includes measurability on G). A proof can be found in Benedek and Panzone [5, pp. 318-319].

It follows readily from the identification of $L^p(I, L^q(\Omega))$ and $L^{\mathbf{r}}(\Omega\times I)$ and from the definitions of the derivatives of scalar and vector-valued distributions ([20]) that if $u \in W^{1,p}(I, L^q(\Omega))$, the function corresponding to $\frac{du}{dt} \in L^p(I, L^q(\Omega))$ in $L^{\mathbf{r}}(\Omega\times I)$ is just u_t (the partial derivative of u as a scalar distribution on $\Omega \times I$). Likewise, if $u \in L^p(I, W^{1,q}(\Omega))$, the "spatial" partial derivative u_{x_j} of u as a scalar distribution on $\Omega\times I$ corresponds to the vector-valued derivative $u_{x_j} \in L^p(I, L^q(\Omega)), 1 \le j \le N$.

Thus, $u \in \mathcal{V}^{p,q}(I,\Omega)$ if and only if $u, u_t, u_{x_j} \in L^{\mathbf{r}}(\Omega \times I), 1 \le j \le N$, when u is viewed as a function of (x,t). In turn, this characterizes the elements of $W^{\mathbf{1},\mathbf{r}}(\Omega \times I))$ where $\mathbf{1} := (1,\dots,1) \in \mathbb{N}^{N+1}$ ([6, p. 165]). That $W^{\mathbf{1},\mathbf{r}}(\Omega \times I)) \hookrightarrow C^0(\overline{\Omega \times I}) \cap L^\infty(\Omega \times I)$ then follows from [6, Theorem 10.4] for $G = \mathbb{R}^{N+1}$ and the extension theorem [6, Theorem 9.6] (the condition $\kappa < 1$ in [6, Theorem 10.4] is exactly $q > Np'$). If I and Ω are bounded, the embedding is compact by [7, Theorem 26.3.5] (not based on Ascoli's theorem) and the estimate in [6, Theorem 10.4]. However, the only Hölder continuity result, [7, Theorem 27.4.2, p. 248], does not allow for $p \neq q$. Therefore, this quite lengthy and technical approach does no prove Theorem 1.1 when $p \neq q$ (and when $p = q$, the much simpler classical theorem of Morrey suffices).

When $I = \mathbb{R}$ and $\Omega = \mathbb{R}^N$, an embedding into $C^0(\mathbb{R}^{N+1})$ was proved by Rao [19] for a related but different space with mixed norm involving a "half-derivative" operator instead of t -differentiation, under the stronger requirement $\frac{2}{p} + \frac{N}{q} < 1$. The proof is by convolution arguments (parabolic Riesz potentials). Still under the same condition $\frac{2}{p}+\frac{N}{q} < 1$ and yet again by other methods, Prüss [18] and Engler [15, Lemma A3] have obtained other embeddings for the smaller space $W^{1,p}(I, L^q(\Omega)) \cap L^p(I, W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega))$.

If both I and Ω are bounded, a more recent abstract theorem of Amann [2, Theorem 1.1] yields the compactness of the embedding of $\mathcal{V}^{p,q}(I,\Omega)$ into $C^{0,\alpha}(I, B^{\frac{N}{q};q,1}(\Omega))$ for every $0 \le \alpha < \frac{1}{p'} - \frac{N}{q}$ (but not for $\alpha = \frac{1}{p'} - \frac{N}{q}$), where $B^{\frac{N}{q};q,1}(\Omega) := [L^q(\Omega), W^{1,q}(\Omega)]_{\frac{N}{q},1}$ is the usual Besov space. (Specifically, let $E_0 = L^q(\Omega), E_1 = W^{1,q}(\Omega), \theta = \frac{N}{q}$ and $p_0 = p_1 = p, s_0 = 1$ in Amann's theorem.) Since $\frac{N}{q}q = N$, it is only true that $B^{\frac{N}{q};q,1}(\Omega) \hookrightarrow C^0_B(\Omega)$ (bounded continuous functions on Ω; see [1, p. 231]), so that this proves the compactness of the embedding of $\mathcal{V}^{p,q}(I,\Omega)$

into $C^{0,\alpha}(I, C^0_B(\Omega))$, but not even its embedding into $C^0(\overline{\Omega \times I})$, let alone into any Hölder space on $\overline{\Omega \times I}$.

In retrospect, Theorem 1.1 is certainly not surprising. Yet, it is apparently not so intuitive, if only to judge by the foregoing review of some of the literature that addresses similar issues but fails to deliver the same property. In addition, the proof given here is elementary, insofar as being based only on widely known results about Sobolev spaces and the Bochner integral. It follows the standard proof of Morrey's theorem in the scalar case ([1], [9]) with of course extra technicalities.

The case when $I = \mathbb{R}$ and $\Omega = \mathbb{R}^N$ is discussed first, in the next section. The general case (Section 3) follows from the existence of a linear extension operator $\mathcal{V}^{p,q}(I,\Omega) \to \mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)$. This is well known when $p = q$ since $\mathcal{V}^{p,p}(I,\Omega) = W^{1,p}(\Omega \times I)$, but must be proved in general.

As pointed out earlier, Theorem 1.1 is useful to establish the well-posedness of nonlinear problems of parabolic type in suitable function spaces, but it also has a direct application to the Hölder continuity, jointly in space and time, of their solutions, even in the linear case. This short application is discussed in Section 4, first for the autonomous equation $u_t = Au$, $u(\cdot,0) = u_0$, where A is the generator of a C_0 holomorphic semigroup on $L^q(\Omega)$ (Theorem 4.1) and next in greater generality. In spite of a multitude of related results in the literature, we have found no evidence that the same Hölder continuity feature has previously been proved by other arguments.

2. Proof of Theorem 1.1 for $\mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)$

We begin with the following denseness lemma.

Lemma 2.1. *If $p, q \in [1,\infty)$, then $C^\infty_0(\mathbb{R}^{N+1})$ is dense in $\mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)$.*

Proof. First, we show that $u \in \mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)$ can be approximated by a sequence $(u_n) \subset \mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)$ such that $u_n(x,t) = 0$ for (x,t) outside a cube in $\mathbb{R}^{N+1}$.

Indeed, let $\zeta \in C^\infty_0(\mathbb{R})$ be such that $\zeta = 1$ on $[-1,1]$ and let $\zeta_n(t) := \zeta(t/n)$. Then, $\zeta_n u \to u$ in $\mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)$, so that we may assume with no loss of generality that u has compact support in t in the first place.

If $\xi_n(x) := \zeta_n(|x|)$, then $\xi_n \in C^\infty_0(\mathbb{R}^N)$ and $\xi_n u(\cdot,t) \to u(\cdot,t)$ in $W^{1,q}(\mathbb{R}^N)$ for a.e. $t \in \mathbb{R}$. It is readily checked that there is a constant $C > 0$ independent of t and n such that $\|\xi_n u(\cdot,t)\|_{W^{1,q}(\mathbb{R}^N)} \leq C\|u(\cdot,t)\|_{W^{1,q}(\mathbb{R}^N)}$. Thus, by dominated convergence, $\xi_n u \to u$ in $L^p(\mathbb{R}, W^{1,q}(\mathbb{R}^N))$. Similar arguments and the remark that $\frac{d}{dt}(\xi_n u) = \xi_n \frac{du}{dt}$ show that $\xi_n u \to u$ in $W^{1,p}(\mathbb{R}, L^q(\mathbb{R}^N))$, so that $\xi_n u \to u$ in $\mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)$. This proves the claim since $\xi_n u$ has compact support in t and x.

From the above, it suffices to show that if $u \in \mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)$ and $u(x,t) = 0$ outside a cube in $\mathbb{R}^{N+1}$, then u can be approximated in $\mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)$ by a sequence from $C^\infty_0(\mathbb{R}^{N+1})$. Let $\Theta \in C^\infty_0(\mathbb{R}^N)$ be such that $\Theta \geq 0$ and $\int_{\mathbb{R}^N} \Theta = 1$ and let $\Theta_n(x) := n^N\Theta(nx)$. Set $u_n := \Theta_n *_x u$, where $*_x$ denotes convolution with respect to the x variable.

For a.e. $t \in \mathbb{R}$, we have $u(\cdot,t) \in W^{1,q}(\mathbb{R}^N)$ and so $u_n(\cdot,t) \to u(\cdot,t)$ in $W^{1,q}(\mathbb{R}^N)$, as is well known. In addition, by Young's inequality, there is a constant $C > 0$ independent of t and n such that $\|u_n(\cdot,t)\|_{W^{1,q}(\mathbb{R}^N)} \leq C\|u(\cdot,t)\|_{W^{1,q}(\mathbb{R}^N)}$. Thus, by dominated convergence, $u_n \to u$ in $L^p(\mathbb{R}; W^{1,q}(\mathbb{R}^N))$. Similar arguments and the remark that $\frac{du_n}{dt} = \Theta_n *_x \frac{du}{dt}$ show that $u_n \to u$ in $W^{1,p}(\mathbb{R}, L^q(\mathbb{R}^N))$, so that $u_n \to u$ in $\mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)$.

Now, let $\theta \in C_0^\infty(\mathbb{R})$ be such that $\theta \geq 0$ and $\int_{\mathbb{R}} \theta = 1$ and let $\theta_m(t) := m\theta(mt)$. With $n \in \mathbb{N}$ being fixed, set $u_{nm}(x,t) := \theta_m *_t u_n(x,t)$, where $*_t$ denotes convolution with respect to the t variable. The standard properties of convolution imply that $u_{nm} \to u_n$ in $W^{1,p}(\mathbb{R}; L^q(\mathbb{R}^N))$ and in $L^p(\mathbb{R}; W^{1,q}(\mathbb{R}^N))$ as $m \to \infty$, so that $u_{nm} \to u_n$ in $\mathcal{V}^{p,q}(\mathbb{R}, \mathbb{R}^N)$ as $m \to \infty$. Thus, u_{nm} is arbitrarily close to u in $\mathcal{V}^{p,q}(\mathbb{R}, \mathbb{R}^N)$ if n and m are large enough. On the other hand, $u_{nm} = \theta_m *_t (\Theta_n *_x u) = (\theta_m \otimes \Theta_n) * u$, where $*$ denotes convolution with respect to the (x,t) variable and u is identified with a function of $L^{\mathbf{r}}(\mathbb{R}^N \times \mathbb{R})$ (see the Introduction). Since u has compact support (so that $u \in L^1(\mathbb{R}^{N+1})$), the same thing is true of u_{nm}. Furthermore, u_{nm} is C^∞ since $\theta_m \otimes \Theta_n$ is C^∞. $\square$

The proof of Theorem 1.1 for $\mathcal{V}^{p,q}(\mathbb{R}, \mathbb{R}^N)$ follows at once from Lemma 2.1 and the following result.

Lemma 2.2. *There is a constant $C > 0$ such that*

$$\|\varphi\|_{L^\infty(\mathbb{R}^{N+1})} \leq C\|\varphi\|_{\mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)} \tag{2.1}$$

and that

$$|\varphi(x,t) - \varphi(y,s)| \leq C|(x,t) - (y,s)|^{\frac{1}{p'} - \frac{N}{q}} \|\varphi\|_{\mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)}, \tag{2.2}$$

for every $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$ and every $(x,t), (y,s) \in \mathbb{R}^{N+1}$.

Proof. Let $J_\rho \subset \mathbb{R}$ be an interval of length $\rho > 0$ containing 0 and $Q_\rho \subset \mathbb{R}^N$ a cube with side ρ containing 0 and parallel to the coordinate axes. For $(x,t) \in Q_\rho \times J_\rho$ and $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$ and since $|t|, |x_j| \leq \rho$,

$$|\varphi(x,t) - \varphi(0)| \leq \rho\Big(\int_0^1 |\varphi_t(\tau x, \tau t)| d\tau + \sum_{j=1}^N \int_0^1 |\varphi_{x_j}(\tau x, \tau t)| d\tau\Big).$$

Upon integrating this inequality over $Q_\rho \times J_\rho$ and with $\overline{\varphi}$ denoting the average of φ on $Q_\rho \times J_\rho$, we obtain

$$\begin{aligned} |\overline{\varphi} - \varphi(0)| \leq \rho^{-N} \int_0^1 \tau^{-N-1} d\tau \Big(& \int_{\tau Q_\rho \times \tau J_\rho} |\varphi_t(y,s)| dy\, ds \\ & + \sum_{j=1}^N \int_{\tau Q_\rho \times \tau J_\rho} |\varphi_{x_j}(y,s)| dy\, ds\Big). \end{aligned} \tag{2.3}$$

Let $\tau \in [0,1]$ and $s \in \tau J_\rho$ be fixed. By Hölder's inequality,

$$\begin{aligned} \int_{\tau Q_\rho} |\varphi_t(y,s)| dy &\leq |\tau Q_\rho|^{1/q'} \Big(\int_{\tau Q_\rho} |\varphi_t(y,s)|^q dy\Big)^{1/q} \\ &\leq |\tau Q_\rho|^{1/q'} \Big(\int_{Q_\rho} |\varphi_t(y,s)|^q dy\Big)^{1/q}, \end{aligned}$$

where $|\tau Q_\rho| = \tau^N |Q_\rho| = \tau^N \rho^N$ is the measure of τQ_ρ. Thus, by using once more Hölder's inequality and since $|\tau J_\rho| = \tau\rho$,

$$\begin{aligned} \int_{\tau Q_\rho \times \tau J_\rho} |\varphi_t(y,s)| dy\, ds &\leq \tau^{\frac{N}{q'}} \rho^{\frac{N}{q'}} \int_{\tau J_\rho} ds \Big(\int_{Q_\rho} |\varphi_t(y,s)|^q dy\Big)^{1/q} \\ &\leq \tau^{\frac{N}{q'} + \frac{1}{p'}} \rho^{\frac{N}{q'} + \frac{1}{p'}} \Big(\int_{J_\rho} ds \Big(\int_{Q_\rho} |\varphi_t(y,s)|^q dy\Big)^{p/q}\Big)^{1/p} \\ &= \tau^{\frac{N}{q'} + \frac{1}{p'}} \rho^{\frac{N}{q'} + \frac{1}{p'}} \|\varphi_t\|_{L^p(J_\rho, L^q(Q_\rho))}. \end{aligned}$$

Altogether and since the norm of $L^p(J_\rho, L^q(Q_\rho))$ is majorized by the norm of $\mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)$, this yields $\int_{\tau Q_\rho\times\tau J_\rho}|\varphi_t(y,s)|dyds \le \tau^{\frac{N}{q'}+\frac{1}{p'}}\rho^{\frac{N}{q'}+\frac{1}{p'}}\|\varphi\|_{\mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)}$.

By the same procedure,

$$\int_{\tau Q_\rho\times\tau J_\rho}|\varphi_{x_j}(y,s)|dyds \le \tau^{\frac{N}{q'}+\frac{1}{p'}}\rho^{\frac{N}{q'}+\frac{1}{p'}}\|\varphi\|_{\mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)}$$

for $1\le j\le N$. Thus, by substitution into (2.3) and a straightforward calculation,

$$|\overline{\varphi}-\varphi(0)| \le C\rho^{\frac{1}{p'}-\frac{N}{q}}\|\varphi\|_{\mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)}, \tag{2.4}$$

where $C := \frac{p'q(N+1)}{q-Np'}$.

By translation invariance, (2.4) remains valid -with the same constant C- when $\varphi(0)$ is replaced by any $\varphi(x,t)$, provided that $\overline{\varphi}$ now denotes the average of φ over $Q_\rho\times J_\rho$ and $J_\rho\subset\mathbb{R}$ is any interval of length ρ containing t while $Q_\rho\subset\mathbb{R}^N$ is any cube with side ρ containing x. In particular, by choosing $\rho=1$ and noticing that, in this case (from Hölder's inequality) $\overline{\varphi}\le\|\varphi\|_{L^1(Q_1\times J_1)}\le\|\varphi\|_{L^p(J_1,L^q(Q_1))}\le\|\varphi\|_{L^p(\mathbb{R},L^q(\mathbb{R}^N))}\le\|\varphi\|_{\mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)}$, we obtain $|\varphi(x,t)|\le C\|\varphi\|_{\mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)}$ after changing C in (2.4) into $C+1$. This proves (2.1).

Next, arbitrary pairs (x,t) and (y,s) in $\mathbb{R}^{N+1}$ are contained in the same cube $Q_\rho\times J_\rho$ with side $\rho=|(x,t)-(y,s)|$. Thus, (2.4) holds with $\varphi(0)$ replaced by $\varphi(x,t)$ or $\varphi(y,s)$ (with the same $\overline{\varphi}$), which implies

$$|\varphi(x,t)-\varphi(y,s)| \le 2C|(x,t)-(y,s)|^{\frac{1}{p'}-\frac{N}{q}}\|\varphi\|_{\mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)}.$$

This proves (2.2) after changing C in (2.4) into $2C$. $\square$

It should be obvious that Lemmas 2.1 and 2.2 imply Theorem 1.1 for $\mathcal{V}^{p,q}(\mathbb{R},\mathbb{R}^N)$.

3. Proof of Theorem 1.1

We shall rely on the following extension property:

Lemma 3.1. *Let $p,q\in[1,\infty)$. There is a bounded linear operator $\mathcal{E}:\mathcal{V}^{p,q}(I,\Omega)\to\mathcal{V}^{p,q}(I,\mathbb{R}^N)$ such that $(\mathcal{E}u)(x,t)=u(x,t)$ for a.e. $(x,t)\in\Omega\times I$.*

Proof. From the assumptions about Ω and by the Stein extension theorem ([1], [22]), there is an extension operator $E\in\mathcal{L}(L^q(\Omega),L^q(\mathbb{R}^N))$ mapping $W^{1,q}(\Omega)$ into $W^{1,q}(\mathbb{R}^N)$ and such that $E\in\mathcal{L}(W^{1,q}(\Omega),W^{1,q}(\mathbb{R}^N))$. Given $u\in L^p(I,L^q(\Omega))$, set

$$(\mathcal{E}u)(\cdot,t) := Eu(\cdot,t), \quad \text{a.e. } t\in I,$$

so that $\mathcal{E}u: I\to L^q(\mathbb{R}^N)$ is strongly measurable and that $\|(\mathcal{E}u)(\cdot,t)\|_{L^q(\mathbb{R}^N)}\le C_0\|u(\cdot,t)\|_{L^q(\Omega)}$ for a.e. $t\in I$, where $C_0:=\|E\|_{\mathcal{L}(L^q(\Omega),L^q(\mathbb{R}^N))}$ is independent of u and t. Thus, $\mathcal{E}u\in L^p(I,L^q(\mathbb{R}^N))$ and $\|\mathcal{E}u\|_{L^p(I,L^q(\mathbb{R}^N))}\le C_0\|u\|_{L^p(I,L^q(\Omega))}$.

As noted in the Introduction, u is measurable on $\Omega\times I$ and $\mathcal{E}u$ is measurable on $\mathbb{R}^N\times I$. As a result, $v(x,t):=(\mathcal{E}u)(x,t)-u(x,t)$ is measurable on $\Omega\times I$. Since also $v(x,t)=0$ for $t\notin S$ and $x\notin\Sigma_t$ where both $S\subset I$ and $\Sigma_t\subset\Omega$ have measure 0, it follows from Tonelli's theorem that $v=0$ a.e. in $\Omega\times I$, so that $(\mathcal{E}u)(x,t)=u(x,t)$ for a.e. $(x,t)\in\Omega\times I$.

If now $u\in L^p(I,W^{1,q}(\Omega))$, then $Eu(\cdot,t)\in W^{1,q}(\mathbb{R}^N)$, i.e., $(\mathcal{E}u)(\cdot,t)\in W^{1,q}(\mathbb{R}^N)$ and $\|(\mathcal{E}u)(\cdot,t)\|_{W^{1,q}(\mathbb{R}^N)}\le C_1\|u(\cdot,t)\|_{W^{1,q}(\Omega)}$ where $C_1:=\|E\|_{\mathcal{L}(W^{1,q}(\Omega),W^{1,q}(\mathbb{R}^N))}$, so that $\mathcal{E}u\in L^p(I,W^{1,q}(\mathbb{R}^N))$ with $\|\mathcal{E}u\|_{L^p(I,W^{1,q}(\mathbb{R}^N))}\le C_1\|u\|_{L^p(I,W^{1,q}(\Omega))}$. This shows that $\mathcal{E} \in \mathcal{L}(L^p(I, L^q(\Omega)), L^p(I, L^q(\mathbb{R}^N)))$ is an extension operator continuously mapping $L^p(I, W^{1,q}(\Omega))$ into $L^p(I, W^{1,q}(\mathbb{R}^N))$.

In particular, if $u \in \mathcal{V}^{p,q}(I, \Omega)$, then $\mathcal{E}u \in L^p(I, W^{1,q}(\mathbb{R}^N))$ and

$$\|\mathcal{E}u\|_{L^p(I,W^{1,q}(\mathbb{R}^N))} \leq C_1 \|u\|_{\mathcal{V}^{p,q}(I,\Omega)}. \tag{3.1}$$

We now show that $\mathcal{E}u \in W^{1,p}(I, L^q(\mathbb{R}^N))$. First, we claim that

$$\frac{d}{dt}(\mathcal{E}u) = \mathcal{E}\frac{du}{dt}, \tag{3.2}$$

where the left-hand side is the derivative of $\mathcal{E}u$ in the sense of distributions on I with values in $L^q(\mathbb{R}^N)$. The right-hand side is defined since $\frac{du}{dt} \in L^p(I, L^q(\Omega))$. For the proof of (3.2), let $\psi \in C_0^\infty(I)$. Then,

$$\langle \frac{d}{dt}(\mathcal{E}u), \psi \rangle = -\langle \mathcal{E}u, \psi' \rangle = -\int_I \psi'(t)(\mathcal{E}u)(\cdot, t)dt \in L^q(\mathbb{R}^N).$$

Now, $\psi'(t)(\mathcal{E}u)(\cdot, t) = \psi'(t)Eu(\cdot, t) = E\left(\psi'(t)u(\cdot, t)\right)$ by the linearity of E. Thus,

$$\langle \frac{d}{dt}(\mathcal{E}u), \psi \rangle = -\int_I E\left(\psi'(t)u(\cdot, t)dt\right) = -E\Big(\int_I \psi'(t)u(\cdot, t)dt\Big),$$

where the second equality follows from the Bochner integral commuting with bounded linear operators ([14, p. 153]). Next, $\int_I \psi'(t)u(\cdot, t)dt = -\int_I \psi(t)\frac{du}{dt}(\cdot, t)dt$ since $u \in W^{1,p}(I, L^q(\Omega))$, so that

$$\langle \frac{d}{dt}(\mathcal{E}u), \psi \rangle = E\Big(\int_I \psi(t)\frac{du}{dt}(\cdot, t)dt\Big).$$

The same linearity and commutativity properties yield

$$\langle \frac{d}{dt}(\mathcal{E}u), \psi \rangle = \int_I \psi(t) E\Big(\frac{du}{dt}(\cdot, t)\Big)dt = \int_I \psi(t)\Big(\mathcal{E}\frac{du}{dt}\Big)(\cdot, t)dt.$$

This proves (3.2).

By (3.2) and the continuity of $\mathcal{E} : L^p(I, L^q(\Omega)) \to L^p(I, L^q(\mathbb{R}^N))$, it follows that $\frac{d}{dt}(\mathcal{E}u) \in L^p(I, L^q(\mathbb{R}^N))$ and that $\|\frac{d}{dt}\mathcal{E}u\|_{L^p(I,L^q(\mathbb{R}^N))} \leq C_0\|\frac{du}{dt}\|_{L^p(I,L^q(\Omega))}$. Thus, $\|\mathcal{E}u\|_{W^{1,p}(I,L^q(\mathbb{R}^N))} \leq C_0\|u\|_{W^{1,p}(I,L^q(\Omega))} \leq C_0\|u\|_{\mathcal{V}^{p,q}(I,\Omega)}$. Together with (3.1), we obtain that $\mathcal{E}u \in \mathcal{V}^{p,q}(I, \mathbb{R}^N)$ with $\|\mathcal{E}u\|_{\mathcal{V}^{p,q}(I,\mathbb{R}^N)} \leq (C_1 + C_0)\|u\|_{\mathcal{V}^{p,q}(I,\Omega)}$. This completes the proof. □

End of the proof of Theorem 1.1. For $u \in \mathcal{V}^{p,q}(I, \Omega)$, let $\mathcal{E}u$ be the extension to $\mathcal{V}^{p,q}(I, \mathbb{R}^N)$ obtained in Lemma 3.1. If $I \neq \mathbb{R}$ is infinite, a reflection of $\mathcal{E}u$ about the endpoint of I yields a (bounded, linear) extension of u to $\mathcal{V}^{p,q}(\mathbb{R}, \mathbb{R}^N)$. If $I = (a, b)$ is bounded, a first reflection about a followed by a reflection about b and multiplication by a smooth function with compact support and equal to 1 on $[a, b]$ produces the same result. Thus, in all cases, we obtain a bounded linear extension $\widetilde{\mathcal{E}} : \mathcal{V}^{p,q}(I, \Omega) \to \mathcal{V}^{p,q}(\mathbb{R}, \mathbb{R}^N)$. It is then obvious that the first part of Theorem 1.1 is implied by the same result for $\mathcal{V}^{p,q}(\mathbb{R}, \mathbb{R}^N)$ proved in Section 2.

If I and Ω are bounded, the compactness of the embedding $\mathcal{V}^{p,q} \hookrightarrow C^{0,\alpha}(\overline{\Omega \times I})$ when $0 \leq \alpha < \frac{1}{p'} - \frac{N}{q}$ follows from the well-known properties of Hölder spaces on bounded domains; see e.g. [1, p.12]. □

Various other embedding theorems can be deduced from Theorem 1.1, for example that $W^{2,p}(I, L^q(\Omega)) \cap W^{1,p}(I, W^{2,q}(\Omega)) \hookrightarrow C^{1,\frac{1}{p'}-\frac{N}{q}}(\overline{\Omega \times I})$ under the same condition $q > Np'$. This follows from $u, u_t, u_{x_j} \in \mathcal{V}^{p,q}(I,\Omega), 1 \leq j \leq N$ and Theorem 1.1 together with elementary properties of distributions.

For full disclosure, it should be mentioned that some embedding theorems for anisotropic Sobolev spaces with mixed norm in [6], [7], closely related to Theorem 1.1, cannot be obtained by the same simple arguments. This happens when $I \times \Omega$ satisfies an "l -horn condition" with $l = (l_1, \dots, l_{N+1})$ and not all the l_k are the same, which however places quite stringent restrictions about the geometry of Ω. For example, if $\Omega = J$ is an interval, $W^{1,p}(I, L^q(J)) \cap L^p(I, W^{2,q}(J)) \hookrightarrow C^0(\overline{J \times I})$ (and the embedding is compact if I and J are bounded) if $q \geq 1$ and $q > \frac{p'}{2}$, because rectangles satisfy a strong $(2,1)$-horn condition [1] ([6, p. 155]). Theorem 1.1 cannot take advantage of the fact that $W^{1,q}(J)$ is replaced by $W^{2,q}(J)$ and so yields the same result (plus Hölder continuity) only when $q > p'$.

Finally, Theorem 1.1 can be generalized when I is replaced by an open subset ω of $\mathbb{R}^M$ satisfying a strong local Lipschitz condition, provided that $\frac{M}{p} + \frac{N}{q} < 1$. The Hölder exponent becomes $1 - \frac{M}{p} - \frac{N}{q}$. The only significant difference occurs in the last step, to extend elements of $\mathcal{V}^{p,q}(\omega, \mathbb{R}^N)$ to $\mathcal{V}^{p,q}(\mathbb{R}^M, \mathbb{R}^N)$. This amounts to finding an extension from $L^p(\omega, W^{1,q}(\mathbb{R}^N))$ to $L^p(\mathbb{R}^M, W^{1,q}(\mathbb{R}^N))$, which is also an extension from $W^{1,p}(\omega, L^q(\mathbb{R}^N))$ to $W^{1,p}(\mathbb{R}^M, L^q(\mathbb{R}^N))$. When $\omega = \mathbb{R}^M_+$, this can be done by reflection. The general case reduces to $\mathbb{R}^M_+$ by localization and bi-Lipschitz change of coordinates.

4. Hölder continuity of the solutions of parabolic equations

Let $\Omega \subset \mathbb{R}^N$ denote once again an open subset satisfying the strong local Lipschitz condition and let A be the generator of a C_0 holomorphic semigroup $S(t)$ on $L^q(\Omega), q \in (1,\infty)$, whose domain $D(A)$ (equipped with the graph norm) is continuously embedded in $W^{1,q}(\Omega)$. This class includes many of the "classical" elliptic operators with various boundary conditions; see the comments after Theorem 4.1. If $u_0 \in L^q(\Omega)$, then

$$u(t) := S(t)u_0$$

is the unique solution of $u_t = Au$ such that $u(\cdot, 0) = u_0$. Basic semigroup theory yields only rather weak continuity properties of u near $t = 0$, that is, $u \in C^1([0,T], L^q(\Omega)) \cap C^0([0,T], D(A))$ if $u_0 \in D(A)$ and only $u \in C^0([0,T], L^q(\Omega))$ if $u_0 \notin D(A)$.

On the other hand, optimal time or space regularity in Sobolev spaces was proved by Di Blasio [13] when $L^q(\Omega)$ is replaced by a general Banach space E. In that setting, "space" regularity is accounted for by the real interpolation spaces between E and $D(A)$ while time regularity is measured by the E-valued Sobolev-Slobodeckii spaces on $(0,T)$.

By combining Di Blasio's results with Theorem 1.1, we shall obtain a more refined and stronger space-time Hölder regularity of u, provided that q is large enough. If X and Y are Banach spaces and $\theta \in (0,1), p \in [1,\infty]$, we denote by $(X,Y)_{\theta,p}$ and $[X,Y]_\theta$ the real and complex interpolation spaces between X and Y, respectively.

Theorem 4.1. *(i) Suppose that $q > N$ and $u_0 \in D(A)$. Then, $u \in C^{0,\alpha - \frac{N}{q}}([0,T] \times \overline{\Omega})$ for every $T > 0$ and every $0 \leq \alpha < 1$.*

[1] But most open subsets, including disks, *do not*.

(ii) Suppose that $p \in (1,\infty)$, $q \in (Np',\infty)$ and $u_0 \in (L^q(\Omega), D(A))_{1/p',p}$. Then, $u \in C^{0,\frac{1}{p'}-\frac{N}{q}}([0,T]\times\overline{\Omega})$ for every $T > 0$.

(iii) Suppose that $\theta \in (0,1)$, $q \in \left(\frac{N}{\theta},\infty\right)$ and $u_0 \in [L^q(\Omega), D(A)]_\theta$. Then, $u \in C^{0,\alpha-\frac{N}{q}}([0,T]\times\overline{\Omega})$ for every $T > 0$ and every $0 \le \alpha < \theta$.

Proof. (i) First, $u \in C^1([0,T], L^q(\Omega)) \cap C^0([0,T], D(A)) \subset \mathcal{V}^{p,q}((0,T),\Omega)$ for every $p \in (1,\infty)$ (because $D(A) \hookrightarrow W^{1,q}(\Omega)$). Next, since $q > N$ and $\alpha < 1$, choose p large enough that $q > Np'$ and $\alpha \le \frac{1}{p'}$ and use Theorem 1.1.

(ii) From [13, Theorem 4 and Theorem 9] (see also [10, Theorem 3.4.2]) the hypothesis $u_0 \in (L^q(\Omega), D(A))_{\frac{1}{p'},p}$ implies $u \in W^{1,p}((0,T), L^q(\Omega)) \cap L^p((0,T), D(A))$, so that $u \in \mathcal{V}^{p,q}((0,T),\Omega)$ and the conclusion follows from Theorem 1.1.

(iii) It is shown in [8, Theorem 4.7.1] that $[L^q(\Omega), D(A)]_\theta \subset (L^q(\Omega), D(A))_{\theta,\infty}$. Furthermore, $(L^q(\Omega), D(A))_{\theta,\infty} \subset (L^q(\Omega), D(A))_{\theta-\varepsilon,p}$ for every $\varepsilon \in (0,\theta)$ and every $p \in [1,\infty]$ since $D(A) \subset L^q(\Omega)$. Thus, if $\varepsilon \in (0,\theta)$ is chosen small enough that $q > \frac{N}{\theta-\varepsilon}$ and that $\alpha \le \theta - \varepsilon$, it suffices to use (ii) with $p = \frac{1}{1-\theta+\varepsilon}$. $\square$

Evidently, (i) is not directly implied by $u \in C^1([0,T], L^q(\Omega)) \cap C^0([0,T], D(A))$. Under the assumptions of (ii), Di Blasio proves the "mixed" regularity [13, Theorem 14] $u \in W^{\varepsilon,p}((0,T), (L^q(\Omega), D(A))_{1-\varepsilon,p})$ for every $\varepsilon \in (0,1)$. Since $D(A) \hookrightarrow W^{1,q}(\Omega)$, it follows that $u \in W^{\varepsilon,p}((0,T), B^{1-\varepsilon;q,p}(\Omega))$, but this does not yield (ii) or (iii). Indeed, by the classical embedding theorems, $u \in C^{0,\varepsilon-\frac{1}{p}}([0,T], C^{0,1-\varepsilon-\frac{N}{q}}(\overline{\Omega}))$ if $\frac{1}{p} < \varepsilon < 1 - \frac{N}{q}$ (hence, $q > Np'$). Thus, (ii) could be recovered if $\min\{\varepsilon - \frac{1}{p}, 1 - \varepsilon - \frac{N}{q}\} \ge \frac{1}{p'} - \frac{N}{q}$; i.e., $1 - \frac{N}{q} \le \varepsilon \le \frac{1}{p}$, but this requires $q \le Np'$ and therefore never happens.

A similar argument shows that no choice of $\varepsilon \in (0,1)$ yields $u \in C^{0,\alpha}(\overline{\Omega}\times[0,T])$ if $\alpha < \frac{1}{p'} - \frac{N}{q}$ is close enough to $\frac{1}{p'} - \frac{N}{q}$, whence (iii) cannot be proved that way either. Actually, even if $D(A) \hookrightarrow W^{k,q}(\Omega)$ with $k > 1$, (ii) and (iii) can be deduced from Di Blasio's mixed regularity result only when $q < kNp'$ (but then $q > \frac{Np'}{k}$ suffices, instead of $q > Np'$).

In connection with the above discussion, it is instructive to notice that if $\alpha \in (0,1)$, then $C^{0,\alpha}([0,T], C^{0,\alpha}(\overline{\Omega})) \subsetneq C^{0,\alpha}(\overline{\Omega}\times[0,T])$ (example: $(0,T) = \Omega$, $u(x,t) := (x+t)^\alpha$), so that membership to the latter space does not require membership to the former.

In the proof of (ii), we saw that the assumption $u_0 \in (L^q(\Omega), D(A))_{\frac{1}{p'},p}$ implies $u \in W^{1,p}((0,T), L^q(\Omega)) \cap L^p((0,T), D(A))$. The converse is true, from the very definition of real interpolation spaces by the trace method. Thus, the assumption $u_0 \in (L^q(\Omega), D(A))_{\frac{1}{p'},p}$ is also necessary for $u \in W^{1,p}((0,T), L^q(\Omega)) \cap L^p((0,T), D(A))$ to solve the nonautonomous problem

$$\begin{cases} u_t = Au + f, \\ u(\cdot,0) = u_0, \end{cases}$$

when $f \in L^p((0,T), L^q(\Omega))$. In turn, the existence and uniqueness of such a solution u is known under additional assumptions about A. See [11, Theorem 4.4] when $u_0 = 0$ (of course, the general case reduces to this case after adding $S(t)u_0$). It may be worth pointing out that the "L^p maximal regularity" in that theorem refers to the aforementioned existence and uniqueness (see Remark 4.2 below) and that, when $q \in (1,\infty)$, the space $L^q(\Omega)$ is a Banach space of class $\mathcal{HT}$ according to the terminology of [11] (more commonly called a UMD Banach space). If so, Theorem 4.1 remains true since the Hölder regularity is solely based on the fact that $u \in \mathcal{V}^{p,q}((0,T),\Omega)$.

Remark 4.2. In [11], L^p maximal regularity is defined (when $u_0 = 0$) by the condition that $u_t \in L^p((0,T), L^q(\Omega))$, which may seem weaker than stated above. However, since $T < \infty$, this implies at once that both $u(t) = \int_0^t u_t(s)ds$ and $Au = u_t - f$ are in $L^p((0,T), L^q(\Omega))$, so that $u \in L^p((0,T), D(A))$ when $D(A)$ is equipped with the graph norm and that $u \in W^{1,p}((0,T), L^q(\Omega))$. If $(0,T)$ is replaced by $(0,\infty)$, this remains true only if the invertibility of A is added to the assumptions of [11, Theorem 4.4].

If $-A$ is sectorial with bounded imaginary powers, part (iii) of Theorem 4.1 holds with $u_0 \in D((-A)^\theta)$ since it is known that $[L^q(\Omega), D(A)]_\theta = D((-A)^\theta)$ in this case [23, p. 103].

Everything can be extended to systems, that is, when $L^q(\Omega)$ is replaced by $(L^q(\Omega))^r, r \in \mathbb{N}$, provided that $D(A) \hookrightarrow (W^{1,q}(\Omega))^r$. Indeed, it suffices to use Theorem 1.1 componentwise.

Of course, the case when A is the realization of a linear elliptic differential operator with homogeneous boundary conditions is of special importance. If so, $D(A) = W^{2m,q}_{\mathcal{B}}(\Omega) := \{v \in W^{2m,q} : \mathcal{B}v = 0\}$ where $m \in \mathbb{N}$ and $\mathcal{B}$ is a system of boundary operators of order less than $2m$. The condition $D(A) \subset W^{1,q}(\Omega)$ always holds, but some regularity assumptions are needed for A to have some or all of the properties listed above. Recall also that $(L^q(\Omega), D(A))_{\theta,p}$ is often explicitly known in this case (Grisvard[2] [16, p. 63]), as is $[L^q(\Omega), D(A)]_\theta$ (Seeley [21]).

Hypotheses about Ω and about the coefficients that ensure L^p maximal regularity are spelled out in [11, Theorem 8.2]. A little more must be assumed for $-A$ to have bounded imaginary powers and even a bounded $\mathcal{H}^\infty$ calculus, which is stronger. Until fairly recently, this was mostly known for operators with constant coefficients, but it has now been proved for a broader class by Denk *et al.* [12].

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